

Research Article

The Optimal Convex Combination Bounds of Arithmetic and Harmonic Means for the Seiffert's Mean

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We find the greatest value α and least value β such that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$. Here $A(a, b)$, $H(a, b)$, and $P(a, b)$ denote the arithmetic, harmonic, and Seiffert's means of two positive numbers a and b , respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$ the Seiffert's mean $P(a, b)$ was introduced by Seiffert [1] as follows:

$$P(a, b) = \frac{a - b}{4 \arctan(\sqrt{a/b}) - \pi}. \quad (1.1)$$

Recently, the inequalities for means have been the subject of intensive research [2–11]. In particular, many remarkable inequalities for the Seiffert's mean can be found in the literature [12–17]. The Seiffert's mean $P(a, b)$ can be rewritten as (see [14, (2.4)])

$$P(a, b) = \frac{a - b}{2 \arcsin((a - b)/(a + b))}. \quad (1.2)$$

Let $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$, $H(a, b) = 2ab/(a + b)$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, and $L(a, b) = (b-a)/(\log b - \log a)$ be the arithmetic, geometric, harmonic, identric, and logarithmic means of two positive real numbers a and b with $a \neq b$. Then

$$\min\{a, b\} < H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b) < \max\{a, b\}. \quad (1.3)$$

In [1], Seiffert proved that

$$L(a, b) < P(a, b) < I(a, b) \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

Later, Seiffert [18] established that

$$\begin{aligned} P(a, b) &> \frac{3A(a, b)G(a, b)}{A(a, b) + 2G(a, b)}, \\ P(a, b) &> \frac{A(a, b)G(a, b)}{L(a, b)}, \\ P(a, b) &> \frac{2}{\pi}A(a, b) \end{aligned} \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

In [19], Sándor proved that

$$\begin{aligned} \frac{1}{2}[A(a, b) + G(a, b)] &< P(a, b) < \sqrt{A(a, b)}\sqrt{\frac{1}{2}[A(a, b) + G(a, b)]}, \\ A^{2/3}(a, b)G^{1/3}(a, b) &< P(a, b) < \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) \end{aligned} \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$.

The following bounds for the Seiffert's mean $P(a, b)$ in terms of the power mean $M_r(a, b) = ((a^r + b^r)/2)^{1/r}$ ($r \neq 0$) were presented by Jagers in [17]:

$$M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b) \quad (1.7)$$

for all $a, b > 0$ with $a \neq b$.

Hästö [13] found the sharp lower power bound for the Seiffert's mean as follows:

$$M_{\log 2 / \log \pi}(a, b) < P(a, b) \quad (1.8)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to find the greatest value α and the least value β such that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Main Result

Theorem 2.1. *The double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.*

Proof. Firstly, we prove that

$$P(a, b) < \frac{5}{6}A(a, b) + \frac{1}{6}H(a, b), \quad (2.1)$$

$$P(a, b) > \frac{2}{\pi}A(a, b) + \left(1 - \frac{2}{\pi}\right)H(a, b) \quad (2.2)$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in \{5/6, 2/\pi\}$. Then (1.1) leads to

$$\begin{aligned} & P(a, b) - [pA(a, b) + (1 - p)H(a, b)] \\ &= \frac{b[p(t^2 + 1)^2 + 4(1 - p)t^2]}{2(t^2 + 1)(4 \arctan t - \pi)} \left[\frac{2(t^4 - 1)}{pt^4 + (4 - 2p)t^2 + p} - 4 \arctan t + \pi \right]. \end{aligned} \quad (2.3)$$

Let

$$f(t) = \frac{2(t^4 - 1)}{pt^4 + (4 - 2p)t^2 + p} - 4 \arctan t + \pi. \quad (2.4)$$

Then simple computations lead to

$$\lim_{t \rightarrow 1} f(t) = 0, \quad (2.5)$$

$$\lim_{t \rightarrow +\infty} f(t) = \frac{2}{p} - \pi, \quad (2.6)$$

$$f'(t) = \frac{4(t - 1)^2}{(t^2 + 1)[pt^4 + (4 - 2p)t^2 + p]^2} g(t), \quad (2.7)$$

where

$$\begin{aligned} g(t) &= -p^2 t^6 + (-2p^2 - 2p + 4)t^5 + (p^2 - 12p + 8)t^4 \\ &\quad + (4p^2 - 20p + 16)t^3 + (p^2 - 12p + 8)t^2 \\ &\quad + (-2p^2 - 2p + 4)t - p^2. \end{aligned} \quad (2.8)$$

We divide the proof into two cases.

