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Research Article

Existence and Asymptotic Behavior of Global Solutions for a Class of Nonlinear Higher-Order Wave Equation

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The initial boundary value problem for a class of nonlinear higher-order wave equation with damping and source term $u_{tt} + Au + a|u_t|^{p-1}u_t = b|u|^{q-1}u$ in a bounded domain is studied, where $A = (-\Delta)^m$, $m \geq 1$ is a nature number, and $a, b > 0$ and $p, q > 1$ are real numbers. The existence of global solutions for this problem is proved by constructing the stable sets and shows the asymptotic stability of the global solutions as time goes to infinity by applying the multiplier method.

1. Introduction

In this paper we consider the existence and asymptotic behavior of global solutions for the initial boundary problem of the nonlinear higher-order wave equation with nonlinear damping and source term:

$$u_{tt} + Au + a|u_t|^{p-1}u_t = b|u|^{q-1}u, \quad x \in \Omega, \ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$D^{\alpha}u(x,t) = 0, \qquad |\alpha| \leq m-1, \quad x \in \partial\Omega, \ t \geq 0, \tag{1.3}$$

where $A = (-\Delta)^m$, $m \geq 1$ is a nature number, $a, b > 0$ and $p, q > 1$ are real numbers, Ω is a bounded domain of R^N with smooth boundary $\partial\Omega$, Δ is the Laplace operator, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $|\alpha| = \sum_{i=1}^{N} |\alpha_i|$, $D^{\alpha} = \prod_{i=1}^{N}(\partial^{\alpha_i}/\partial x_i^{\alpha_i})$, $x = (x_1, x_2, \ldots, x_N)$.

When $m = 1$, the existence and uniqueness, as well as decay estimates, of global solutions and blow up of solutions for the initial boundary value problem and Cauchy problem of (1.1) have been investigated by many people through various approaches and assumptive conditions [1–8]. Rammaha [9] deals with wave equations that feature two competing forces and analyzes the influence of these forces on the long-time behavior of solutions. Barbu et al. [10] study the following initial-boundary value problem:

$$\begin{gathered} u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,T) \equiv Q_T, \\ u(x,0) = u_0(x) \in H_0^1(\Omega), \quad u_t(x,0) = u_1(x) \in L^2(\Omega), \\ u = 0, \quad (x,t) \in \Gamma \times (0,T), \end{gathered} \tag{1.4}$$

where Ω is a bounded domain in R^N with a smooth boundary Γ, $j(s)$ is a C^1 convex, real value function defined on R, and j' denotes the derivative of j. They prove that every generalized solution to the above problem and additional regularity blows up in finite time, whenever the exponent p is greater than the critical value $k + m$, and the initial energy is negative.

For the following model of semilinear wave equation with a nonlinear boundary dissipation and nonlinear boundary(interior) sources,

$$\begin{gathered} u_{tt} = \Delta u + f(u), \quad (x,t) \in \Omega \times (0,\infty), \\ \partial_\nu u + u + g(u_t) = h(u), \quad (x,t) \in \Gamma \times (0,\infty), \\ u(0) = u_0(x) \in H^1(\Omega), \quad u_t(0) \in u_1(x) \in L^2(\Omega), \end{gathered} \tag{1.5}$$

where the operators $f(u), g(u_t)$, and $h(u)$ are Nemytskii operators associated with scalar, continuous functions $f(s), g(s)$, and $h(s)$ defined for $s \in R$. The function $g(s)$ is assumed monotone. The paper [11, 12] proves the existence and uniqueness of both local and global solutions of this system on the finite energy space and derive uniform decay rates of the energy when $t \to \infty$.

