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Research Article

Existence of Solutions for a Weighted $p(t)$-Laplacian Impulsive Integrodifferential System with Multipoint and Integral Boundary Value Conditions

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By the Leray-Schauder's degree, the existence of solutions for a weighted $p(t)$-Laplacian impulsive integro-differential system with multi-point and integral boundary value conditions is considered. The sufficient results for the existence are given under the resonance and nonresonance cases, respectively. Moreover, we get the existence of nonnegative solutions at nonresonance.

1. Introduction

In this paper, we consider the existence of solutions for the following weighted $p(t)$-Laplacian integrodifferential system:

$$-\Delta_{p(t)}u + f\Big(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big) = 0, \quad t \in (0,1),\ t \neq t_i, \tag{1.1}$$

where $u : [0,1] \to \mathbb{R}^N$, $f(\cdot,\cdot,\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $t_i \in (0,1)$, $i = 1,\dots,k$, with the following impulsive boundary value conditions

$$\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k, \tag{1.2}$$

$$\begin{aligned}&\lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t)-\lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t)\\&\quad=B_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right),\quad i=1,\dots,k,\end{aligned}\tag{1.3}$$

$$\lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t)=\sum_{\ell=1}^{m-2}\alpha_\ell\lim_{t\to\eta_\ell^-} w(t)\left|u'\right|^{p(t)-2}u'(t),\qquad u(0)=\int_0^1 g(t)u(t)dt,\tag{1.4}$$

where $p\in C([0,1],\mathbb{R})$ and $p(t)>1$, $-\Delta_{p(t)}u:=-(w(t)|u'|^{p(t)-2}u')'$ is called the weighted $p(t)$-Laplacian; $0<t_1<t_2<\cdots<t_k<1$, $0<\eta_1<\cdots<\eta_{m-2}<1$; $\alpha_\ell\ge 0$, $(\ell=1,\dots,m-2)$ and $0\le\sum_{\ell=1}^{m-2}\alpha_\ell\le 1$; $g\in L^1[0,1]$ is nonnegative, $\int_0^1 g(t)dt=\sigma$ with $\sigma\in[0,1]$; $A_i,B_i\in C(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$; T and S are linear operators defined by $(Tu)(t)=\int_0^t k_*(t,s)u(s)ds$, $(Su)(t)=\int_0^1 h_*(t,s)u(s)ds$, $t\in[0,1]$, where $k_*,h_*\in C([0,1]\times[0,1],\mathbb{R})$.

If $\sum_{\ell=1}^{m-2}\alpha_\ell<1$ and $\sigma<1$, we say the problem is nonresonant, but if $\sum_{\ell=1}^{m-2}\alpha_\ell=1$ and $\sigma=1$, we say the problem is resonant.

Throughout the paper, $o(1)$ means function which uniformly convergent to 0 (as $n\to+\infty$); for any $v\in\mathbb{R}^N$, v^j will denote the jth component of v; the inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$; $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Denote $J=[0,1]$, $J'=(0,1)\setminus\{t_1,\dots,t_k\}$, $J_0=[t_0,t_1]$, $J_i=(t_i,t_{i+1}]$, $i=1,\dots,k$, where $t_0=0$, $t_{k+1}=1$. Denote J_i^o the interior of J_i, $i=0,1,\dots,k$. Let

$$\mathrm{PC}\left(J,\mathbb{R}^N\right)=\left\{x:J\longrightarrow\mathbb{R}^N\;\middle|\;\begin{array}{l}x\in C\left(J_i,\mathbb{R}^N\right),\ i=0,1,\dots,k,\\ \lim\limits_{t\to t_i^+}x(t)\ \text{exists for}\ i=1,\dots,k\end{array}\right\},\tag{1.5}$$

$w\in\mathrm{PC}(J,\mathbb{R})$ satisfies $0<w(t)$, for all $t\in(0,1)\setminus\{t_1,\dots,t_k\}$, and $(w(t))^{-1/(p(t)-1)}\in L^1(0,1)$;

$$\mathrm{PC}^1\left(J,\mathbb{R}^N\right)=\left\{x\in\mathrm{PC}\left(J,\mathbb{R}^N\right)\;\middle|\;\begin{array}{l}x'\in C\left(J_i^o,\mathbb{R}^N\right),\lim\limits_{t\to t_i^+}(w(t))^{1/(p(t)-1)}x'(t),\\ \lim\limits_{t\to t_{i+1}^-}(w(t))^{1/(p(t)-1)}x'(t)\ \text{exists for}\ i=0,1,\dots,k\end{array}\right\}.\tag{1.6}$$

For any $x=(x^1,\dots,x^N)\in\mathrm{PC}(J,\mathbb{R}^N)$, denote $|x^i|_0=\sup\{|x^i(t)|\mid t\in J'\}$. Obviously, $\mathrm{PC}(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_0=(\sum_{i=1}^N|x^i|_0^2)^{1/2}$, and $\mathrm{PC}^1(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_1=\|x\|_0+\|(w(t))^{1/(p(t)-1)}x'\|_0$. Denote $L^1=L^1(J,\mathbb{R}^N)$ with the norm $\|x\|_{L^1}=(\sum_{i=1}^N|x^i|_{L^1}^2)^{1/2}$, for all $x\in L^1$, where $|x^i|_{L^1}=\int_0^1|x^i(t)|dt$.

For simplicity, we denote $\mathrm{PC}(J,\mathbb{R}^N)$ and $\mathrm{PC}^1(J,\mathbb{R}^N)$ by PC and PC^1, respectively, and denote

$$u(t_i^+)=\lim_{t\to t_i^+}u(t),\quad u(t_i^-)=\lim_{t\to t_i^-}u(t),$$

$$w(0)\left|u'\right|^{p(0)-2}u'(0)=\lim_{t\to 0^+}w(t)\left|u'\right|^{p(t)-2}u'(t),$$

$$w(1)\left|u'\right|^{p(1)-2}u'(1)=\lim_{t\to 1^-}w(t)\left|u'\right|^{p(t)-2}u'(t),$$

$$A_i = A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k,$$
$$B_i = B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k. \tag{1.7}$$

In recent years, there has been an increasing interest in the study of differential equations with nonstandard $p(t)$-growth conditions. These problems have many interesting applications (see [1–4]). Many results have been obtained on these kinds of problems, for example [5–17]. If $p(t) \equiv p$ (a constant), (1.1)–(1.4) becomes the well known p-Laplacian problem. If $p(t)$ is a general function, one can see easily that $-\Delta_{p(t)}cu \neq c^{p(t)}(-\Delta_{p(t)}u)$ in general, while $-\Delta_p cu = c^p(-\Delta_p u)$, so $-\Delta_{p(t)}$ represents a non-homogeneity and possesses more nonlinearity, thus $-\Delta_{p(t)}$ is more complicated than $-\Delta_p$. For example, we have the following.

(a) In general, the infimum $\lambda_{p(x)}$ of eigenvalues for the $p(x)$-Laplacian Dirichlet problems is zero, and $\lambda_{p(x)} > 0$ only under some special conditions (see [10]). When $\Omega \subset \mathbb{R}$ ($N = 1$) is an interval, the results in [10] show that $\lambda_{p(x)} > 0$ if and only if $p(x)$ is monotone. But the property of $\lambda_p > 0$ is very important in the study of p-Laplacian problems, for example, in [18], the authors use this property to deal with the existence of solutions.

(b) If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant) and $-\Delta_p u > 0$, then u is concave, this property is used extensively in the study of one-dimensional p-Laplacian problems (see [19]), but it is invalid for $-\Delta_{p(t)}$. It is another difference between $-\Delta_p$ and $-\Delta_{p(t)}$.

Recently, there are many works devoted to the existence of solutions to the Laplacian impulsive differential equation boundary value problems, for example [20–28]. Many methods had been applied to deal with these problems, for example sub-super-solution method, fixed point theorem, monotone iterative method, coincidence degree, variational principles (see [29]), and so forth. Because of the nonlinearity of $-\Delta_p$, results about the existence of solutions for p-Laplacian impulsive differential equation boundary value problems are rare (see [30]). In [31], using coincidence degree method, the present author investigate the existence of solutions for $p(r)$-Laplacian impulsive differential equation with multipoint boundary value conditions. Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, there are many papers on the differential equations with integral boundary value problems, for example, [32–35].

In this paper, when $p(t)$ is a general function, we investigate the existence of solutions and nonnegative solutions for the weighted $p(t)$-Laplacian impulsive integrodifferential system with multipoint and integral boundary value conditions. Our results contain both the cases of resonance and nonresonance, and the method is based upon Leray-Schauder's degree. Moreover, this paper will consider the existence of (1.1) with (1.2), (1.4) and the following impulsive condition:

$$\begin{aligned}\lim_{t\to t_i^+} (w(t))^{1/(p(t)-1)} u'(t) &- \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t) \\ &= D_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k,\end{aligned} \tag{1.8}$$

where $D_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, the impulsive condition (1.8) is called linear impulsive condition (LI for short), and (1.3) is called nonlinear impulsive condition (NLI for short). In generaly, p-Laplacian impulsive problems have two kinds of impulsive conditions, that is, LI and NLI.

Let $N \geq 1$, the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean the following:

(i) for almost every $t \in J$ the function $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous;

(ii) for each $(x, y, s, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y, s, z)$ is measurable on J;

(iii) for each $R > 0$ there is a $\alpha_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y, s, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, $|s| \leq R$, $|z| \leq R$, one has

$$\left| f(t, x, y, s, z) \right| \leq \alpha_R(t). \tag{1.9}$$

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1.1) if $u \in \mathrm{PC}^1$ with $w(t)|u'|^{p(t)-2}u'$ absolutely continuous on J_i^o, $i = 0, 1, \ldots, k$, which satisfies (1.1) a.e. on J.

