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Research Article

Convolution Properties of Classes of Analytic and Meromorphic Functions

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General classes of analytic functions defined by convolution with a fixed analytic function are introduced. Convolution properties of these classes which include the classical classes of starlike, convex, close-to-convex, and quasiconvex analytic functions are investigated. These classes are shown to be closed under convolution with prestarlike functions and the Bernardi-Libera integral operator. Similar results are also obtained for the classes consisting of meromorphic functions in the punctured unit disk.

1. Motivation and Definitions

Let $\mathcal{H}(\mathcal{U})$ be the set of all analytic functions defined in the unit disk $\mathcal{U} := \{z : |z| < 1\}$. Denote by $\mathcal{A}$ the class of normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined in $\mathcal{U}$. For two functions f and g in $\mathcal{A}$, the convolution or Hadamard product of f and g is the function $f * g$ defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. A function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. If the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

The classes of starlike and convex analytic functions and other related subclasses of analytic functions can be put in the form

$$\mathcal{S}^*(g, h) := \left\{ f \in \mathcal{A} \mid \frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z) \right\}, \tag{1.1}$$

where g is a fixed function and h is a suitably normalized function with positive real part. In particular, let $\mathcal{S}^*(h) := \mathcal{S}^*(z/(1-z), h)$ and $\mathcal{K}(h) := \mathcal{S}^*(z/(1-z)^2, h)$. For $h(z) = (1+(1-2\alpha)z)/(1-z)$, $0 \leq \alpha < 1$, $\mathcal{S}^*(h)$ and $\mathcal{K}(h)$ are, respectively, the familiar classes $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{K}(\alpha)$ consisting of convex functions of order α. Analogous to the class $\mathcal{S}^*(g,h)$, the class $\mathcal{K}(g,h)$ is defined by

$$\mathcal{K}(g,h) := \left\{ f \in \mathcal{A} \mid 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z) \right\}. \tag{1.2}$$

Let f and $g \in \mathcal{A}$ satisfy

$$\operatorname{Re}\frac{zf'(z)}{f(z)+g(z)} > 0, \qquad \operatorname{Re}\frac{zg'(z)}{f(z)+g(z)} > 0. \tag{1.3}$$

By adding the two inequalities, it is evident that the function $(f(z)+g(z))/2$ is starlike and hence both f and g are close-to-convex and univalent. This motivates us to consider the following classes of functions.

It is assumed in the sequel that $m \geq 1$ is a fixed integer, g is a fixed function in $\mathcal{A}$, and h is a convex univalent function with positive real part in $\mathcal{U}$ satisfying $h(0) = 1$.

Definition 1.1. The class $\mathcal{S}_m^*(h)$ consists of $\widehat{f} := \langle f_1, f_2, \ldots, f_m \rangle$, $f_k \in \mathcal{A}$, $1 \leq k \leq m$, satisfying $\sum_{j=1}^m f_j(z)/z \neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{mzf_k'(z)}{\sum_{j=1}^m f_j(z)} \prec h(z), \quad k = 1, \ldots, m. \tag{1.4}$$

The class $\mathcal{S}_m^*(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g := \langle f_1 * g, f_2 * g, \ldots, f_m * g \rangle \in \mathcal{S}_m^*(h)$. The class $\mathcal{K}_m(h)$ consists of $\widehat{f}$ for which $z\widehat{f}' \in \mathcal{S}_m^*(h)$, where $\widehat{f}' := \langle f_1', f_2', \ldots, f_m' \rangle$ and $z\widehat{f}' := \langle zf_1', zf_2', \ldots, zf_m' \rangle$. Equivalently, $\widehat{f} \in \mathcal{K}_m(h)$ if $\widehat{f}$ satisfies the condition $\sum_{j=1}^m f_j'(z) \neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{m(zf_k')'(z)}{\sum_{j=1}^m f_j'(z)} \prec h(z), \quad k = 1, \ldots, m. \tag{1.5}$$

The class $\mathcal{K}_m(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g \in \mathcal{K}_m(h)$.