When $m = 2$, Guesmia [13] considered the equation

$$u_{tt} + \Delta^2 u + q(x)u + g(u_t) = 0, \quad x \in \Omega, \ t > 0 \tag{1.6}$$

with initial boundary value conditions (1.2) and (1.3), where g is a continuous and increasing function with $g(0) = 0$, and $q : \Omega \to [0, +\infty)$ is a bounded function. He prove a global existence and a regularity result of the problem (1.6), (1.2), and (1.3). Under suitable growth conditions on g, he also established decay results for weak and strong solutions. Precisely, In [13], Guesmia showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Results similar to the above system, coupled with a semilinear wave equation, have been established by Guesmia [14]. As $q(x)u + g(u_t)$ in (1.6) is replaced by $\Delta^2 u_t + \Delta g(\Delta u)$. Aassila and Guesmia [15] have obtained a exponential decay theorem through the use of an important lemma of Komornik [16]. Moreover, Messaoudi [17] sets up an existence result of this problem and shows that the solution continues to exist globally if $p \geq q$; however, it blows up in finite time if $p < q$.

Nakao [18] has used Galerkin method to present the existence and uniqueness of the bounded solutions, and periodic and almost periodic solutions to the problem (1.1)–(1.3) as the dissipative term is a linear function νu_t. Nakao and Kuwahara [19] studied decay estimates of global solutions to the problem (1.1)–(1.3) by using a difference inequality when the dissipative term is a degenerate case $a(x)u_t$. When there is no dissipative term in (1.1), Brenner and von Wahl [20] proved the existence and uniqueness of classical solutions to the initial boundary problem for (1.1) in Hilbert space. Pecher [21] investigated the existence and uniqueness of Cauchy problem for (1.1) by the use of the potential well method due to Payne and Sattinger [6] and Sattinger [22].

When $a = 0$, for the semilinear higher-order wave equation (1.1), Wang [23] shows that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s. Miao [24] obtains the scattering theory at low energy using time-space estimates and nonlinear estimates. Meanwhile, he also gives the global existence and uniqueness of solutions under the condition of low energy.

The proof of global existence for problem (1.1)–(1.3) is based on the use of the potential well theory [6, 22]. See also Todorova [7, 25] for more recent work. And we study the asymptotic behavior of global solutions by applying the lemma of Komornik [16].

We adopt the usual notation and convention. Let $H^k(\Omega)$ denote the Sobolev space with the norm $\|u\|_{H^k(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}$, let $H_0^k(\Omega)$ denote the closure in $H^k(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_r$ the Lebesgue space $L^r(\Omega)$ norm and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, we write equivalent norm $\|A^{1/2}\cdot\|$ instead of $H_0^m(\Omega)$ norm $\|\cdot\|_{H_0^m(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants and may be different at each appearance.

This paper is organized as follows. In the next section, we will study the existence of global solutions of problem (1.1)–(1.3). Then in Section 3, we are devoted to the proof of decay estimate.

We conclude this introduction by stating a local existence result, which is known as a standard one (see [17]).

Theorem 1.1. *Suppose that $p, q > 1$ satisfies*

$$1 < q < +\infty, \quad N \leq 2m; \quad 1 < q \leq \frac{N}{N-2m}, \quad N > 2m, \tag{1.7}$$

$$1 < p < +\infty, \quad N \leq 2m; \quad 1 < p \leq \frac{N+2m}{N-2m}, \quad N > 2m, \tag{1.8}$$

and $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$, then there exists $T > 0$ such that the problem (1.1)–(1.3) *has a unique local solution $u(t)$ in the class*

$$u \in C\left([0,T); H_0^m(\Omega)\right), \quad u_t \in C\Big([0,T); L^2(\Omega)\Big) \cap L^{p+1}(\Omega \times [0,T)). \tag{1.9}$$

Theorem 1.2. *Under the assumptions in Theorem 1.1, if*

$$\sup_{0\leq t\leq T_{\max}} \left(\|u_t(t)\|^2 + \left\|A^{1/2}u(t)\right\|^2\right) < +\infty, \tag{1.10}$$

then $T_{\max} = +\infty$, *where* $[0, T_{\max}]$ *is the maximum time interval on which the solution* $u(x,t)$ *of problem* (1.1)–(1.3) *exists.*