In this paper, we always use C_i to denote positive constants, if it cannot lead to confusion. Denote

$$z^- = \inf_{t \in J} z(t), \quad z^+ = \sup_{t \in J} z(t), \quad \text{for any } z \in \mathrm{PC}(J, \mathbb{R}). \tag{1.10}$$

We say f satisfies sub-$(p^- - 1)$ growth condition, if f satisfies

$$\lim_{|u|+|v|+|s|+|z| \to +\infty} \frac{f(t, u, v, s, z)}{(|u| + |v| + |s| + |z|)^{q(t)-1}} = 0, \quad \text{for } t \in J \text{ uniformly}, \tag{1.11}$$

where $q(t) \in \mathrm{PC}(J, \mathbb{R})$, and $1 < q^- \leq q^+ < p^-$.

This paper is organized as four sections. In Section 2, we present some preliminary and give the operator equation which has the same solutions of (1.1)–(1.4). In Section 3, we give the existence of solutions and nonnegative solutions for system (1.1)–(1.4) at nonresonance. Finally, in Section 4, we give the existence of solutions for system (1.1)–(1.4) at resonance.

2. Preliminary

For any $(t, x) \in J \times \mathbb{R}^N$, denote $\varphi(t, x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties.

Lemma 2.1 (see [31]). *φ is a continuous function and satisfies the following.*

(i) *For any $t \in [0, 1]$, $\varphi(t, \cdot)$ is strictly monotone, satisfying*

$$\langle \varphi(t, x_1) - \varphi(t, x_2), x_1 - x_2 \rangle > 0, \quad \text{for any } x_1, x_2 \in \mathbb{R}^N, \ x_1 \neq x_2. \tag{2.1}$$

(ii) *There exists a function $\alpha : [0, +\infty) \to [0, +\infty)$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi(t, x), x \rangle \geq \alpha(|x|)|x|, \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

It is well known that $\varphi(t,\cdot)$ is an homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in J$. Denote

$$\varphi^{-1}(t,x) = |x|^{(2-p(t))/(p(t)-1)}x, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\},\ \varphi^{-1}(t,0) = 0,\ \forall t \in J. \tag{2.3}$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and sends bounded sets to bounded sets.

In this section, we will do some preparation and give the operator equation which has the same solutions of (1.1)–(1.4). At first, let us now consider the following simple impulsive problem with boundary value condition (1.4)

$$\begin{gathered} -\Delta_{p(t)}u + f(t) = 0, \quad t \in (0,1),\ t \neq t_i, \\ \lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = a_i, \quad i = 1,\dots,k, \\ \lim_{t\to t_i^+} w(t)\left|u'\right|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)\left|u'\right|^{p(t)-2}u'(t) = b_i, \quad i = 1,\dots,k, \end{gathered} \tag{2.4}$$

where $a_i, b_i \in \mathbb{R}^N$; $f \in L^1$.

We will discuss (2.4) with (1.4) in the cases of resonance and nonresonance, respectively.

2.1. The Case of Nonresonance

Suppose $0 \le \sum_{\ell=1}^{m-2}\alpha_\ell < 1$ and $0 \le \sigma < 1$. If u is a solution of (2.4) with (1.4), we have

$$w(t)\varphi\left(t,u'(t)\right) = w(0)\varphi\left(0,u'(0)\right) + \sum_{t_i<t} b_i + \int_0^t f(s)ds, \quad \forall t \in J'. \tag{2.5}$$

Denote $a = (a_1,\dots,a_k) \in \mathbb{R}^{kN}$, $b = (b_1,\dots,b_k) \in \mathbb{R}^{kN}$, $\rho_1 = w(0)\varphi(0,u'(0))$. It is easy to see that ρ_1 is dependent on a, b and $f(t)$. Define operator $F : L^1 \to \mathrm{PC}$ as

$$F(f)(t) = \int_0^t f(s)ds, \quad \forall t \in J,\ \forall f \in L^1. \tag{2.6}$$

By solving for u' in (2.5) and integrating, we find

$$u(t) = u(0) + \sum_{t_i<t} a_i + F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1 + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t), \quad \forall t \in J, \tag{2.7}$$

which together with the boundary value condition (1.4) implies

$$\begin{gathered} \rho_1 = \frac{\left\{\sum_{\ell=1}^{m-2}\alpha_\ell\left[\sum_{t_i<\eta_\ell} b_i + F(f)(\eta_\ell)\right]\right\}}{1-\sum_{\ell=1}^{m-2}\alpha_\ell}, \\ u(0) = \frac{1}{(1-\sigma)}\int_0^1 g(t)\left\{F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1 + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t) + \sum_{t_i<t} a_i\right\}dt. \end{gathered} \tag{2.8}$$

Denote $W = \mathbb{R}^{2kN} \times L^1$ with the norm $\|\omega\| = \sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} |b_i| + \|h\|_{L^1}$, for all $\omega = (a,b,h) \in W$, then W is a Banach space.

We define $\widetilde{\rho_1} : W \to \mathbb{R}^N$ as

$$\widetilde{\rho_1}(\omega) = \frac{\left\{\sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} b_i + F(h)(\eta_\ell)\right]\right\}}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell}, \quad \forall \omega = (a,b,h) \in W, \tag{2.9}$$

then $\widetilde{\rho_1}(\cdot)$ is continuous. Throughout the paper, we denote $E = \int_0^1 (w(t))^{-1/(p(t)-1)} dt$. It is easy to see the following.

Lemma 2.2. *The function $\widetilde{\rho_1} : W \to \mathbb{R}^N$ is continuous and sends bounded sets to bounded sets. Moreover, for any $\omega = (a,b,h) \in W$, we have*

$$\left|\widetilde{\rho_1}(\omega)\right| \leq \frac{\sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{i=1}^{k} |b_i| + \|h\|_{L^1}\right]}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell}. \tag{2.10}$$

We denote $N_f(u) : [0,1] \times PC^1 \to L^1$ the Nemytskii operator associated to f defined by

$$N_f(u)(t) = f\left(t, u(t), (w(t))^{1/(p(t)-1)} u'(t), S(u), T(u)\right), \quad \text{a.e. on } J. \tag{2.11}$$

We define $\rho_1 : PC^1 \to \mathbb{R}^N$ as

$$\rho_1(u) = \widetilde{\rho_1}(A, B, N_f)(u), \tag{2.12}$$

where $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_k)$.

It is clear that $\rho_1(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is compact continuous.

If u is a solution of (2.4) with (1.4), we have

$$u(t) = u(0) + \sum_{t_i<t} a_i + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(\omega) + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t), \quad \forall t \in [0,1]. \tag{2.13}$$

For fixed $a, b \in \mathbb{R}^{kN}$, we define $K_{(a,b)} : L^1 \to \mathrm{PC}^1$ as

$$K_{(a,b)}(h)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(a,b,h) + \sum_{t_i<t} b_i + F(h)(t)\right)\right]\right\}(t), \quad \forall t \in J. \tag{2.14}$$

Define $K_1 : \mathrm{PC}^1 \to \mathrm{PC}^1$ as

$$K_1(u)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F\left(N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t \in J. \tag{2.15}$$

Lemma 2.3. (i) *The operator $K_{(a,b)}$ is continuous and sends equiintegrable sets in L^1 to relatively compact sets in PC^1.*

(ii) *The operator K_1 is continuous and sends bounded sets in PC^1 to relatively compact sets in PC^1.*

Proof. (i) It is easy to check that $K_{(a,b)}(h)(\cdot) \in \mathrm{PC}^1$, for all $h \in L^1$, for all $a, b \in \mathbb{R}^{kN}$. Since $(w(t))^{-1/(p(t)-1)} \in L^1$ and

$$K_{(a,b)}(h)'(t) = \varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(a,b,h) + \sum_{t_i<t} b_i + F(h)\right)\right], \quad \forall t \in [0,1], \tag{2.16}$$

it is easy to check that $K_{(a,b)}(\cdot)$ is a continuous operator from L^1 to PC^1.

Let U be an equiintegrable set in L^1, then there exists $\tau \in L^1$, such that

$$|u(t)| \le \tau(t) \quad \text{a.e. in } J, \text{ for any } u \in L^1. \tag{2.17}$$

We want to show that $\overline{K_{(a,b)}(U)} \subset \mathrm{PC}^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K_{(a,b)}(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = K_{(a,b)}(h_n)$. For any $t_1, t_2 \in J$, we have

$$|F(h_n)(t_1) - F(h_n)(t_2)| = \left|\int_0^{t_1} h_n(t)dt - \int_0^{t_2} h_n(t)dt\right| = \left|\int_{t_1}^{t_2} h_n(t)dt\right| \le \left|\int_{t_1}^{t_2} \tau(t)dt\right|. \tag{2.18}$$

Hence the sequence $\{F(h_n)\}$ is uniformly bounded and equicontinuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{F(h_n)\}$ (which we rename the same) which is convergent in PC. According to the bounded continuous of the operator $\widetilde{\rho_1}$, we can choose a subsequence of $\{\widetilde{\rho_1}(a,b,h_n) + F(h_n)\}$ (which we still denote by $\{\widetilde{\rho_1}(a,b,h_n) + F(h_n)\}$) which is convergent in PC, then $w(t)^{1/(p(t)-1)} K_{(a,b)}(h_n)'(t) = \varphi^{-1}(t, \widetilde{\rho_1}(a,b,h_n) + \sum_{t_i<t} b_i + F(h_n))$ is convergent in PC.

Since

$$K_{(a,b)}(h_n)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\widetilde{\rho_1}(a,b,h_n) + \sum_{t_i<t} b_i + F(h_n)\right)\right]\right\}(t), \quad \forall t \in [0,1], \tag{2.19}$$

it follows from the continuity of φ^{-1} and the integrability of $w(t)^{-1/(p(t)-1)}$ in L^1 that $K_{(a,b)}(h_n)$ is convergent in PC. Thus $\{u_n\}$ is convergent in PC^1.

(ii) It is easy to see from (i) and Lemma 2.2.

This completes the proof. □

Let us define $P_1 : \mathrm{PC}^1 \to \mathrm{PC}^1$ as $P_1(u) = \{\int_0^1 g(t)[K_1(u)(t) + \sum_{t_i<t} A_i]dt\}/(1-\sigma)$. It is easy to see that P_1 is compact continuous.