Now let $\widehat{f} \in \mathcal{S}_m^*(h)$ and $F(z) = \sum_{j=1}^m f_j(z)/m$. From (1.4), it follows that

$$\frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}), \quad k = 1, \ldots, m. \tag{1.6}$$

The convexity of $h(\mathcal{U})$ implies that

$$\frac{1}{m}\frac{z\sum_{k=1}^{m} f_k'(z)}{F(z)} \in h(\mathcal{U}), \tag{1.7}$$

which shows that the function F is starlike in $\mathcal{U}$. Thus, it follows from (1.4) that the component function f_k of $\widehat{f}$ is close-to-convex in $\mathcal{U}$, and hence univalent. Similarly, the component function f_k of $\widehat{f} \in \mathcal{K}_m(h)$ is univalent.

If $m = 1$, then the classes $\mathcal{S}_m^*(g,h)$ and $\mathcal{K}_m(g,h)$ are reduced, respectively, to $\mathcal{S}^*(g,h)$ and $\mathcal{K}(g,h)$ introduced and investigated in [1]; these classes were denoted there by $\mathcal{S}_g(h)$ and $\mathcal{K}_g(h)$, respectively. If $g = k_a$, where

$$k_a(z) := \frac{z}{(1-z)^a}, \quad a > 0, \tag{1.8}$$

then the class $\mathcal{S}_m^*(g,h)$ coincides with the class studied in [2], which there was denoted by $\mathcal{S}_a(h)$, and $\mathcal{K}_m(g,h)$ reduces to a class introduced in [3] which there was denoted by $\mathcal{K}_a(h)$. It is evident that the classes $\mathcal{S}_m^*(g,h)$ and $\mathcal{K}_m(g,h)$ extend the classical classes of starlike and convex functions, respectively.

Definition 1.2. The class $\mathcal{C}_m(h)$ consists of $\widehat{f} := \langle f_1, f_2, \ldots, f_m\rangle$, $f_k \in \mathcal{A}$, $1 \le k \le m$, satisfying the subordination

$$\frac{mzf_k'(z)}{\sum_{j=1}^{m}\varphi_j(z)} \prec h(z), \quad k = 1,\ldots,m, \tag{1.9}$$

for some $\widehat{\varphi} \in \mathcal{S}_m^*(h)$. In this case, we say that $\widehat{f} \in \mathcal{C}_m(h)$ with respect to $\widehat{\varphi} \in \mathcal{S}_m^*(h)$. The class $\mathcal{C}_m(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g := \langle f_1 * g, f_2 * g, \ldots, f_m * g\rangle \in \mathcal{C}_m(h)$. The class $\mathcal{Q}_m(h)$ consists of $\widehat{f}$ for which $z\widehat{f}' \in \mathcal{C}_m(h)$ or equivalently satisfying the subordination

$$\frac{m(zf_k')'(z)}{\sum_{j=1}^{m}\varphi_j'(z)} \prec h(z), \quad k = 1,\ldots,m, \tag{1.10}$$

for some $\widehat{\varphi} \in \mathcal{K}_m(h)$ with $z\widehat{\varphi}' = \widehat{\psi}$, $\widehat{\psi} \in \mathcal{S}_m^*(h)$. In this case, we say that $\widehat{f} \in \mathcal{Q}_m(h)$ with respect to $\widehat{\varphi} \in \mathcal{K}_m(h)$. The class $\mathcal{Q}_m(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g \in \mathcal{Q}_m(h)$.

When $m = 1$, the classes $\mathcal{C}_m(g,h)$ and $\mathcal{Q}_m(g,h)$ reduce, respectively, to $\mathcal{C}_g(h)$ and $\mathcal{Q}_g(h)$ introduced and investigated in [1]. If $g = k_a$, where k_a is defined by (1.8), then the class $\mathcal{C}_m(g,h)$ coincides with $\mathcal{C}_a(h)$ studied in [2]. Clearly the classes $\mathcal{C}_m(g,h)$ and $\mathcal{Q}_m(g,h)$ extend the classical classes of close-to-convex and quasiconvex functions, respectively.

For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} \mid f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha) \right\}, \tag{1.11}$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$.

The well-known result that the classes of starlike functions of order α and convex functions of order α are closed under convolution with prestarlike functions of order α follows from the following.