Please notice that in [17], we can also construct the following space X_T in proving the existence of local solution by using contraction mapping principle:

$$X_T = \left\{ u \in C\left([0,T]; H_0^m(\Omega)\right),\ u_t \in C\Big([0,T]; L^2(\Omega)\Big) \right\}, \tag{1.11}$$

which is equipment with norm

$$\|u(t)\|_{X_T} = \sup_{0\le t\le T} \frac{1}{2}\left(\|u_t(t)\|^2 + \left\| A^{1/2} u(t) \right\|^2 \right). \tag{1.12}$$

Let $\varepsilon > 0$, and

$$X_{\varepsilon,T} = \left\{ u \in X_T : \|u\|_{X_T} \le \varepsilon \right\}. \tag{1.13}$$

We define a distance $d(u,v) = \|u - v\|_{X_T}$ on $X_{\varepsilon,T}$, and then $X_{\varepsilon,T}$ is a complete distance space. This show that, for small enough ε, there exists an unique fixed point on $X_{\varepsilon,T}$ and T only depends on ε. Therefore, with the standard extension method of solution, we obtain $T_{\max} = +\infty$ for

$$\sup_{0\le t\le T_{\max}} \left(\|u_t(t)\|^2 + \left\| A^{1/2} u(t) \right\|^2 \right) < +\infty. \tag{1.14}$$

Here we omit the detailed proof of extension.

2. The Global Existence

In order to state and prove our main results, we first define the following functionals:

$$\begin{aligned} I(u) = I(u(t)) &= \left\| A^{1/2} u(t) \right\|^2 - b\|u(t)\|_{q+1}^{q+1}, \\ J(u) = J(u(t)) &= \frac{1}{2}\left\| A^{1/2} u(t) \right\|^2 - \frac{b}{q+1}\|u(t)\|_{q+1}^{q+1}, \end{aligned} \tag{2.1}$$

and according to paper [18, 24] we put

$$d = \inf\left\{ \sup_{\lambda>0} J(\lambda u),\ u \in H_0^m(\Omega)/\{0\} \right\}. \tag{2.2}$$

Then, for the problem (1.1)–(1.3), we are able to define the stable set

$$W = \left\{ u \in H_0^m(\Omega), I(u) > 0 \right\} \cup \{0\}. \tag{2.3}$$

We denote the total energy related to (1.1) by

$$E(u(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left\|A^{1/2}u(t)\right\|^2 - \frac{b}{q+1}\|u(t)\|_{q+1}^{q+1} = \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) \tag{2.4}$$

for $u \in H_0^m(\Omega)$, $t \geq 0$, and $E(u(0)) = (1/2)\|u_1\|^2 + J(u_0)$ is the total energy of the initial data.

Lemma 2.1. *Let r be a number with $2 \leq r < +\infty$, $N \leq 2m$ or $2 \leq r \leq 2N/(N-2m)$, $N > 2m$. Then there is a constant C depending on Ω and r such that*

$$\|u\|_r \leq C\left\|A^{1/2}u\right\|, \quad \forall u \in H_0^m(\Omega). \tag{2.5}$$

Lemma 2.2. *Assume that $u \in H_0^m(\Omega)$; if (1.7) holds, then*

$$d = \frac{q-1}{2(q+1)}\frac{1}{\left(bC_*^{q+1}\right)^{2/(q-1)}} \tag{2.6}$$

is a positive constant, where C_* is the most optimal constant in Lemma 2.1, namely, $C_* = \sup(\|u\|_{q+1}/\|A^{1/2}u\|)$.

Proof. Since

$$J(\lambda u) = \frac{\lambda^2}{2}\left\|A^{1/2}u\right\|^2 - \frac{b\lambda^{q+1}}{q+1}\|u\|_{q+1}^{q+1}, \tag{2.7}$$

so, we get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\left\|A^{1/2}u\right\|^2 - b\lambda^q\|u\|_{q+1}^{q+1}. \tag{2.8}$$

Let $(d/d\lambda)J(\lambda u) = 0$, which implies that

$$\lambda_1 = b^{-1/(q-1)}\left(\frac{\|u\|_{q+1}^{q+1}}{\left\|A^{1/2}u\right\|^2}\right)^{-1/(q-1)}. \tag{2.9}$$