Lemma 2.4. *Suppose* $0 \leq \sum_{\ell=1}^{m-2} \alpha_\ell < 1$ *and* $0 \leq \sigma < 1$, *then* u *is a solution of* (1.1)–(1.4) *if and only if* u *is a solution of the following abstract operator equation*

$$u = P_1(u) + \sum_{t_i<t} A_i + K_1(u). \tag{2.20}$$

Proof. Suppose u is a solution of (1.1)–(1.4). From the definition of $\rho_1(\cdot)$ and $P_1(\cdot)$, similar to the discussion before Lemma 2.2, we know that u is a solution of (2.20).

Conversely, if u is a solution of (2.20), then (1.2) is satisfied.

From (2.20), we have

$$\begin{gathered} w(t)\varphi(t,u'(t)) = \rho_1(u) + \sum_{t_i<t} B_i + F(N_f(u))(t), \quad t \in (0,1),\ t \neq t_i, \\ (w(t)\varphi(t,u'))' = N_f(u)(t), \quad t \in (0,1),\ t \neq t_i. \end{gathered} \tag{2.21}$$

It follows from (2.21) that (1.3) is satisfied.

From (2.21) and the definition of ρ_1, we have

$$\lim_{t\to 0^+} w(t)|u'|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t\to\eta_\ell^-} w(t)|u'|^{p(t)-2}u'(t). \tag{2.22}$$

From (2.20) and the definition of P_1, it is easy to check that

$$u(0) = \int_0^1 g(t)u(t)dt. \tag{2.23}$$

It follows from (2.22) and (2.23) that (1.4) is satisfied.

Hence u is a solutions of (1.1)–(1.4). This completes the proof. □

2.2. The Case of Resonance

Suppose $\sum_{\ell=1}^{m-2} \alpha_\ell = 1$ and $\sigma = 1$. If u is a solution of (2.4) with (1.4), we have

$$w(t)\varphi(t,u'(t)) = w(0)\varphi(0,u'(0)) + \sum_{t_i<t} b_i + \int_0^t f(s)ds, \quad \forall t \in J'. \tag{2.24}$$

Denote $a = (a_1,\ldots,a_k) \in \mathbb{R}^{kN}$, $b = (b_1,\ldots,b_k) \in \mathbb{R}^{kN}$, $\rho_2 = w(0)\varphi(0,u'(0))$. It is easy to see that ρ_2 is dependent on a, b and $f(t)$.

The boundary value condition (1.4) implies that

$$\sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} b_i + F(f)(\eta_\ell) \right] = 0,$$
$$\int_0^1 g(t) \left\{ F \left\{ \varphi^{-1} \left[t, (w(t))^{-1} \left(\rho_2 + \sum_{t_i<t} b_i + F(f)(t) \right) \right] \right\}(t) + \sum_{t_i<t} a_i \right\} dt = 0. \tag{2.25}$$

For any $\omega \in W$, we denote

$$\Lambda_\omega(\rho_2) = \int_0^1 g(t) \left\{ F \left\{ \varphi^{-1} \left[t, (w(t))^{-1} \left(\rho_2 + \sum_{t_i<t} b_i + F(h)(t) \right) \right] \right\}(t) + \sum_{t_i<t} a_i \right\} dt. \tag{2.26}$$

Lemma 2.5. *The function $\Lambda_\omega(\cdot)$ has the following properties.*

(i) *For any fixed $\omega \in W$, the equation*

$$\Lambda_\omega(\rho_2) = 0 \tag{2.27}$$

has unique solution $\widetilde{\rho_2}(\omega) \in \mathbb{R}^N$.

(ii) *The function $\widetilde{\rho_2} : W \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover, for any $\omega = (a, b, h) \in W$, we have*

$$\left|\widetilde{\rho_2}(\omega)\right| \leq 3N \left[(2N)^{p^+} \left(\sum_{i=1}^k |a_i| \right)^{p^\#-1} + \sum_{i=1}^k |b_i| + \|h\|_{L^1} \right], \tag{2.28}$$

where

$$M^{p^\#-1} = \begin{cases} M^{p^+-1}, & M > 1 \\ M^{p^--1}, & M \leq 1. \end{cases} \tag{2.29}$$

Proof. (i) From Lemma 2.1, it is immediate that

$$\langle \Lambda_\omega(x_1) - \Lambda_\omega(x_2), x_1 - x_2 \rangle > 0, \quad \text{for } x_1 \neq x_2,\ \forall x_1, x_2 \in \mathbb{R}^N, \tag{2.30}$$

and hence, if (2.27) has a solution, then it is unique.

Set

$$R_0 = 3N \left[(2N)^{p^+} \left(\sum_{i=1}^k |a_i| \right)^{p^\#-1} + \sum_{i=1}^k |b_i| + \|h\|_{L^1} \right]. \tag{2.31}$$

Suppose $|\rho_2| > R_0$, it is easy to see that there exists some $j_0 \in \{1, \ldots, N\}$ such that, the absolute value of the j_0th component $\rho_2^{j_0}$ of ρ_2 satisfies

$$\left|\rho_2^{j_0}\right| \geq \frac{1}{N}|\rho_2| > \frac{1}{N}R_0 = 3\left[(2N)^{p^+}\left(\sum_{i=1}^{k}|a_i|\right)^{p^\#-1} + \sum_{i=1}^{k}|b_i| + \|h\|_{L^1}\right]. \tag{2.32}$$

Thus the j_0th component of $\rho_2 + \sum_{t_i<t} b_i + F(h)(t)$ keeps sign on J, then it is not hard to check that the j_0th component of $\Lambda_\omega(\rho_2)$ keeps the same sign of $\rho_2^{j_0}$.

Thus $\Lambda_\omega(\rho_2) \neq 0$. Let us consider the equation

$$\lambda\Lambda_\omega(\rho_2) + (1-\lambda)\rho_2 = 0, \quad \lambda \in [0,1]. \tag{2.33}$$

According to the preceding discussion, all the solutions of (2.33) belong to $b(R_0+1) = \{x \in \mathbb{R}^N \mid |x| < R_0 + 1\}$. Therefore

$$d_B[\Lambda_\omega(\rho_2), b(R_0+1), 0] = d_B[I, b(R_0+1), 0] \neq 0, \tag{2.34}$$

it means the existence of solutions of $\Lambda_\omega(\rho_2) = 0$.

In this way, we define a function $\widetilde{\rho_2}(\omega) : W \to \mathbb{R}^N$, which satisfies $\Lambda_\omega(\widetilde{\rho_2}(\omega)) = 0$.

(ii) By the proof of (i), we also obtain $\widetilde{\rho_2}$ sends bounded sets to bounded sets, and

$$\left|\widetilde{\rho_2}(\omega)\right| \leq 3N\left[(2N)^{p^+}\left(\sum_{i=1}^{k}|a_i|\right)^{p^\#-1} + \sum_{i=1}^{k}|b_i| + \|h\|_{L^1}\right]. \tag{2.35}$$

It only remains to prove the continuity of $\widetilde{\rho_2}$. Let $\{\omega_n\}$ is a convergent sequence in W and $\omega_n \to \omega$, as $n \to +\infty$. Since $\{\widetilde{\rho_2}(\omega_n)\}$ is a bounded sequence, it contains a convergent subsequence $\{\widetilde{\rho_2}(\omega_{n_j})\}$. Suppose $\widetilde{\rho_2}(\omega_{n_j}) \to \rho_2^0$ as $j \to +\infty$. Since $\Lambda_{\omega_{n_j}}(\widetilde{\rho_2}(\omega_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_\omega(\rho_2^0) = 0$, which together with (i) implies $\rho_2^0 = \widetilde{\rho_2}(\omega)$, it means $\widetilde{\rho_2}$ is continuous. This completes the proof. □

We define $\rho_2 : \mathrm{PC}^1 \to \mathbb{R}^N$ as

$$\rho_2(u) = \widetilde{\rho_2}(A, B, N_f)(u), \tag{2.36}$$

where $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_k)$.

It is clear that $\rho_2(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is a compact continuous mapping.

Let us define

$$P_2 : \mathrm{PC}^1 \longrightarrow \mathrm{PC}^1, u \longrightarrow u(0); \quad Q : L^1 \longrightarrow L^1, \quad h \longrightarrow \sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} b_i + F(h)(\eta_\ell) \right],$$
$$\Theta : L^1 \longrightarrow L^1, h \longrightarrow h - \frac{1}{\sum_{\ell=1}^{m-2} \alpha_\ell \eta_\ell} Qh, \tag{2.37}$$

and $K^*_{(a,b)} : L^1 \to \mathrm{PC}^1$ as

$$K^*_{(a,b)}(h)(t) = F\left\{ \varphi^{-1}\left[t, (w(t))^{-1} \left(\widetilde{\rho_2}(a,b,h) + \sum_{t_i<t} b_i + F(h)(t) \right) \right] \right\}(t), \quad \forall t \in J. \tag{2.38}$$

Similar to the proof of Lemma 2.3, we have the following lemma.

Lemma 2.6. *The operator $(K^*_{(a,b)} \circ \Theta)(\cdot)$ is continuous and sends equiintegrable sets in L^1 to relatively compact sets in PC^1.*

Denote

$$Q_{N_f}(u) = \sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} B_i + F(N_f(u))(\eta_\ell) \right],$$
$$\Theta_f(u) = N_f(u) - \frac{1}{\sum_{\ell=1}^{m-2} \alpha_\ell \eta_\ell} Q_{N_f}(u),$$
$$\rho_2(u) = \widetilde{\rho_2}(A, B, \Theta_f)(u), \tag{2.39}$$
$$K_2(u)(t) = F\left\{ \varphi^{-1}\left[t, (w(t))^{-1} \left(\rho_2(u) + \sum_{t_i<t} B_i + F(\Theta_f(u))(t) \right) \right] \right\}(t), \quad \forall t \in J.$$

Lemma 2.7. *Suppose $\sum_{\ell=1}^{m-2} \alpha_\ell = 1$ and $\sigma = 1$, then u is a solution of* (1.1)–(1.4) *if and only if u is a solution of the following abstract operator equation*

$$u = P_2(u) + \sum_{t_i<t} A_i + Q_{N_f}(u) + K_2(u). \tag{2.40}$$

Proof. Suppose u is a solution of (1.1)–(1.4), it is clear that u is a solution of (2.40).