Theorem 1.3 (see [4, Theorem 2.4]). *Let $\alpha \leq 1$, $\phi \in \mathcal{R}_\alpha$, and $f \in \mathcal{S}^*(\alpha)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\operatorname{co}}(H(\mathcal{U})), \tag{1.12}$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$, where $\overline{\operatorname{co}}(H(\mathcal{U}))$ denotes the closed convex hull of $H(\mathcal{U})$.

In the following section, by using the methods of convex hull and differential subordination, convolution properties of functions belonging to the four classes $\mathcal{S}_m^*(g,h)$, $\mathcal{K}_m(g,h)$, $\mathcal{C}_m(g,h)$ and $\mathcal{Q}_m(g,h)$, are investigated. It would be evident that various earlier works, see, for example, [5–10], are special instances of our work.

In Section 3, new subclasses of meromorphic functions are introduced. These subclasses extend the classical subclasses of meromorphic starlike, convex, close-to-convex, and quasiconvex functions. Convolution properties of these newly defined subclasses will be investigated. Simple consequences of the results obtained will include the work of Bharati and Rajagopal [6] involving the function $k_a(z) := 1/(z(1-z)^a)$, $a > 0$, as well as the work of Al-Oboudi and Al-Zkeri [5] on the modified Salagean operator.

2. Convolution of Analytic Functions

Our first result shows that the classes $\mathcal{S}_m^*(g,h)$ and $\mathcal{K}_m(g,h)$ are closed under convolution with prestarlike functions.

Theorem 2.1. *Let $m \geq 1$ be a fixed integer and g a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$, and $\phi \in \mathcal{R}_\alpha$.*

(1) *If $\hat{f} \in \mathcal{S}_m^*(g,h)$, then $\hat{f} * \phi \in \mathcal{S}_m^*(g,h)$.*

(2) *If $\hat{f} \in \mathcal{K}_m(g,h)$, then $\hat{f} * \phi \in \mathcal{K}_m(g,h)$.*

Proof. (1) It is sufficient to prove that $\hat{f} * \phi \in \mathcal{S}_m^*(h)$ whenever $\hat{f} \in \mathcal{S}_m^*(h)$. Once this is established, the general result for $\hat{f} \in \mathcal{S}_m^*(g,h)$ follows from the fact that

$$\hat{f} \in \mathcal{S}_m^*(g,h) \Longleftrightarrow \hat{f} * g \in \mathcal{S}_m^*(h). \tag{2.1}$$

For $k = 1, 2, \ldots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z), \qquad H_k(z) = \frac{z f_k'(z)}{F(z)}. \tag{2.2}$$

It will first be proved that F belongs to $\mathcal{S}^*(\alpha)$. For $\widehat{f} \in \mathcal{S}_m^*(h)$ and $z \in \mathcal{U}$, clearly

$$\frac{z f_k'(z)}{F(z)} \in h(\mathcal{U}), \quad k = 1, \ldots, m. \tag{2.3}$$

Since $h(\mathcal{U})$ is a convex domain, it follows that

$$\frac{1}{m}\sum_{k=1}^{m} \frac{z f_k'(z)}{F(z)} \in h(\mathcal{U}), \tag{2.4}$$

or

$$\frac{zF'(z)}{F(z)} \prec h(z). \tag{2.5}$$

Since $\operatorname{Re} h(z) > \alpha$, the subordination (2.5) yields

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) > \alpha, \tag{2.6}$$

and hence $F \in \mathcal{S}^*(\alpha)$.

A computation shows that

$$\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * f_j)(z)} = \frac{(\phi * z f_k')(z)}{\left(\phi * (1/m)\sum_{j=1}^{m} f_j\right)(z)} = \frac{(\phi * H_k F)(z)}{(\phi * F)(z)}. \tag{2.7}$$

Since $\phi \in R_\alpha$ and $F \in \mathcal{S}^*(\alpha)$, Theorem 1.3 yields

$$\frac{(\phi * H_k F)(z)}{(\phi * F)(z)} \in \overline{\text{co}}(H_k(\mathcal{U})), \tag{2.8}$$

and because $H_k(z) \prec h(z)$, we deduce that

$$\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * f_j)(z)} \prec h(z), \quad k = 1, \ldots, m. \tag{2.9}$$

Thus $\widehat{f} * \phi \in \mathcal{S}_m^*(h)$.