Case 1. If $p = 5/6$, then it follows from (2.8) that

$$g(t) = -\frac{1}{36} \left(25t^4 + 16t^3 + 54t^2 + 16t + 25 \right) (t-1)^2 < 0 \quad (2.9)$$

for $t > 1$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and (2.7) together with (2.9).

Case 2. If $p = 2/\pi$, then from (2.8) we have

$$g(1) = 8(5 - 6p) = 8 \left(5 - \frac{12}{\pi} \right) > 0, \quad (2.10)$$

$$\lim_{t \rightarrow +\infty} g(t) = -\infty, \quad (2.11)$$

$$\begin{aligned} g'(t) &= -6p^2 t^5 + (-10p^2 - 10p + 20)t^4 + (4p^2 - 48p + 32)t^3 \\ &\quad + (12p^2 - 60p + 48)t^2 + (2p^2 - 24p + 16)t - 2p^2 - 2p + 4, \end{aligned} \quad (2.12)$$

$$g'(1) = 24(5 - 6p) = 24 \left(5 - \frac{12}{\pi} \right) > 0, \quad (2.13)$$

$$\lim_{t \rightarrow +\infty} g'(t) = -\infty, \quad (2.14)$$

$$\begin{aligned} g''(t) &= -30p^2 t^4 + (-40p^2 - 40p + 80)t^3 + (12p^2 - 144p + 96)t^2 \\ &\quad + (24p^2 - 120p + 96)t + 2p^2 - 24p + 16, \end{aligned} \quad (2.15)$$

$$g''(1) = 8(36 - 41p - 4p^2) = 8 \left(36 - \frac{82}{\pi} - \frac{16}{\pi^2} \right) > 0, \quad (2.16)$$

$$\lim_{t \rightarrow +\infty} g''(t) = -\infty, \quad (2.17)$$

$$\begin{aligned} g'''(t) &= -120p^2 t^3 + (-120p^2 - 120p + 240)t^2 \\ &\quad + (24p^2 - 288p + 192)t + 24p^2 - 120p + 96, \end{aligned} \quad (2.18)$$

$$g'''(1) = 48(11 - 11p - 4p^2) = 48 \left(11 - \frac{22}{\pi} - \frac{16}{\pi^2} \right) > 0, \quad (2.19)$$

$$\lim_{t \rightarrow +\infty} g'''(t) = -\infty, \quad (2.20)$$

$$g^{(4)}(t) = -360p^2 t^2 + (-240p^2 - 240p + 480)t + 24p^2 - 288p + 192, \quad (2.21)$$

$$g^{(4)}(1) = 48(14 - 11p - 12p^2) = 48 \left(14 - \frac{22}{\pi} - \frac{48}{\pi^2} \right) > 0, \quad (2.22)$$

$$\lim_{t \rightarrow +\infty} g^{(4)}(t) = -\infty, \quad (2.23)$$

$$g^{(5)}(t) = -720p^2t - 240p^2 - 240p + 480, \quad (2.24)$$

$$g^{(5)}(1) = 240\left(2 - p - 4p^2\right) = 240\left(2 - \frac{2}{\pi} - \frac{16}{\pi^2}\right) < 0. \quad (2.25)$$

From (2.24) and (2.25) we clearly see that $g^{(5)}(t) < 0$ for $t \geq 1$, hence $g^{(4)}(t)$ is strictly decreasing in $[1, +\infty)$. It follows from (2.22) and (2.23) together with the monotonicity of $g^{(4)}(t)$ that there exists $\lambda_1 > 1$ such that $g^{(4)}(t) > 0$ for $t \in [1, \lambda_1)$ and $g^{(4)}(t) < 0$ for $t \in (\lambda_1, +\infty)$, hence $g'''(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$.