Conversely, if u is a solution of (2.40), then (1.2) is satisfied and

$$Q_{N_f}(u) = 0. \tag{2.41}$$

Thus $\Theta_f(u) = N_f(u)$.

From (2.40) and (2.41), we have

$$
\begin{gathered}
w(t)\varphi(t,u'(t)) = \rho_2(u) + \sum_{t_i<t} B_i + F(\Theta_f(u))(t), \quad t \in (0,1),\ t \neq t_i, \\
(w(t)\varphi(t,u'))' = N_f(u)(t), \quad t \in (0,1),\ t \neq t_i.
\end{gathered}
\tag{2.42}
$$

According to (2.42), we get that (1.3) is satisfied. Since $Q_{N_f}(u) = 0$, we have

$$
\lim_{t\to 0^+} w(t)|u'|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t\to \eta_\ell^-} w(t)|u'|^{p(t)-2}u'(t). \tag{2.43}
$$

It follows from the definition of ρ_2 that

$$
\int_0^1 g(t)\left\{F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_2(u) + \sum_{t_i<t} B_i + F(\Theta_f(u))(t)\right)\right]\right\}(t) + \sum_{t_i<t} A_i\right\}dt = 0, \tag{2.44}
$$

then $u(0) = \int_0^1 g(t)u(t)dt$.

Hence u is a solutions of (1.1)–(1.4). This completes the proof. □

3. Existence of Solutions in the Case of Nonresonance

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions and nonnegative solutions for system (1.1)–(1.4) at nonresonance.

When f satisfies sub-$(p^- - 1)$ growth condition, we have the following.

Theorem 3.1. *Suppose* $0 \le \sum_{\ell=1}^{m-2} \alpha_\ell < 1$ *and* $0 \le \sigma < 1$, f *satisfies sub-*$(p^- - 1)$ *growth condition, and operators* A *and* B *satisfy the following condition*

$$
\begin{aligned}
&\sum_{i=1}^{k} |A_i(u,v)| \le C_1(1+|u|+|v|)^{(q^+-1)/(p^+-1)}, \\
&\sum_{i=1}^{k} |B_i(u,v)| \le C_2(1+|u|+|v|)^{q^+-1},
\end{aligned}
\qquad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.1}
$$

then problem (1.1)–(1.4) *has at least one solution.*

Proof. First we consider the following problem:

$$\begin{gathered}
-\Delta_{p(t)}u+\lambda N_f(u)(t)=0, \quad t\in(0,1),\ t\neq t_i,\\
\lim_{t\to t_i^+}u(t)-\lim_{t\to t_i^-}u(t)=\lambda A_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right),\quad i=1,\dots,k,\\
\lim_{t\to t_i^+}w(t)|u'|^{p(t)-2}u'(t)-\lim_{t\to t_i^-}w(t)|u'|^{p(t)-2}u'(t)\\
=\lambda B_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right),\quad i=1,\dots,k,\\
\lim_{t\to 0^+}w(t)\left|u'\right|^{p(t)-2}u'(t)=\sum_{\ell=1}^{m-2}\alpha_\ell\lim_{t\to\eta_\ell^-}w(t)\left|u'\right|^{p(t)-2}u'(t),\quad u(0)=\int_0^1 g(t)u(t)dt.
\end{gathered}\tag{S_1}$$

Denote

$$\begin{gathered}
\rho_{1,\lambda}(u)=\widetilde{\rho_1}(\lambda A,\lambda B,\lambda N_f)(u),\\
K_{1,\lambda}(u)=F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_{1,\lambda}(u)+\lambda\sum_{t_i<t}B_i+F\left(\lambda N_f(u)\right)(t)\right)\right]\right\},\\
P_{1,\lambda}(u)=\frac{1}{(1-\sigma)}\int_0^1 g(t)\left[K_{1,\lambda}(u)(t)+\sum_{t_i<t}\lambda A_i\right]dt,\\
\Psi_f(u,\lambda)=P_{1,\lambda}(u)+\lambda\sum_{t_i<t}A_i+K_{1,\lambda}(u),
\end{gathered}\tag{3.2}$$

where $N_f(u)$ is defined in (2.11).

We know that (S_1) has the same solution of the following operator equation when $\lambda=1$,

$$u=\Psi_f(u,\lambda).\tag{3.3}$$

It is easy to see that operator $\rho_{1,\lambda}$ is compact continuous for any $\lambda\in[0,1]$. It follows from Lemmas 2.2 and 2.3 that $\Psi_f(\cdot,\lambda)$ is compact continuous from PC^1 to PC^1 for any $\lambda\in[0,1]$.

We claim that all the solutions of (3.3) are uniformly bounded for $\lambda\in[0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n,\lambda_n)\}$ for (3.3) such that $\|u_n\|_1\to+\infty$ as $n\to+\infty$, and $\|u_n\|_1>1$ for any $n=1,2,\dots$.

From Lemma 2.2, we have

$$\left|\rho_{1,\lambda}(u)\right|\le C_3\left[\sum_{i=1}^k|B_i|+\left\|N_f(u)\right\|_{L^1}\right]\le C_4\left(1+\|u\|_1^{q^+-1}\right).\tag{3.4}$$

Thus

$$\left|\rho_{1,\lambda}(u)+\sum_{t_i<t}\lambda B_i+F\left(\lambda N_f\right)\right|\leq\left|\rho_{1,\lambda}(u)\right|+\left|\sum_{t_i<t}B_i\right|+\left|F\left(N_f\right)\right|\leq C_5\left(1+\|u\|_1^{q^+-1}\right). \tag{3.5}$$

From (S_1), we have

$$w(t)\left|u_n'(t)\right|^{p(t)-2}u_n'(t)=\rho_{1,\lambda}(u_n)+\sum_{t_i<t}\lambda B_i+\int_0^t\lambda N_f(u_n)(s)ds,\quad\forall t\in J'. \tag{3.6}$$

It follows from (2.12) and Lemma 2.2 that

$$w(t)\left|u_n'(t)\right|^{p(t)-1}\leq\left|\rho_{1,\lambda}(u_n)\right|+\sum_{i=1}^{k}|B_i|+\int_0^1\left|N_f(u_n)(s)\right|ds\leq C_6+C_7\|u_n\|_1^{q^+-1},\quad\forall t\in J'. \tag{3.7}$$

Denote $\alpha=(q^+-1)/(p^--1)$. The above inequality holds

$$\left\|(w(t))^{1/(p(t)-1)}u_n'(t)\right\|_0\leq C_8\|u_n\|_1^{\alpha},\quad n=1,2,\ldots. \tag{3.8}$$

It follows from (3.1) and (3.5) that

$$|u_n(0)|\leq C_9\|u_n\|_1^{\alpha},\qquad\text{where }\alpha=\frac{q^+-1}{p^--1}. \tag{3.9}$$

For any $j=1,\ldots,N$, we have

$$\begin{aligned}\left|u_n^j(t)\right|&=\left|u_n^j(0)+\sum_{t_i<t}A_i^j+\int_0^t\left(u_n^j\right)'(s)ds\right|\\&\leq\left|u_n^j(0)\right|+\left|\sum_{t_i<t}A_i\right|+\left|\int_0^t(w(s))^{-1/(p(s)-1)}\sup_{t\in(0,1)}\left|(w(t))^{1/(p(t)-1)}\left(u_n^j\right)'(t)\right|ds\right|\\&\leq\|u_n\|_1^{\alpha}[C_{10}+C_8E]+\left|\sum_{t_i<t}A_i\right|\leq C_{11}\|u_n\|_1^{\alpha},\quad\forall t\in J,\ n=1,2,\ldots,\end{aligned} \tag{3.10}$$

which implies that $|u_n^j|_0\leq C_{12}\|u_n\|_1^{\alpha}$, $j=1,\ldots,N$; $n=1,2,\ldots$. Thus

$$\|u_n\|_0\leq NC_{12}\|u_n\|_1^{\alpha},\quad n=1,2,\ldots. \tag{3.11}$$

It follows from (3.8) and (3.11) that $\{\|u_n\|_1\}$ is uniformly bounded.

Thus, we can choose a large enough $R_0 > 0$ such that all the solutions of (3.3) belong to $B(R_0) = \{u \in PC^1 \mid \|u\|_1 < R_0\}$. Therefore the Leray-Schauder degree $d_{LS}[I - \Psi_f(\cdot, \lambda), B(R_0), 0]$ is well defined for $\lambda \in [0,1]$, and

$$d_{LS}\left[I - \Psi_f(\cdot, 1), B(R_0), 0\right] = d_{LS}\left[I - \Psi_f(\cdot, 0), B(R_0), 0\right]. \tag{3.12}$$

It is easy to see that u is a solution of $u = \Psi_f(u, 0)$ if and only if u is a solution of the following usual differential equation

$$\begin{gathered}
-\Delta_{p(t)} u = 0, \quad t \in (0,1), \\
\lim_{t \to 0^+} w(t)\left|u'\right|^{p(t)-2} u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t \to \eta_\ell^-} w(t)\left|u'\right|^{p(t)-2} u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{S_2}$$

Obviously, system (S_2) possesses a unique solution u_0. Since $u_0 \in B(R_0)$, we have

$$d_{LS}\left[I - \Psi_f(\cdot, 1), B(R_0), 0\right] = d_{LS}\left[I - \Psi_f(\cdot, 0), B(R_0), 0\right] \neq 0, \tag{3.13}$$

which implies that (1.1)–(1.4) has at least one solution. This completes the proof. □