As $\lambda = \lambda_1$, an elementary calculation shows that

$$\frac{d^2}{d\lambda^2}J(\lambda u) < 0. \tag{2.10}$$

Thus, we have from Lemma 2.1 that

$$\begin{aligned}\sup_{\lambda\geq 0} J(\lambda u) &= J(\lambda_1 u) = \frac{q-1}{2(q+1)} b^{-2/(q-1)} \left(\frac{\|u\|_{q+1}}{\|A^{1/2}u\|} \right)^{-2(q+1)/(q-1)} \\ &\geq \frac{q-1}{2(q+1)} \frac{1}{b^{2/(q-1)}} C^{2(q+1)/(q-1)} > 0.\end{aligned} \tag{2.11}$$

We get from the definition of d

$$d = \frac{q-1}{2(q+1)} \frac{1}{\left(bC_*^{q+1}\right)^{2/(q-1)}} > 0. \tag{2.12}$$

□

Lemma 2.3. *Let $u(t)$ be a solution of the problem (1.1)–(1.3). Then $E(u(t))$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt} E(u(t)) = -a\|u_t(t)\|_{p+1}^{p+1}. \tag{2.13}$$

Proof. By multiplying (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt} E(u(t)) = -a\|u_t(t)\|_{p+1}^{p+1} \leq 0. \tag{2.14}$$

Therefore, $E(u(t))$ is a nonincreasing function on t. □

Theorem 2.4. *Suppose that (1.7) holds. If $u_0 \in W, u_1 \in L^2(\Omega)$ and the initial energy satisfies $E(u(0)) < d$, then $u \in W$, for each $t \in [0, T)$.*

Proof. Assume that there exists a number $t^* \in [0, T)$ such that $u(t) \in W$ on $[0, t^*)$ and $u(t^*) \notin W$. Then we have

$$I(u(t^*)) = 0, \qquad u(t^*) \neq 0. \tag{2.15}$$

Since $u(t) \in W$ on $[0, t^*)$, so it holds that

$$\begin{aligned}J(u(t)) &= \frac{1}{2}\left\|A^{1/2}u(t)\right\|^2 - \frac{b}{q+1}\|u(t)\|_{q+1}^{q+1} \\ &\geq \frac{1}{2}\left\|A^{1/2}u(t)\right\|^2 - \frac{1}{q+1}\left\|A^{1/2}u(t)\right\|^2 = \frac{q-1}{2(q+1)}\left\|A^{1/2}u(t)\right\|^2,\end{aligned} \tag{2.16}$$

it follows from $I(u(t^*)) = 0$ that

$$J(u(t^*)) = \frac{1}{2}\left\|A^{1/2}u(t^*)\right\|^2 - \frac{b}{q+1}\|u(t^*)\|_{q+1}^{q+1} = \frac{q-1}{2(q+1)}\left\|A^{1/2}u(t^*)\right\|^2, \tag{2.17}$$

and therefore, we have from (2.16) and (2.17) that

$$\left\| A^{1/2}u(t) \right\|^2 \le \frac{2(q+1)}{q-1} J(u(t)) \le \frac{2(q+1)}{q-1} E(u(t)) \le \frac{2(q+1)}{q-1} E(u(0)) \tag{2.18}$$

for all $t \in [0, t^*]$.

We obtain from Lemma 2.2 and $E(u(0)) < d$ that

$$E(u(0)) < \frac{q-1}{2(q+1)} \frac{1}{\left(bC_*^{q+1}\right)^{2/(q-1)}}, \tag{2.19}$$

which implies that

$$bC_*^{q+1} \left(\frac{2(q+1)}{q-1} E(u(0)) \right)^{(q-1)/2} < 1. \tag{2.20}$$