From (2.19) and (2.20) together with the monotonicity of $g'''(t)$ we know that there exists $\lambda_2 > 1$ such that $g'''(t) > 0$ for $t \in [1, \lambda_2)$ and $g'''(t) < 0$ for $t \in (\lambda_2, +\infty)$, hence, $g''(t)$ is strictly increasing in $[1, \lambda_2]$ and strictly decreasing in $[\lambda_2, \infty)$.

From (2.16) and (2.17) together with the monotonicity of $g''(t)$ we clearly see that there exists $\lambda_3 > 1$ such that $g'(t)$ is strictly increasing in $[1, \lambda_3]$ and strictly decreasing in $[\lambda_3, \infty)$. It follows from (2.13) and (2.14) together with the monotonicity of $g'(t)$ that there exists $\lambda_4 > 1$ such that $g(t)$ is strictly increasing in $[1, \lambda_4]$ and strictly decreasing in $[\lambda_4, \infty)$. Then (2.7), (2.10) and (2.11) imply that there exists $\lambda_5 > 1$ such that $f(t)$ is strictly increasing in $(1, \lambda_5]$ and strictly decreasing in $[\lambda_5, \infty)$.

Note that (2.6) becomes

$$\lim_{t \rightarrow +\infty} f(t) = 0 \quad (2.26)$$

for $p = 2/\pi$.

It follows from (2.5) and (2.26) together with the monotonicity of $f(t)$ that

$$f(t) > 0 \quad (2.27)$$

for $t > 1$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with (2.27).

Secondly, we prove that $(5/6)A(a, b) + (1/6)H(a, b)$ is the best possible upper convex combination bound of arithmetic and harmonic means for the Seiffert's mean $P(a, b)$.

For any $t > 1$ and $\beta \in \mathbb{R}$, from (1.1) we have

$$\begin{aligned} P(1, t^2) - \left[\beta A(1, t^2) + (1 - \beta)H(1, t^2) \right] &= \frac{t^2 - 1}{4 \arctan t - \pi} - \frac{\beta}{2}(1 + t^2) - 2(1 - \beta) \frac{t^2}{1 + t^2} \\ &= \frac{h(t)}{(4 \arctan t - \pi)(1 + t^2)}, \end{aligned} \quad (2.28)$$

where

$$h(t) = (t^4 - 1) - \frac{\beta}{2}(t^2 + 1)^2(4 \arctan t - \pi) - 2(1 - \beta)t^2(4 \arctan t - \pi). \quad (2.29)$$

It follows from (2.29) that

$$h(1) = h'(1) = h''(1) = 0, \quad (2.30)$$

$$h'''(1) = 4(5 - 6\beta). \quad (2.31)$$

If $\beta < 5/6$, then (2.31) leads to

$$h'''(1) > 0. \quad (2.32)$$

From (2.32) and the continuity of $h'''(t)$ we clearly see that there exists $\delta = \delta(\beta) > 0$ such that

$$h'''(t) > 0 \quad (2.33)$$

for $t \in [1, 1 + \delta)$. Then (2.30) and (2.33) imply that

$$h(t) > 0 \quad (2.34)$$

for $t \in (1, 1 + \delta)$.

Therefore, $P(1, t^2) > \beta A(1, t^2) + (1 - \beta)H(1, t^2)$ for $t \in (1, 1 + \delta)$ follows from (2.28) and (2.34).

Finally, we prove that $(2/\pi)A(a, b) + (1 - 2/\pi)H(a, b)$ is the best possible lower convex combination bound of arithmetic and harmonic means for the Seiffert's mean $P(a, b)$.

For $\alpha > 2/\pi$, then from (1.1) one has

$$\lim_{x \rightarrow +\infty} \frac{\alpha A(1, x) + (1 - \alpha)H(1, x)}{P(1, x)} = \frac{\pi}{2}\alpha > 1. \quad (2.35)$$

Inequality (2.35) implies that for any $\alpha > 2/\pi$ there exists $X = X(\alpha) > 1$ such that $\alpha A(1, x) + (1 - \alpha)H(1, x) > P(1, x)$ for $x \in (X, +\infty)$. \square

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