Theorem 3.2. *Suppose* $0 \leq \sum_{\ell=1}^{m-2} \alpha_\ell < 1$ *and* $0 \leq \sigma < 1$, f *satisfies sub-*$(p^- - 1)$ *growth condition, and operators* A *and* D *satisfy the following*

$$\begin{aligned}
&\sum_{i=1}^{k} |A_i(u,v)| \leq C_1(1 + |u| + |v|)^{(q^+-1)/(p^+-1)}, \\
& \hspace{18em} \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \\
&\sum_{i=1}^{k} |D_i(u,v)| \leq C_2(1 + |u| + |v|)^{\alpha_i^+},
\end{aligned} \tag{3.14}$$

where $\alpha_i \leq (q^+ - 1)/(p(r_i) - 1)$, *and* $p(r_i) - 1 \leq q^+ - \alpha_i$, $i = 1, \ldots, k$, *then problem* (1.1) *with* (1.2), (1.4), *and* (1.8) *has at least one solution.*

Proof. Obviously,

$$B_i(u,v) = \varphi(r_i, v + D_i(u,v)) - \varphi(r_i, v). \tag{3.15}$$

From Theorem 3.1, it suffices to show that

$$\sum_{i=1}^{k} |B_i(u,v)| \leq C_2(1 + |u| + |v|)^{q^+-1}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{3.16}$$

(a) Suppose $|v| \leq M^*|D_i(u,v)|$, where M^* is a large enough positive constant. From the definition of D, we have

$$|B_i(u,v)| \leq C_1|D_i(u,v)|^{p(r_i)-1} \leq C_2(1+|u|+|v|)^{\alpha_i(p(r_i)-1)}. \tag{3.17}$$

Since $\alpha_i < (q^+ - 1)/(p(r_i)-1)$, we have $\alpha_i(p(r_i)-1) \leq q^+ - 1$. Thus (3.16) is valid.

(b) Suppose $|v| > M^*|D_i(u,v)|$, we have

$$|B_i(u,v)| \leq C_3|v|^{p(r_i)-1}\frac{|D_i(u,v)|}{|v|} = C_4|v|^{p(r_i)-2}|D_i(u,v)|. \tag{3.18}$$

There are two cases.

Case 1 ($p(r_i)-1 \geq 1$). Since $p(r_i)-1 \leq q^+ - \alpha_i$, we have $p(r_i)-2+\alpha_i \leq q^+ - 1$, and then

$$|B_i(u,v)| \leq C_5|v|^{p(r_i)-2}|D_i(u,v)| \leq C_6(1+|u|+|v|)^{p(r_i)-2+\alpha_i} \leq C_6(1+|u|+|v|)^{q^+-1}. \tag{3.19}$$

Thus (3.16) is valid.

Case 2 ($p(r_i)-1 < 1$). Since $\alpha_i < (q^+ - 1)/(p(r_i)-1)$, we have $\alpha_i(p(r_i)-1) \leq q^+ - 1$, and

$$|B_i(u,v)| \leq C_7|v|^{p(r_i)-2}|D_i(u,v)| \leq C_8|D_i(u,v)|^{p(r_i)-1} \leq C_9(1+|u|+|v|)^{\alpha_i(p(r_i)-1)}. \tag{3.20}$$

Thus (3.16) is valid. Thus problem (1.1) with (1.2), (1.4), and (1.8) has at least one solution. This completes the proof. □

Let us consider

$$-\Delta_{p(t)}u + \phi\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\right) = 0, \quad t \in (0,1),\ t \neq t_i, \tag{3.21}$$

where δ is a parameter, and

$$\begin{aligned} &\phi\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\right) \\ &\quad = f\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\right) + \delta h\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\right), \end{aligned} \tag{3.22}$$

where $h,f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory.

We have the following.

Theorem 3.3. *Suppose* $0 \leq \sum_{\ell=1}^{m-2} \alpha_\ell < 1$ *and* $0 \leq \sigma < 1$, f *satisfies sub-*$(p^- - 1)$ *growth condition, and we assume that*

$$\begin{aligned} &\sum_{i=1}^{k} |A_i(u,v)| \leq C_1(1+|u|+|v|)^{(q^+-1)/(p^+-1)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \\ &\sum_{i=1}^{k} |B_i(u,v)| \leq C_2(1+|u|+|v|)^{q^+-1}, \end{aligned} \tag{3.23}$$

then problem (3.21) *with* (1.2)–(1.4) *has at least one solution when the parameter* δ *is small enough.*

Proof. Denote

$$\begin{aligned} &\phi_\lambda\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\Big) \\ &\quad = f\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big) + \lambda\delta h\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big). \end{aligned} \tag{3.24}$$

We consider the existence of solutions of the following equation with (1.2)–(1.4)

$$-\Delta_{p(t)}u + \phi_\lambda\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\Big) = 0, \quad t \in (0,1),\ t \neq t_i. \tag{3.25}$$

Denote

$$\begin{aligned} \rho^{\#}_{1,\lambda}(u,\delta) &= \widetilde{\rho_1}(A,B,N_{\phi_\lambda})(u), \\ K^{\#}_{1,\lambda}(u,\delta) &= F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho^{\#}_{1,\lambda}(u,\delta) + \sum_{t_i<t} B_i + F(N_{\phi_\lambda}(u))(t)\right)\right]\right\}, \\ P^{\#}_{1,\lambda}(u,\delta) &= \frac{1}{(1-\sigma)}\int_0^1 g(t)\left[K^{\#}_{1,\lambda}(u,\delta)(t) + \sum_{t_i<t} A_i\right]dt, \\ \Phi_\delta(u,\lambda) &= P^{\#}_{1,\lambda}(u,\delta) + \sum_{t_i<t} A_i + K^{\#}_{1,\lambda}(u,\delta), \end{aligned} \tag{3.26}$$

where $N_{\phi_\lambda}(u)$ is defined in (2.11).

We know that (3.25) with (1.2)–(1.4) has the same solution of $u = \Phi_\delta(u,\lambda)$.

Obviously, $\phi_0 = f$. So $\Phi_\delta(u,0) = \Psi_f(u,1)$. As in the proof of Theorem 3.1, we know that all the solutions of $u = \Phi_\delta(u,0)$ are uniformly bounded, then there exists a large enough $R_0 > 0$ such that all the solutions of $u = \Phi_\delta(u,0)$ belong to $B(R_0) = \{u \in \mathrm{PC}^1 \mid \|u\|_1 < R_0\}$. Since $\Phi_\delta(\cdot,0)$ is compact continuous from PC^1 to PC^1, we have

$$\inf_{u \in \partial B(R_0)} \|u - \Phi_\delta(u,0)\|_1 > 0. \tag{3.27}$$

Since f, h are Caratheodory, we have

$$\begin{aligned}
\left\|F(N_{\phi_\lambda}(u)) - F(N_{\phi_0}(u))\right\|_0 &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\
\left|\rho^{\#}_{1,\lambda}(u,\delta) - \rho^{\#}_{1,0}(u,\delta)\right| &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\
\left\|K^{\#}_{1,\lambda}(u,\delta) - K^{\#}_{1,0}(u,\delta)\right\|_1 &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\
\left|\rho^{\#}_{1,\lambda}(u,\delta) - P^{\#}_{1,0}(u,\delta)\right| &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0.
\end{aligned} \tag{3.28}$$

Thus

$$\|\Phi_\delta(u,\lambda) - \Phi_0(u,\lambda)\|_1 \longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \tag{3.29}$$

Obviously, $\Phi_0(u,\lambda) = \Phi_\delta(u,0) = \Phi_0(u,0)$. We obtain

$$\|\Phi_\delta(u,\lambda) - \Phi_\delta(u,0)\|_1 \longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \tag{3.30}$$

Thus, when δ is small enough, we can conclude that

$$\begin{aligned}
&\inf_{(u,\lambda)\in\partial B(R_0)\times[0,1]} \|u - \Phi_\delta(u,\lambda)\|_1 \\
&\quad\geq \inf_{u\in\partial B(R_0)} \|u - \Phi_\delta(u,0)\|_1 - \sup_{(u,\lambda)\in\overline{B(R_0)}\times[0,1]} \|\Phi_\delta(u,0) - \Phi_\delta(u,\lambda)\|_1 > 0.
\end{aligned} \tag{3.31}$$

Thus $u = \Phi_\delta(u,\lambda)$ has no solution on $\partial B(R_0)$ for any $\lambda \in [0,1]$, when δ is small enough. It means that the Leray-Schauder degree $d_{\mathrm{LS}}[I - \Phi_\delta(\cdot,\lambda), B(R_0), 0]$ is well defined for any $\lambda \in [0,1]$, and

$$d_{\mathrm{LS}}[I - \Phi_\delta(u,\lambda), B(R_0), 0] = d_{\mathrm{LS}}[I - \Phi_\delta(u,0), B(R_0), 0]. \tag{3.32}$$

Since $\Phi_\delta(u,0) = \Psi_f(u,1)$, from the proof of Theorem 3.1, we can see that the right hand side is nonzero. Thus (3.21) with (1.2)–(1.4) has at least one solution. This completes the proof. □

Theorem 3.4. *Suppose* $0 \leq \sum_{\ell=1}^{m-2} \alpha_\ell < 1$ *and* $0 \leq \sigma < 1$, f *satisfies sub-*$(p^- - 1)$ *growth condition, and we assume that*

$$\left.\begin{aligned}
&\sum_{i=1}^{k} |A_i(u,v)| \leq C_1(1 + |u| + |v|)^{(q^+-1)/(p^+-1)}, \\
&\sum_{i=1}^{k} |D_i(u,v)| \leq C_2(1 + |u| + |v|)^{\alpha_i^+},
\end{aligned}\right\} \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.33}$$

where $\alpha_i \le (q^+ - 1)/(p(r_i) - 1)$, *and* $p(r_i) - 1 \le q^+ - \alpha_i$, $i = 1, \ldots, k$, *then problem* (3.21) *with* (1.2), (1.4), *and* (1.8) *has at least one solution when the parameter* δ *is small enough.*

Proof. As it is similar to the proof of Theorems 3.2 and 3.3, we omit it here. □

In the following, we will consider the existence of nonnegative solutions. For any $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ means $x^j \ge 0$ for any $j = 1, \ldots, N$.