By exploiting Lemma 2.1, (2.18), and (2.20), we easily arrive at

$$\begin{aligned} b\|u\|_{q+1}^{q+1} &\le bC^{q+1} \left\| A^{1/2}u \right\|^{q+1} = bC^{q+1} \left\| A^{1/2}u \right\|^{q-1} \left\| A^{1/2}u \right\|^2 \\ &\le bC_*^{q+1} \left(\left(\frac{2(q+1)}{q-1} E(u(0)) \right)^{(q-1)/2} \right) \left\| A^{1/2}u \right\|^2 < \left\| A^{1/2}u \right\|^2 \end{aligned} \tag{2.21}$$

for all $t \in [0, t^*]$. Therefore, we obtain

$$I(u(t^*)) = \left\| A^{1/2}u(t^*) \right\|^2 - b\|u(t^*)\|_{q+1}^{q+1} > 0, \tag{2.22}$$

which contradicts (2.15). Thus, we conclude that $u(t) \in W$ on $[0, T)$. □

Theorem 2.5. *Assume that* (1.7) *and* (1.8) *hold,* $u(t)$ *is a local solution of problem* (1.1)–(1.3). *If* $u_0 \in W$, $u_1 \in L^2(\Omega)$, *and* $E(u(0)) < d$, *then the solution* $u(t)$ *is a global solution of problem* (1.1)–(1.3).

Proof. We obtain from (2.18) that

$$\begin{aligned} d > E(u(0)) \ge E(u(t)) &= \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) \\ &\ge \frac{1}{2}\|u_t(t)\|^2 + \frac{q-1}{2(q+1)} \left\| A^{1/2}u \right\|^2 \ge \frac{q-1}{2(q+1)} \left(\|u_t(t)\|^2 + \left\| A^{1/2}u \right\|^2 \right) \end{aligned} \tag{2.23}$$

Therefore,

$$\|u_t(t)\|^2 + \left\|A^{1/2}u\right\|^2 \le \frac{2(q+1)}{q-1}d < +\infty. \tag{2.24}$$

It follows from Theorem 1.2 that $u(x,t)$ is the global solution of problem (1.1)–(1.3). □

3. Decay Estimate

The following two lemmas play an important role in studying the decay estimate of global solutions for the problem (1.1)–(1.3).

Lemma 3.1 (see [16]). *Let $F : R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $\beta \ge 1$ and $A > 0$ such that*

$$\int_S^{+\infty} F(t)^{(\beta+1)/2} dt \le AF(S), \quad 0 \le S < +\infty, \tag{3.1}$$

then $F(t) \le CF(0)(1+t)^{-2/(\beta-1)}$, for all $t \ge 0$, if $\beta > 1$, and $F(t) \le CF(0)e^{-\omega t}$, for all $t \ge 0$, if $\beta = 1$, where C and ω are positive constants independent of $F(0)$.

Lemma 3.2. *If the hypotheses in Theorem 2.4 hold, then*

$$b\|u(t)\|_{q+1}^{q+1} \le (1-\theta)\left\|A^{1/2}u(t)\right\|^2, \quad \forall t \in [0,+\infty), \tag{3.2}$$

where

$$\theta = 1 - bC_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u(0))\right)^{(q-1)/2} > 0. \tag{3.3}$$

Moreover, one has

$$I(u(t)) \ge \theta\left\|A^{1/2}u(t)\right\|^2 \ge \frac{\theta}{1-\theta}b\|u(t)\|_{q+1}^{q+1}, \quad \forall t \in [0,+\infty). \tag{3.4}$$

Proof. We get from Lemma 2.1 and (2.23) that

$$\begin{aligned} b\|u\|_{q+1}^{q+1} &\le bC^{q+1}\left\|A^{1/2}u\right\|^{q+1} = bC^{q+1}\left\|A^{1/2}u\right\|^{q-1}\left\|A^{1/2}u\right\|^2 \\ &\le bC_*^{q+1}\left(\frac{2(q+1)}{q-1}E(u(0))\right)^{(q-1)/2}\left\|A^{1/2}u\right\|^2. \end{aligned} \tag{3.5}$$