(2) The function $\widehat{f}$ is in $\mathcal{K}_m(g,h)$ if and only if $z\widehat{f}'$ is in $\mathcal{S}_m^*(g,h)$ and by the first part above, it follows that $\phi * z\widehat{f}' = z(\phi * \widehat{f})' \in \mathcal{S}_m^*(g,h)$. Hence $\phi * \widehat{f} \in \mathcal{K}_m(g,h)$. □

Remark 2.2. The above theorem can be expressed in the following equivalent forms:

$$\mathcal{S}_m^*(g,h) \subset \mathcal{S}_m^*(\phi * g,h), \qquad \mathcal{K}_m(g,h) \subset \mathcal{K}_m(\phi * g,h). \tag{2.10}$$

When $m = 1$, various known results are easily obtained as special cases of Theorem 2.1. For instance, [1, Theorem 3.3, page 336] is easily deduced from Theorem 2.1(1), while [1, Corollary 3.1, page 336] follows from Theorem 2.1(2). If $g(z) = k_a$ is defined by (1.8), then [3, Theorem 4, page 110] follows from Theorem 2.1(1), and [3, Corollary 4.1, page 111] follows from Theorem 2.1(2).

Corollary 2.3. *Let $m \geq 1$ be a fixed integer and g a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$. *Define*

$$F_k(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f_k(t)dt \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1,\ldots,m). \tag{2.11}$$

If $\widehat{f} \in \mathcal{S}_m^*(g,h)$, then $\widehat{F} = \langle F_1,\ldots,F_m\rangle \in \mathcal{S}_m^*(g,h)$. Similarly, if $\widehat{f} \in \mathcal{K}_m(g,h)$, then $\widehat{F} \in \mathcal{K}_m(g,h)$.

Proof. Define the function ϕ by

$$\phi(z) = z + \sum_{n=2}^{\infty}\frac{\gamma+1}{\gamma+n}z^n. \tag{2.12}$$

For $\operatorname{Re}\gamma \geq 0$, the function ϕ is a convex function [11], and hence $\phi \in \mathcal{R}_\alpha$ ([4, Theorem 2.1, page 49]). It is clear from the definition of F_k that

$$F_k(z) = f_k(z) * \left(z + \sum_{n=2}^{\infty}\frac{\gamma+1}{\gamma+n}z^n\right) = (f_k * \phi)(z), \tag{2.13}$$

so that $\widehat{F} = \widehat{f} * \phi$. By Theorem 2.1(1), it follows that $\widehat{F} = \widehat{f} * \phi \in \mathcal{S}_m^*(g,h)$.

The second result is proved in a similar manner. □

Remark 2.4. If $g(z) = k_a(z)$ is defined by (1.8), then Corollary 2.3 reduces to [2, Theorem 2, page 324].

Theorem 2.5. *Let $m \geq 1$ be a fixed integer and g a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$, *and $\phi \in \mathcal{R}_\alpha$.*

(1) *If $\widehat{f} \in \mathcal{C}_m(g,h)$ with respect to $\widehat{\varphi} \in \mathcal{S}_m^*(g,h)$, then $\widehat{f} * \phi \in \mathcal{C}_m(g,h)$ with respect to $\widehat{\varphi} * \phi \in \mathcal{S}_m^*(g,h)$.*

(2) *If $\widehat{f} \in \mathcal{Q}_m(g,h)$ with respect to $\widehat{\varphi} \in \mathcal{K}_m(g,h)$, then $\widehat{f} * \phi \in \mathcal{Q}_m(g,h)$ with respect to $\widehat{\varphi} * \phi \in \mathcal{K}_m(g,h)$.*

Proof. (1) In view of the fact that

$$\widehat{f} \in \mathcal{C}_m(g,h) \Longleftrightarrow \widehat{f} * g \in \mathcal{C}_m(h), \tag{2.14}$$

we well only prove that $\widehat{f} * \phi \in \mathcal{C}_m(h)$ when $\widehat{f} \in \mathcal{C}_m(h)$. Let $\widehat{f} \in \mathcal{C}_m(h)$. For $k = 1, 2, \dots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} \varphi_j(z), \qquad H_k(z) = \frac{z f_k'(z)}{F(z)}. \tag{2.15}$$

Since $\widehat{\varphi} \in \mathcal{S}_m^*(h)$, it is evident from (2.6) that $F \in \mathcal{S}^*(\alpha)$.