Theorem 3.5. *Suppose* $0 \le \sum_{\ell=1}^{m-2} \alpha_\ell < 1$, $0 \le \sigma < 1$, *we also assume*

(1^0) $f(t, x, y, s, z) \ge 0$, *for all* $(t, x, y, s, z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;

(2^0) *for any* $i = 1, \ldots, k$, $B_i(u, v) \ge 0$, *for all* $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then every solution of (1.1)–(1.4) *is nonnegative.*

Proof. Let u be a solution of (1.1)–(1.4), integrating (1.1) from 0 to t, we have

$$w(t)\varphi(t, u'(t)) = \rho_1(u) + \sum_{t_i < t} B_i + F(N_f(u))(t), \quad \forall t \in (0,1), \ t \ne t_1, \ldots, t_k, \tag{3.34}$$

where $\rho_1 = w(0)\varphi(0, u'(0))$. The boundary value condition holds

$$\rho_1 = \frac{\left\{\sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i < \eta_\ell} B_i + F(N_f(u))(\eta_\ell)\right]\right\}}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell}. \tag{3.35}$$

Conditions (1^0)-(2^0) mean $\rho_1(u) \ge 0$. Obviously, for any for all $t \in J'$, we have

$$w(t)\varphi(t, u'(t)) = \rho_1(u) + \sum_{t_i < t} B_i + F(N_f(u))(t) \ge 0. \tag{3.36}$$

It follows from conditions (1^0)-(2^0) and (3.36) that $u(t)$ is increasing on J, namely $u(t') - u(t'') \ge 0$, for all $t', t'' \in J$ with $t' \ge t''$. Thus the boundary value condition holds $u(0) = \int_0^1 g(t)u(t)dt \ge \int_0^1 g(t)u(0)dt = \sigma u(0)$, then $u(0) \ge 0$.

Since $u(t)$ is increasing and $u(0) \ge 0$, we have $u(t) \ge 0$, for all $t \in J$.

Thus every solution of (1.1)–(1.4) is nonnegative. The proof is completed. □

Corollary 3.6. *Under the conditions of Theorem 3.1, we also assume*

(1^0) $f(t, x, y, s, z) \ge 0$, *for all* $(t, x, y, s, z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, s, z \ge 0$;

(2^0) *for any* $i = 1, \ldots, k$, $B_i(u, v) \ge 0$, *for all* $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ *with* $u \ge 0$;

(3^0) *for any* $t \in [0,1]$ *and* $s \in [0,1]$, $k_*(t, s) \ge 0$, $h_*(t, s) \ge 0$.

Then (1.1)–(1.4) *has a nonnegative solution.*

Proof. Define $M(u) = (M_*(u^1), \ldots, M_*(u^N))$, where

$$M_*(u) = \begin{cases} u, & u \ge 0, \\ 0, & u < 0. \end{cases} \tag{3.37}$$

Denote

$$\widetilde{f}(t,u,v,S(u),T(u)) = f(t,M(u),v,S(M(u)),T(M(u))), \quad \forall (t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N, \tag{3.38}$$

then $\widetilde{f}(t,u,v,S(u),T(u))$ satisfies Caratheodory condition, and $\widetilde{f}(t,u,v,S(u),T(u)) \geq 0$ for any $(t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $i = 1,\dots,k$, we denote

$$\widetilde{A}_i(u,v) = A_i(M(u),v), \quad \widetilde{B}_i(u,v) = B_i(M(u),v), \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.39}$$

then $\widetilde{A}_i$ and $\widetilde{B}_i$ are continuous, and satisfy

$$\widetilde{B}_i(u,v) \geq 0, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N \text{ for any } i = 1,\dots,k. \tag{3.40}$$

It is not hard to check that

$(2^0)'$ $\lim_{|u|+|v|\to+\infty}(\widetilde{f}(t,u,v,S(u),T(u))/(|u|+|v|)^{q(t)-1}) = 0$, for $t \in J$ uniformly, where $q(t) \in C(J,\mathbb{R})$, and $1 < q^- \leq q^+ < p^-$;

$(3^0)'$ $\sum_{i=1}^k |\widetilde{A}_i(u,v)| \leq C_1(1+|u|+|v|)^{(q^+-1)/(p^+-1)}$, for all $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;

$(4^0)'$ $\sum_{i=1}^k |\widetilde{B}_i(u,v)| \leq C_2(1+|u|+|v|)^{q^+-1}$, for all $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Let us consider

$$\begin{gathered}
-\Delta_{p(t)}u + \widetilde{f}\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right) = 0, \quad t \in J', \\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t_i) = \widetilde{A}_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to t_i^+} w(t)\varphi(t,u'(t)) - \lim_{t\to t_i^-} w(t)\varphi(t,u'(t)) \\
= \widetilde{B}_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t\to \eta_\ell^-} w(t)\left|u'\right|^{p(t)-2}u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{3.41}$$

It follows from Theorems 3.1 and 3.5 that (3.41) have a nonnegative solution u. Since $u \geq 0$, we have $M(u) = u$. Thus u is a nonnegative solution of (1.1)–(1.4). This completes the proof. □

4. Existence of Solutions in the Case of Resonance

In the following, we will consider the existence of solutions for system (1.1)–(1.4) at resonance.

Theorem 4.1. *Suppose* $\sum_{\ell=1}^{m-2}\alpha_\ell = 1$ *and* $\sigma = 1$, Ω *is an open bounded set in* PC^1 *such that the following conditions hold.*

(1^0) *For each* $\lambda \in (0,1)$ *the problem*

$$\begin{gathered}
-\Delta_{p(t)}u + \lambda N_f(u)(t) = 0, \quad t \in (0,1),\ t \neq t_i, \\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = \lambda A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t) \\
= \lambda B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2}\alpha_\ell \lim_{t\to\eta_\ell^-} w(t)\left|u'\right|^{p(t)-2}u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{4.1}$$

has no solution on $\partial\Omega$.

(2^0) *The equation*

$$\omega(l) := \left\{\sum_{\ell=1}^{m-2}\alpha_\ell\left[\sum_{t_i<\eta_\ell} B_i(l,0) + \int_0^{\eta_\ell} f(t,l,0,S(l),T(l))dt\right]\right\} = 0, \tag{4.2}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3^0) *The Brouwer degree* $d_B[\omega, \Omega\cap\mathbb{R}^N, 0] \neq 0$.

Then problem (1.1)–(1.4) *have a solution on* $\overline{\Omega}$.

Proof. Let us consider the following impulsive equation

$$\begin{gathered}
-\Delta_{p(t)}u + \lambda N_f(u)(t) + \frac{(1-\lambda)\left[Q_{N_f}(u)\right]}{\sum_{\ell=1}^{m-2}\alpha_\ell\eta_\ell} = 0, \quad t \in (0,1),\ t \neq t_i, \\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = \lambda A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t) \\
= \lambda B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2}\alpha_\ell \lim_{t\to\eta_\ell^-} w(t)\left|u'\right|^{p(t)-2}u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{4.3}$$

For any $\lambda \in (0,1]$, if u is a solution to (4.1) or u is a solution to (4.3), we have necessarily

$$Q_{N_f}(u) = 0. \tag{4.4}$$

It means that (4.1) and (4.3) have the same solutions for $\lambda \in (0,1]$.

We denote $N(\cdot,\cdot) : \mathrm{PC}^1 \times [0,1] \to L^1$ defined by

$$N(u,\lambda) = \lambda N_f(u) + \frac{(1-\lambda)\left[Q_{N_f}(u)\right]}{\sum_{\ell=1}^{m-2} \alpha_\ell \eta_\ell}, \tag{4.5}$$

where $N_f(u)$ is defined by (2.11). Denote

$$\begin{gathered}
Q_\lambda : L^1 \longrightarrow L^1, u \longrightarrow \sum_{\ell=1}^{m-2} \alpha_\ell \left[\lambda \sum_{t_i<\eta_\ell} B_i + F(N(u,\lambda))(\eta_\ell)\right], \\
\Theta_\lambda : L^1 \longrightarrow L^1, u \longrightarrow N(u,\lambda) - \frac{Q_\lambda(u)}{\sum_{\ell=1}^{m-2} \alpha_\ell \eta_\ell}, \\
\rho_{2,\lambda}(u) = \widetilde{\rho_2}(\lambda A, \lambda B, \Theta_\lambda), \\
K_{2,\lambda}(u)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_{2,\lambda}(u) + \lambda \sum_{t_i<t} B_i + F(\Theta_\lambda(u))(t)\right)\right]\right\}(t), \quad \forall t \in J.
\end{gathered} \tag{4.6}$$

Set

$$\Psi_f^*(u,\lambda) = P_2(u) + \lambda \sum_{t_i<t} A_i + Q_\lambda(u) + K_{2,\lambda}(u), \tag{4.7}$$

then the fixed point of $\Psi_f^*(u,1)$ is a solution for (1.1)–(1.4). Also problem (4.3) can be rewritten in the equivalent form

$$u = \Psi_f^*(u,\lambda). \tag{4.8}$$

Since f is Caratheodory, it is easy to see that $N(\cdot,\cdot)$ is continuous and sends bounded sets into equiintegrable sets. It is easy to see that P_2 is compact continuous. From Lemma 2.6, we can conclude that $\Psi_f^*(u,\lambda)$ is continuous and compact for any $\lambda \in [0,1]$. We assume that (4.8) does not have a solution on $\partial\Omega$ for $\lambda = 1$, otherwise we complete the proof. Now from hypothesis (1^0) it follows that (4.8) has no solutions for $(u,\lambda) \in \partial\Omega \times (0,1]$. For $\lambda = 0$, (4.3) is equivalent to the following usual problem

$$\begin{gathered}
-\Delta_{p(t)} u + \frac{[QN_f(u)]}{\sum_{\ell=1}^{m-2} \alpha_\ell \eta_\ell} = 0, \quad t \in (0,1), \\
\lim_{t\to 0^+} w(t)|u'|^{p(t)-2} u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t\to \eta_\ell^-} w(t)|u'|^{p(t)-2} u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{4.9}$$