Let

$$\theta = 1 - bC_*^{q+1}\left(\frac{2(q+1)}{p}E(u(0))\right)^{(q-1)/2}, \tag{3.6}$$

then we have from (2.20) that $0 < \theta < 1$. Thus, it follows that from (3.5)

$$b\|u\|_{q+1}^{q+1} \leq (1-\theta)\left\|A^{1/2}u\right\|^2. \tag{3.7}$$

Meanwhile, we conclude from (3.7) that

$$I(u) = \left\|A^{1/2}u\right\|^2 - b\|u\|_{q+1}^{q+1} \geq \left\|A^{1/2}u\right\|^2 - (1-\theta)\left\|A^{1/2}u\right\|^2 = \theta\left\|A^{1/2}u\right\|^2 \geq \frac{\theta b}{1-\theta}\|u\|_{q+1}^{q+1}. \tag{3.8}$$

This complete the proof of Lemma 3.2. □

Theorem 3.3. *If the hypotheses in Theorem 2.5 are valid, then the global solutions of problem* (1.1)–(1.3) *have the following asymptotic behavior:*

$$\lim_{t\to+\infty}\|u_t(t)\| = 0, \quad \lim_{t\to+\infty}\left\|A^{1/2}u(t)\right\| = 0. \tag{3.9}$$

Let $E(t) = E(u(t))$. If one can prove that the energy of the global solution satisfies the estimate

$$\int_S^T E(t)^{(p+1)/2}dt \leq ME(S) \tag{3.10}$$

for all $0 \leq S < T < +\infty$, then Theorem 3.3 will be proved by Lemma 3.1. The proof of Theorem 3.3 is composed of the following propositions.

Proposition 3.4. *Suppose that $u(x,t)$ is the global solutions of* (1.1)–(1.3), *then one has*

$$\begin{aligned}
&\int_S^T\int_\Omega E(t)^{(p-1)/2}\left(|u_t|^2 + \left|A^{1/2}u\right|^2 - b|u|^{q+1}\right)dx\,dt \\
&\quad\leq \int_S^T\int_\Omega E(t)^{(p-1)/2}\left[2|u_t|^2 - a|u_t|^{p-1}u_tu\right]dx\,dt \\
&\qquad + \frac{p-1}{2}\int_S^T\int_\Omega E(t)^{(p-3)/2}E'(t)uu_tdxdt + ME(S)^{(p+1)/2}
\end{aligned} \tag{3.11}$$

for all $0 \leq S < T < +\infty$.

Proof. Multiplying by $E(t)^{(p-1)/2}u$ on both sides of (1.1) and integrating over $\Omega \times [S,T]$, we obtain that

$$\int_S^T \int_\Omega E(t)^{(p-1)/2} u \left[u_{tt} + Au + a|u_t|^{p-1}u_t - bu|u|^{q-1} \right] dx\, dt = 0, \tag{3.12}$$

where $0 \leq S < T < +\infty$.

Since

$$\begin{aligned}\int_S^T \int_\Omega E(t)^{(p-1)/2} u u_{tt} dx\, dt &= \int_\Omega E(t)^{(p-1)/2} u u_t dx \bigg|_S^T \\ &- \int_S^T \int_\Omega E(t)^{(p-1)/2} |u_t|^2 dx\, dt - \frac{p-1}{2} \int_S^T \int_\Omega E(t)^{(p-3)/2} E'(t) u u_t dx\, dt,\end{aligned} \tag{3.13}$$

so, substituting (3.13) into the left-hand side of (3.12), we get that

$$\begin{aligned}&\int_S^T \int_\Omega E(t)^{(p-1)/2} \left(|u_t|^2 + \left| A^{1/2} u \right|^2 - b|u|^{q+1} \right) dx\, dt \\ &\quad= \int_S^T \int_\Omega E(t)^{(p-1)/2} \left[2|u_t|^2 - a|u_t|^{p-1} u_t u \right] dx\, dt \\ &\qquad+ \frac{p-1}{2} \int_S^T \int_\Omega E(t)^{(p-3)/2} E'(t) u u_t dx\, dt - \int_\Omega E(t)^{(p-1)/2} u u_t dx \bigg|_S^T .\end{aligned} \tag{3.14}$$