That $\widehat{\varphi} * \phi \in \mathcal{S}_m^*(h)$ follows from Theorem 2.1(1). Now a computation shows that

$$\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * \varphi_j)(z)} = \frac{(\phi * z f_k')(z)}{\left(\phi * (1/m)\sum_{j=1}^{m}\varphi_j\right)(z)} = \frac{(\phi * H_k F)(z)}{(\phi * F)(z)}. \tag{2.16}$$

Since $\phi \in \mathcal{R}_\alpha$ and $F \in \mathcal{S}^*(\alpha)$, Theorem 1.3 yields

$$\frac{(\phi * H_k F)(z)}{(\phi * F)(z)} \in \overline{\mathrm{co}}(H_k(\mathcal{U})), \tag{2.17}$$

and because $H_k(z) \prec h(z)$, it follows that

$$\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * \varphi_j)(z)} \prec h(z), \quad k = 1, \dots, m. \tag{2.18}$$

Thus $\widehat{f} * \phi \in \mathcal{C}_m(h)$.

(2) The function $\widehat{f}$ is in $\mathcal{Q}_m(g,h)$ if and only if $z\widehat{f}'$ is in $\mathcal{C}_m(g,h)$ and by the first part, clearly $\phi * z\widehat{f}' = z(\phi * \widehat{f})' \in \mathcal{C}_m(g,h)$. Hence $\phi * \widehat{f} \in \mathcal{Q}_m(g,h)$. □

Remark 2.6. Again when $m = 1$, known results are easily obtained as special cases of Theorem 2.5. For instance, [1, Theorem 3.5, page 337] follows from Theorem 2.5(1), and [1, Theorem 3.9, page 339] is a special case of Theorem 2.5(2).

Corollary 2.7. *Let $m \geq 1$ be a fixed integer and g a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$. Let F_k be the Bernardi-Libera integral transform of f_k defined by* (2.11). *If $\widehat{f} \in \mathcal{C}_m(g,h)$, then $\widehat{F} = \langle F_1, \dots, F_m \rangle \in \mathcal{C}_m(g,h)$.*

The proof is similar to the proof of Corollary 2.3 and is therefore omitted.

Remark 2.8. If $g(z) = k_a(z)$ is defined by (1.8), then Corollary 2.7 reduces to [2, Theorem 4, page 326].

3. Convolution of Meromorphic Functions

Let Σ denote the class of functions f of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n, \tag{3.1}$$

that are analytic in the punctured unit disk $\mathcal{U}^*:=\{z:0<|z|<1\}$. The convolution of two meromorphic functions f and g, where f is given by (3.1) and $g(z)=(1/z)+\sum_{n=0}^{\infty}b_nz^n$, is given by

$$(f*g)(z):=\frac{1}{z}+\sum_{n=0}^{\infty}a_nb_nz^n. \tag{3.2}$$

In this section, several subclasses of meromorphic functions in the punctured unit disk are introduced by means of convolution with a given fixed meromorphic function. First we take note that the familiar classes of meromorphic starlike and convex functions and other related subclasses of meromorphic functions can be put in the form

$$\Sigma^s(g,h):=\left\{f\in\Sigma\mid-\frac{z(f*g)'(z)}{(f*g)(z)}\prec h(z)\right\}, \tag{3.3}$$

where g is a fixed function in Σ and h is a suitably normalized analytic function with positive real part. For instance, the class of meromorphic starlike functions of order α, $0\le\alpha<1$, defined by

$$\Sigma^s:=\left\{f\in\Sigma\mid-\operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha\right\} \tag{3.4}$$

is a particular case of $\Sigma^s(g,h)$ with $g(z)=1/(z(1-z))$ and $h(z)=(1+(1-2\alpha)z)/(1-z)$.