If u is a solution to this problem, we must have

$$0=\sum_{\ell=1}^{m-2}\alpha_\ell\left[\sum_{t_i<\eta_\ell}B_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right)+\int_0^{\eta_\ell}N_f(u)dt\right]. \tag{4.10}$$

As this problem is a usual differential equation, we have

$$w(t)\left|u'\right|^{p(t)-2}u'\equiv\rho_2, \tag{4.11}$$

where $\rho_2\in\mathbb{R}^N$ is a constant. Therefore $(u^i)'$ keeps the same sign of ρ_2^i. From $u(0)=\int_0^1 g(t)u(t)dt$, we have $\int_0^1 g(t)[u(0)-u(t)]dt=0$. From the continuity of u, there exist $t_i\in(0,1)$, such that $(u^i)'(t_i)=0$, $i=1,\dots,N$. Hence $u'\equiv 0$, it holds $u\equiv d$, a constant. Thus (4.10) holds

$$\sum_{\ell=1}^{m-2}\alpha_\ell\left[\sum_{t_i<\eta_\ell}B_i(l,0)+\int_0^{\eta_\ell}f(t,l,0,S(l),T(l))dt\right]=0, \tag{4.12}$$

which together with hypothesis (2^0) implies that $u=d\notin\partial\Omega$. Thus we have proved that (4.8) has no solution (u,λ) on $\partial\Omega\times[0,1]$. Therefore the Leray-Schauder degree $d_{\mathrm{LS}}[I-\Psi_f^*(\cdot,\lambda),\Omega,0]$ is well defined for $\lambda\in[0,1]$, and from the homotopy invariant property of that degree we have

$$d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,1),\Omega,0\right]=d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,0),\Omega,0\right]. \tag{4.13}$$

Now it is clear that the problem

$$u=\Psi_f^*(u,1) \tag{4.14}$$

is equivalent to problem (1.1)–(1.4), and (4.13) tells us that problem (4.14) will have a solution if we can show that

$$d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,0),\Omega,0\right]\neq 0. \tag{4.15}$$

It is not hard to check that $K_{2,0}(\cdot)\equiv 0$. Thus

$$\begin{aligned}\Psi_f^*(u,0)&=P_2u+QN_f(u)+K_{2,0}(u)=P_2u+QN_f(u),\\ u-\Psi_f^*(u,0)&=u-P_2u-QN_f(u)=-QN_f(u),\quad \text{on } \overline{\Omega}.\end{aligned} \tag{4.16}$$

By the properties of the Leray-Schauder degree, we have

$$d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,0),\Omega,0\right]=(-1)^N d_B\left[\omega,\Omega\cap\mathbb{R}^N,0\right], \tag{4.17}$$

where the function ω is defined in (4.2) and d_B denotes the Brouwer degree. By hypothesis (3^0), this last degree is different from zero. This completes the proof. □

Our next theorem is a consequence of Theorem 4.1. As an application of Theorem 4.1, let us consider the following system

$$-\Delta_{p(t)}u+\gamma\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big)+e\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big)=0,\quad t\in J', \tag{4.18}$$

with (1.2), (1.3), and (1.4), where $e: J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is Caratheodory, $\gamma=(\gamma^1,\dots,\gamma^N): J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is continuous, and for any fixed $y_0\in\mathbb{R}^N$, $y_0^i\neq 0$ holds $\gamma^i(t,y_0,0,S(y_0),T(y_0))\neq 0$, for all $t\in J$, $i=1,\dots,N$.

Theorem 4.2. *Suppose that the following conditions hold*

(1^0) $\gamma(t,kx,ky,ks,kz)=k^{q(t)-1}\gamma(t,x,y,s,z)$ *for all* $k>0$ *and all* $(t,x,y,s,z)\in J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$, *where* $q(t)\in C(J,\mathbb{R})$ *satisfies* $1<q^-\le q^+<p^-$;

(2^0) $\lim_{|u|+|v|+|s|+|z|\to+\infty}(e(t,u,v,s,z)/(|u|+|v|+|s|+|z|)^{q(t)-1})=0$, *for* $t\in J$ *uniformly;*

(3^0) $\sum_{i=1}^k|A_i(u,v)|\le C_1(1+|u|+|v|)^{\theta}$, *for all* $(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$, *where* $0<\theta<(p^--1)/(p^+-1)$;

(4^0) $\sum_{i=1}^k|B_i(u,v)|\le C_2(1+|u|+|v|)^{\beta-1}$, *for all* $(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$, *where* $1\le\beta<q^+$;

(5^0) *for large enough* $R_0>0$, *the equation*

$$\omega_\gamma(l):=\left\{\sum_{e=1}^{m-2}\alpha_e\left[\sum_{t_i<\eta_e}B_i(l,0)+\int_0^{\eta_e}\gamma(t,l,0,S(l),T(l))dt\right]\right\}=0, \tag{4.19}$$

has no solution on $\partial B(R_0)\cap\mathbb{R}^N$, *where* $B(R_0)=\{u\in PC^1\mid \|u\|_1<R_0\}$;

(6^0) *the Brouwer degree* $d_B[\omega_\gamma,b(R_0),0]\neq 0$ *for large enough* $R_0>0$, *where* $b(R_0)=\{x\in\mathbb{R}^N\mid |x|<R_0\}$.

Then problem (4.18) with (1.2), (1.3), and (1.4) has at least one solution.

Proof. For any $u\in \mathrm{PC}^1$ and $\lambda\in[0,1]$, we denote

$$N_{f_\lambda}(u)=\gamma\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big)+\lambda e\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big). \tag{4.20}$$

At first, we consider the following problem

$$\begin{gathered}
-\Delta_{p(t)}u + N_{f_\lambda}(u)(t) = 0, \quad t \in (0,1),\ t \neq t_i, \\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = A_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t) \\
= B_i\left(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{1/(p(t)-1)} u'(t)\right), \quad i = 1,\dots,k, \\
\lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{\ell=1}^{m-2} \alpha_\ell \lim_{t\to \eta_\ell^-} w(t)\left|u'\right|^{p(t)-2}u'(t), \qquad u(0) = \int_0^1 g(t)u(t)dt.
\end{gathered} \tag{4.21}$$

As in the proof of Theorem 4.1, we know that (4.21) has the same solutions of

$$u = \Psi_f^*(u,\lambda) = P_2(u) + \sum_{t_i<t} A_i + QN_{f_\lambda}(u) + K_2\left(\Theta_{f_\lambda}(u)\right), \tag{4.22}$$

where Θ_{f_λ} is defined in (2.39).

We claim that all the solutions of (4.21) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n, \lambda_n)\}$ for (4.21) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$, and $\|u_n\|_1 > 1$ for any $n = 1, 2, \dots$.

Since (u_n, λ_n) are solutions of (4.21), we have

$$w(t)\varphi\left(t, u_n'(t)\right) = \rho_2(u_n) + \sum_{t_i<t} B_i + F\left(N_{f_{\lambda_n}}(u_n)\right)(t), \tag{4.23}$$

$u_n(t) = u_n(0) + \sum_{t_i<t} A_i + F\{\varphi^{-1}[t, (w(t))^{-1}(\rho_2(u_n) + \sum_{t_i<t} B_i + F(N_{f_{\lambda_n}}(u_n))(t))]\}(t)$.Since $u_n(0) = \int_0^1 g(t)u_n(t)dt$, we have

$$\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_2(u_n) + \sum_{t_i<t} B_i + F\left(N_{f_{\lambda_n}}(u_n)\right)(t)\right)\right]\right\}(t) + \sum_{t_i<t} A_i\right)dt = 0. \tag{4.24}$$

It follows from Lemma 2.5 that

$$\left|\rho_2(u_n)\right| \le 3NC\left(1 + \|u_n\|_1^{\theta(p^+-1)} + \|u_n\|_1^{q^+-1}\right). \tag{4.25}$$

From (3^0), (4^0), (4.23) and (4.25), we can see that

$$\left\|(w(t))^{1/(p(t)-1)} u_n'(t)\right\|_0 \le o(1)\|u_n\|_1. \tag{4.26}$$

From (4.26), we have

$$\lim_{n\to+\infty}\frac{\|u_n\|_0}{\|u_n\|_1}=1. \tag{4.27}$$

Denote $\delta_n=(|u_n^1|_0/\|u_n\|_0, |u_n^2|_0/\|u_n\|_0,\ldots,|u_n^N|_0/\|u_n\|_0)$, then $\delta_n\in\mathbb{R}^N$ and $|\delta_n|=1$ $(n=1,2,\ldots)$. Thus $\{\delta_n\}$ possesses a convergent subsequence (which still denoted by δ_n), then there exists a vector $\delta_0=(\delta_0^1,\delta_0^2,\ldots,\delta_0^N)\in\mathbb{R}^N$ such that $|\delta_0|=1$ and $\lim_{n\to+\infty}\delta_n=\delta_0$. Without loss of generality, we assume that $\delta_0^1>0$. Since $u_n\in \mathrm{PC}^1$, there exist $\eta_n^i\in J$ such that

$$\left|u_n^i\left(\eta_n^i\right)\right|\geq\left(1-\frac{1}{n}\right)\left|u_n^i\right|_0,\quad i=1,2,\ldots,N;\ n=1,2,\ldots. \tag{4.28}$$

Obviously

$$\left|u_n^1(t)-u_n^1\left(\eta_n^1\right)\right|=\left|\int_{\eta_n^1}^t\left(u_n^1\right)'(t)dt+\sum_{\eta_n^1<t_i<t}A_i^1\right|\leq o(1)\|u_n\|_1\int_0^1(w(t))^{-1/(p(t)-1)}dt+\sum_{i=1}^k\left|A_i^1\right|. \tag{4.29}$$

Note that $\|u_n\|_1\to+\infty$ (as $n\to+\infty$) and $\delta_0^1>0$, it follows from (4.27), (4.28), and (3^0) that

$$\lim_{n\to+\infty}\frac{\left\{o(1)\|u_n\|_1\int_0^1(w(t))^{-1/(p(t)-1)}dt+\sum_{i=1}^k\left|A_i^1\right|\right\}}{\left|u_n^1(\eta_n^1)\right|}=0. \tag{4.30}$$

By (4.27), (4.29), and (4.30) we have $\lim_{n\to+\infty}u_n^1(t)/u_n^1(\eta_n^1)=1$ for $t\in J$ uniformly, which implies

$$\lim_{n\to+\infty}\frac{u_n(t)}{\|u_n\|_1}=\delta_*,\quad \lim_{n\to+\infty}\frac{(w(t))^{1/(p(t)-1)}u_n'(t)}{\|u_n\|_1}=0,\quad \text{for } t\in J \text{ uniformly}, \tag{4.31}$$

where $\delta_*\in\mathbb{R}^N$, satisfies $|\delta_*|=1$, $|\delta_*^i|=\delta_0^i$.