Next we observe from (2.23) that

$$\begin{aligned}\left| - \int_\Omega E(t)^{(p-1)/2} u u_t dx \bigg|_S^T \right| &\leq E(t)^{(p-1)/2} \left(\frac{1}{2} \|u\|^2 + \frac{1}{2} \|u_t\|^2 \right) \bigg|_S^T \\ &\leq E(t)^{(p-1)/2} \left(\frac{C^2}{2} \left\| A^{1/2} u \right\|^2 + \frac{1}{2} \|u_t\|^2 \right) \bigg|_S^T \\ &\leq E(t)^{(p-1)/2} \left(\frac{(q+1)C^2}{q-1} \frac{q-1}{2(q+1)} \left\| A^{1/2} u \right\|^2 + \frac{1}{2} \|u_t\|^2 \right) \bigg|_S^T \\ &\leq \max\left(\frac{(q+1)C^2}{q-1}, 1 \right) E(t)^{(p+1)/2} \bigg|_S^T \leq M E(S)^{(p+1)/2}.\end{aligned} \tag{3.15}$$

Therefore we conclude from (3.14) and (3.15) that the estimate (3.11) holds. □

Proposition 3.5. *If $u(x,t)$ is the global solutions of the problem* (1.1)–(1.3)*, then one has the following estimate:*

$$\int_S^T E(t)^{(p+1)/2} dt \leq M \int_S^T \int_\Omega E(t)^{(p-1)/2} \left[2|u_t|^2 - a|u_t|^{p-1} u_t u\right] dx\, dt + ME(S)^{(p+1)/2}. \tag{3.16}$$

Proof. It follows from Lemma 3.2 and $0 < \theta < 1$ that

$$\begin{aligned}
&\int_S^T \int_\Omega E(t)^{(p-1)/2} \left(|u_t|^2 + \left|A^{1/2}u\right|^2 - b|u|^{q+1}\right) dx\, dt \\
&\quad = \int_S^T E(t)^{(p-1)/2} \left(\|u_t\|^2 + I(u(t))\right) dt \geq \int_S^T E(t)^{(p-1)/2} \left(\|u_t\|^2 + \theta \left\|A^{1/2}u\right\|^2\right) dt \\
&\quad \geq 2\theta \int_S^T E(t)^{(p-1)/2} \left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left\|A^{1/2}u\right\|^2\right) dt \geq 2\theta \int_S^T E(t)^{(p+1)/2} dt.
\end{aligned} \tag{3.17}$$

We have from Lemma 2.1 and (2.23) that

$$\begin{aligned}
&\left|\frac{p-1}{2} \int_S^T \int_\Omega E(t)^{(p-3)/2} E'(t) u u_t dx\, dt\right| \\
&\quad \leq \frac{p-1}{2} \int_S^T E(t)^{(p-3)/2} |E'(t)| \left(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2\right) dt \\
&\quad \leq -\frac{p-1}{2} \int_S^T E(t)^{(p-3)/2} E'(t) \left(\frac{C^2}{2}\left\|A^{1/2}u\right\|^2 + \frac{1}{2}\|u_t\|^2\right) dt \\
&\quad \leq -\frac{p-1}{2} \int_S^T E(t)^{(p-3)/2} E'(t) \left(\frac{(q+1)C^2}{q-1} \frac{q-1}{2(q+1)} \left\|A^{1/2}u\right\|^2 + \frac{1}{2}\|u_t\|^2\right) dt \\
&\quad \leq -\frac{p-1}{2} \max\left(\frac{(q+1)C^2}{q-1}, 1\right) \int_S^T E(t)^{(p-1)/2} E'(t) dt \\
&\quad = -\frac{p-1}{p+1} \max\left(\frac{(q+1)C^2}{q-1}, 1\right) E(t)^{(p+1)/2} \Big|_S^T \leq ME(S)^{(p+1)/2}.
\end{aligned} \tag{3.18}$$