Here four classes $\Sigma_m^s(g,h)$, $\Sigma_m^k(g,h)$, $\Sigma_m^c(g,h)$, and $\Sigma_m^q(g,h)$ of meromorphic functions are introduced and the convolution properties of these new subclasses are investigated. As before, it is assumed that $m\ge1$ is a fixed integer, g is a fixed function in Σ, and h is a convex univalent function with positive real part in $\mathcal{U}$ satisfying $h(0)=1$.

Definition 3.1. The class $\Sigma_m^s(h)$ consists of $\widehat{f}:=\langle f_1,f_2,\dots,f_m\rangle$, $f_k\in\Sigma$, $1\le k\le m$, satisfying $\sum_{j=1}^m f_j(z)\neq0$ in $\mathcal{U}^*$ and the subordination

$$-\frac{mzf_k'(z)}{\sum_{j=1}^m f_j(z)}\prec h(z),\quad k=1,\dots,m. \tag{3.5}$$

The class $\Sigma_m^s(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g := \langle f_1 * g, f_2 * g, \dots, f_m * g\rangle \in \Sigma_m^s(h)$. The class $\Sigma_m^k(h)$ consists of $\widehat{f}$ for which $-z\widehat{f}' \in \Sigma_m^s(h)$ or equivalently satisfying the condition $\sum_{j=1}^m f_j'(z) \neq 0$ in $\mathcal{U}^*$ and the subordination

$$-\frac{m(zf_k')'(z)}{\sum_{j=1}^m f_j'(z)} \prec h(z), \quad k = 1, \dots, m. \tag{3.6}$$

The class $\Sigma_m^k(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g \in \Sigma_m^k(h)$.

Various subclasses of meromorphic functions investigated in earlier works are special instances of the above defined classes. For instance, if $g(z) := (1/z) + (1/(1-z))$, then $\Sigma_m^s(g,h)$ coincides with $\Sigma_m^s(h)$. By putting $g = p_\mu * q_{\beta,\lambda}$, where

$$p_\mu(z) := \frac{1}{z(1-z)^\mu}, \qquad q_{\beta,\lambda}(z) := \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{\lambda}{k+1+\lambda}\right)^\beta z^k, \tag{3.7}$$

the class $\Sigma_m^s(g,h)$ reduces to the class $\Sigma_{\lambda,\mu}^\beta(m,h)$ investigated in [9]. If $g = k_n$, where

$$k_n(z) := \frac{1}{z} + \sum_{k=1}^{\infty}[1+\lambda(k+1)]^n z^k, \tag{3.8}$$

then the class of $\Sigma_m^s(g,h)$ is the class $\Sigma_m(n,\lambda,h)$ studied in [5]. If $g = k_a$, where

$$k_a(z) := \frac{1}{z(1-z)^a}, \quad a > 0, \tag{3.9}$$

then the class $\Sigma_m^s(g,h)$ coincides with $\Sigma_m(a,h)$ investigated in [6].

Definition 3.2. The class $\Sigma_m^c(h)$ consists of $\widehat{f} := \langle f_1, f_2, \dots, f_m\rangle$, $f_k \in \Sigma$, $1 \le k \le m$, satisfying the subordination

$$-\frac{mzf_k'(z)}{\sum_{j=1}^m \varphi_j(z)} \prec h(z), \quad k = 1, \dots, m, \tag{3.10}$$

for some $\widehat{\varphi} \in \Sigma_m^s(h)$. In this case, we say that $\widehat{f} \in \Sigma_m^c(h)$ with respect to $\widehat{\varphi} \in \Sigma_m^s(h)$. The class $\Sigma_m^c(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g := \langle f_1 * g, f_2 * g, \dots, f_m * g\rangle \in \Sigma_m^c(h)$. The class $\Sigma_m^q(h)$ consists of $\widehat{f}$ for which $-z\widehat{f}' \in \Sigma_m^c(h)$ or equivalently satisfying the subordination

$$-\frac{m(zf_k')'(z)}{\sum_{j=1}^m \varphi_j'(z)} \prec h(z), \quad k = 1, \dots, m, \tag{3.11}$$

for some $\widehat{\varphi} \in \mathcal{K}_m(h)$ with $-z\widehat{\varphi}' = \widehat{\psi}$ and $\widehat{\psi} \in \mathcal{S}_m^*(h)$. The class $\Sigma_m^q(g,h)$ consists of $\widehat{f}$ for which $\widehat{f} * g \in \Sigma_m^q(h)$.