From (1.4), we have

$$\begin{aligned}0=\sum_{\ell=1}^{m-2}\alpha_\ell\Bigg\{\sum_{t_i<\eta_\ell}B_i+\int_0^{\eta_\ell}\Big[&\gamma\Big(t,u_n,(w(t))^{1/(p(t)-1)}u_n',S(u_n),T(u_n)\Big)\\&+e\left(t,u_n,(w(t))^{1/(p(t)-1)}u_n',S(u_n),T(u_n)\right)\Big]dt\Bigg\}.\end{aligned} \tag{4.32}$$

Note that $\gamma^1(t,\delta_*,0,S(\delta_*),T(\delta_*))\neq 0$, it follows from (4.31), (4^0) and the continuity of γ^1 that

$$\sum_{\ell=1}^{m-2}\alpha_\ell\left\{\sum_{t_i<\eta_\ell}B_i^1+\int_0^{\eta_\ell}\|u_n\|_1^{q(t)-1}\left\{\gamma^1[t,\delta_*,0,S(\delta_*),T(\delta_*)]+o(1)\right\}dt\right\}\neq 0, \tag{4.33}$$

which contradicts to (4.32). This implies that there exists a big enough $R_0>0$ such that all the solutions of (4.21) belong to $B(R_0)$, then we have

$$d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,1),B(R_0),0\right]=d_{\mathrm{LS}}\left[I-\Psi_f^*(\cdot,0),B(R_0),0\right]. \tag{4.34}$$

In order to obtaining the existence of solutions (4.18) with (1.2), (1.3), and (1.4), we only need to prove that $d_{\mathrm{LS}}[I-\Psi_f^*(\cdot,0),B(R_0),0]\neq 0$.

Now we consider the following equation

$$\begin{gathered}
-\Delta_{p(t)}u+\lambda N_\gamma(u)(t)+\frac{(1-\lambda)\left[Q_{N_\gamma}(u)\right]}{\sum_{\ell=1}^{m-2}\alpha_\ell\eta_\ell}=0,\quad t\in(0,1),\ t\neq t_i,\\
\lim_{t\to t_i^+}u(t)-\lim_{t\to t_i^-}u(t)=\lambda A_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right),\quad i=1,\ldots,k,\\
\lim_{t\to t_i^+}w(t)|u'|^{p(t)-2}u'(t)-\lim_{t\to t_i^-}w(t)|u'|^{p(t)-2}u'(t)\\
=\lambda B_i\left(\lim_{t\to t_i^-}u(t),\lim_{t\to t_i^-}(w(t))^{1/(p(t)-1)}u'(t)\right),\quad i=1,\ldots,k,\\
\lim_{t\to 0^+}w(t)\left|u'\right|^{p(t)-2}u'(t)=\sum_{\ell=1}^{m-2}\alpha_\ell\lim_{t\to\eta_\ell^-}w(t)\left|u'\right|^{p(t)-2}u'(t),\quad u(0)=\int_0^1 g(t)u(t)dt,
\end{gathered} \tag{4.35}$$

where $N_\gamma(u)=\gamma(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u))$.

Similar to the preceding discussion, for any $\lambda\in(0,1]$, all the solutions of (4.35) are uniformly bounded.

If u is a solution of the following usual equation with (1.4)

$$\left(w(t)\left|u'\right|^{p(t)-2}u'\right)'=\frac{\left[Q_{N_\gamma}(u)\right]}{\sum_{\ell=1}^{m-2}\alpha_\ell\eta_\ell},\quad t\in(0,1), \tag{4.36}$$

we have

$$Q_{N_\gamma}(u)=0,\qquad w(t)\left|u'\right|^{p(t)-2}u'\equiv c. \tag{4.37}$$

As $u(0) = \int_0^1 g(t)u(t)dt$, we have $w(t)|u'|^{p(t)-2}u' \equiv 0$, it means that u is a solution of

$$\omega_\gamma(l) = \left\{ \sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} B_i(l,0) + \int_0^{\eta_\ell} \gamma(t,l,0,S(l),T(l))dt \right] \right\} = 0. \tag{4.38}$$

By hypothesis (5^0), (4.35) has no solutions on $\partial B(R_0) \times [0,1]$, from Theorem 4.1, we obtain that (4.18) with (1.2), (1.3), and (1.4) has at least one solution. This completes the proof. □

Corollary 4.3. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, conditions (2^0), (3^0) and (4^0) of Theorem 4.2 are satisfied, condition (3^0) of Corollary 3.6 is also satisfied, $\gamma(t,u,v,S(u),T(u)) = \beta(t)(|u|^{q(t)-2}u+|v|^{q(t)-2}v+|S(u)|^{q(t)-2}S(u)+|T(u)|^{q(t)-2}T(u))$, where $\beta(t), q(t) \in C(J,\mathbb{R})$ are positive functions satisfying $1 < q^- \le q^+ < p^-$; then* (4.18) *with* (1.2), (1.3), *and* (1.4) *has at least one solution.*

Proof. Denote

$$G(l,\lambda) = \left\{ \sum_{\ell=1}^{m-2} \alpha_\ell \left[\sum_{t_i<\eta_\ell} \lambda B_i(l,0) + \int_0^{\eta_\ell} \gamma(t,l,0,S(l),T(l))dt \right] \right\}. \tag{4.39}$$

From condition (4^0), we have

$$|B_i(l,0)| \le C(1+|l|)^{\beta-1}, \quad 1 \le \beta < q^+. \tag{4.40}$$

Note that k_* and h_* are nonnegative. From the above inequality, we can see that all the solutions of $G(l,\lambda) = 0$ are uniformly bounded for $\lambda \in [0,1]$. Thus $d_B[G(l,\lambda), b(R_0), 0]$ is well defined for $\lambda \in [0,1]$ and

$$\begin{gathered} d_B[\omega_\gamma, b(R_0), 0] = d_B[G(l,1), b(R_0), 0] = d_B[G(l,0), b(R_0), 0], \\ G(l,0) = \left\{ \sum_{\ell=1}^{m-2} \alpha_\ell \left(\int_0^{\eta_\ell} \beta(t) \left[|l|^{q(t)-2}l + |S(l)|^{q(t)-2}S(l) + |T(l)|^{q(t)-2}T(l) \right] dt \right) \right\}, \end{gathered} \tag{4.41}$$

and it is easy to see that $G(l,0) = 0$ has a unique solution in $\mathbb{R}^N$ and

$$d_B[\omega_\gamma, b(R_0), 0] = d_B[I, b(R_0), 0] \ne 0. \tag{4.42}$$

According to Theorem 4.2, we get that (4.18) with (1.2), (1.3), and (1.4) has at least a solution. This completes the proof. □

Let us consider

$$-\Delta_{p(t)}u + f\left(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\right) = 0, \quad t \in (0,1), \ t \ne t_i, \tag{4.43}$$

where δ is a parameter, and

$$\begin{aligned} &f\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u),\delta\Big) \\ &\quad = \varsigma\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big) + \delta h\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big), \end{aligned} \tag{4.44}$$

where $h,\varsigma : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory.

From Theorem 4.2, similar to the proof of Theorem 3.3, we have the following.

Theorem 4.4. *If conditions of (1^0) and (3^0)–(6^0) of Theorem 4.2 are satisfied, then problem* (4.43) *with* (1.2), (1.3), *and* (1.4) *has at least one solution when the parameter δ is small enough.*

Theorem 4.5. *If conditions of (1^0)–(3^0) and (5^0)-(6^0) of Theorem 4.2 are satisfied, and D satisfy*

$$\sum_{i=1}^{k} |D_i(u,v)| \leq C(1+|u|+|v|)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{4.45}$$

where

$$\alpha_i \leq \frac{q^+ - 1}{p(r_i) - 1}, \quad p(r_i) - 1 \leq q^+ - \alpha_i, \quad i = 1, \dots, k, \tag{4.46}$$

then problem (4.18) *with* (1.2), (1.3), *and* (1.8) *has at least one solution.*

Proof. Similar to the proof of Theorem 3.2, the condition (4^0) of Theorem 4.2 is satisfied. Thus problem (4.18) with (1.2), (1.3) and (1.8) has at least a solution. □

Similar to the proof of Theorem 3.2 and Corollary 4.3, we have the following.

Corollary 4.6. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory,* (4.45), (4.46) *and conditions (2^0) and (3^0) of Theorem 4.2 are satisfied, condition (3^0) of Corollary 3.6 is also satisfied, $\gamma(t,u,v,S(u),T(u)) = \beta(t)(|u|^{q(t)-2}u + |v|^{q(t)-2}v + |S(u)|^{q(t)-2}S(u) + |T(u)|^{q(t)-2}T(u))$, where $\beta(t), q(t) \in C(J,\mathbb{R})$ are positive functions satisfying $1 < q^- \leq q^+ < p^-$; then* (4.43) *with* (1.2), (1.3), *and* (1.8) *has at least one solution when the parameter δ is small enough.*

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