We get from (3.11), (3.17), and (3.18) that

$$2\theta \int_S^T E(t)^{(p+1)/2} dt \leq \int_S^T \int_\Omega E(t)^{(p-1)/2} \left[2|u_t|^2 - a|u_t|^{p-1} u_t u\right] dx\, dt + ME(S)^{(p+1)/2}. \tag{3.19}$$

Therefore we conclude the estimate (3.16) from (3.19). □

Proposition 3.6. *Let $u(x,t)$ be the global solutions of the initial boundary problem* (1.1)–(1.3), *then the following estimate holds:*

$$\int_S^T E(t)^{(p+1)/2} dt \le M(1+E(0))^{(p-1)/2} E(S). \tag{3.20}$$

Proof. We get from Young inequality and (2.13) that

$$\begin{aligned}
2\int_S^T\int_\Omega E(t)^{(p-1)/2}|u_t|^2 dx\,dt &\le \int_S^T\int_\Omega \left(\varepsilon_1 E(t)^{(p+1)/2} + M(\varepsilon_1)|u_t|^{p+1}\right) dx\,dt \\
&\le M\varepsilon_1 \int_S^T E(t)^{(p+1)/2} dt + M(\varepsilon_1)\int_S^T \|u_t\|_{p+1}^{p+1} dt \\
&= M\varepsilon_1 \int_S^T E(t)^{(p+1)/2} dt - \frac{M(\varepsilon_1)}{a}(E(T)-E(S)) \\
&\le M\varepsilon_1 \int_S^T E(t)^{(p+1)/2} dt + ME(S).
\end{aligned} \tag{3.21}$$

We receive from Young inequality, Lemma 2.1, (2.13), and (2.23) that

$$\begin{aligned}
&- a\int_S^T\int_\Omega E(t)^{(p-1)/2} u u_t |u_t|^{p-1} dx dt \\
&\quad\le a\int_S^T E(t)^{(p-1)/2}\left(\varepsilon_2\|u\|_{p+1}^{p+1} + M(\varepsilon_2)\|u_t\|_{p+1}^{p+1}\right) dt \\
&\quad\le aC^{p+1}\varepsilon_2 E(0)^{(p-1)/2}\int_S^T \left\|A^{1/2}u\right\|^{p+1} dt + aM(\varepsilon_2)E(S)^{(p-1)/2}\int_S^T \|u_t\|_{p+1}^{p+1} dt \\
&\quad= aC^{p+1}\varepsilon_2 E(0)^{(p-1)/2}\int_S^T \left(\frac{2(q+1)}{q-1}E(t)\right)^{(p+1)/2} dt + M(\varepsilon_2)E(S)^{(p-1)/2}(E(S)-E(T)) \\
&\quad\le aC^{p+1}\varepsilon_2 E(0)^{(p-1)/2}\left(\frac{2(q+1)}{q-1}\right)^{(p+1)/2}\int_S^T E(t)^{(p+1)/2} dt + M(\varepsilon_2)E(S)^{(p+1)/2},
\end{aligned} \tag{3.22}$$

where $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are positive constants depending on ε_1 and ε_2.

ε_1 and ε_2 are small enough such that

$$M\varepsilon_1 + aE(0)^{(p-1)/2}\left(\frac{2(q+1)}{q-1}C^2\right)^{(p+1)/2}\varepsilon_2 < 1, \tag{3.23}$$

and then, substituting (3.21) and (3.22) into (3.16), we get

$$\int_S^T E(t)^{(p+1)/2} dt \le ME(S) + ME(S)^{(p+1)/2} \le M(1+E(0))^{(p-1)/2} E(S). \tag{3.24}$$

Therefore, we have from Lemma 3.1 and Proposition 3.6 that

$$E(t) \le M(E(0))(1+t)^{-(p-1)/2}, \quad t \in [0, +\infty). \tag{3.25}$$

Here $M(E(0)) > 0$ is a constant depending on $E(0)$.

It follows from (2.23) and (3.25) that

$$\lim_{t \to +\infty} \|u_t(t)\| = \lim_{t \to +\infty} \left\| A^{1/2} u(t) \right\| = 0. \tag{3.26}$$

The proof of Theorem 3.3 is thus finished. □

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