If $g(z) := (1/z) + (1/(1-z))$, then $\Sigma_m^c(g,h)$ coincides with $\Sigma_m^c(h)$. If $g(z) = k_n(z)$ is defined by (3.8), then $\Sigma_m^c(g,h)$ reduces to $Q_m(n,\lambda,h)$ investigated in [5]. If $g(z) = k_a(z)$ is defined by (3.9), then the class $\Sigma_m^c(g,h)$ is the class $C_m^*(a,h)$ studied in [6].

We shall require the theorem below which is a simple modification of Theorem 1.3.

Theorem 3.3. *Let $\alpha \le 1$, $f,\phi \in \Sigma$, $z^2\phi \in \mathcal{R}_\alpha$, and $z^2 f \in \mathcal{S}^*(\alpha)$. Then, for any analytic function $H \in \mathcal{H}(\mathcal{U})$,*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\mathrm{co}}(H(\mathcal{U})). \tag{3.12}$$

Theorem 3.4. *Assume that $m \ge 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2-\alpha$, $0 \le \alpha < 1$, *and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{R}_\alpha$.*

(1) *If $\widehat{f} \in \Sigma_m^s(g,h)$, then $\widehat{f} * \phi \in \Sigma_m^s(g,h)$.*

(2) *If $\widehat{f} \in \Sigma_m^k(g,h)$, then $\widehat{f} * \phi \in \Sigma_m^k(g,h)$.*

Proof. (1) It is enough to prove the result for $g(z) = 1/z(1-z)$. For $k = 1,2,\ldots,m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z), \qquad H_k(z) = -\frac{zf_k'(z)}{F(z)}. \tag{3.13}$$

We show that F satisfies the condition $z^2F \in \mathcal{S}^*(\alpha)$. For $\widehat{f} \in \Sigma_m^s(h)$ and $z \in \mathcal{U}$, clearly

$$-\frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}), \quad k = 1,\ldots,m. \tag{3.14}$$

Since $h(\mathcal{U})$ is a convex domain, it follows that

$$-\frac{1}{m}\sum_{k=1}^{m}\frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}), \tag{3.15}$$

or

$$-\frac{zF'(z)}{F(z)} \prec h(z). \tag{3.16}$$

Since $\operatorname{Re} h(z) < 2-\alpha$, the subordination (3.16) yields

$$-\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) < 2-\alpha, \tag{3.17}$$

and thus

$$\operatorname{Re}\left(\frac{z(z^2F)'(z)}{z^2F(z)}\right) = \operatorname{Re}\frac{zF'(z)}{F(z)} + 2 > \alpha. \tag{3.18}$$

Inequality (3.18) shows that $z^2F \in \mathcal{S}^*(\alpha)$.

A routine computation now gives

$$-\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * f_j)(z)} = \frac{(\phi * (-zf_k'))(z)}{\left(\phi * (1/m)\sum_{j=1}^{m} f_j\right)(z)} = \frac{(\phi * H_kF)(z)}{(\phi * F)(z)}. \tag{3.19}$$

Since $z^2\phi \in R_\alpha$ and $z^2F \in \mathcal{S}^*(\alpha)$, Theorem 3.3 yields

$$\frac{(\phi * H_kF)(z)}{(\phi * F)(z)} \in \overline{\mathrm{co}}(H_k(\mathcal{U})), \tag{3.20}$$

and because $H_k(z) \prec h(z)$, it is clear that

$$-\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * f_j)(z)} \prec h(z), \quad k = 1, \ldots, m. \tag{3.21}$$

Thus $\widehat{f} * \phi \in \Sigma_m^s(h)$.

(2) The function $\widehat{f}$ is in $\Sigma_m^k(g,h)$ if and only if $-z\widehat{f}'$ is in $\Sigma_m^s(g,h)$ and the result of part (1) shows that $\phi * (-z\widehat{f}') = -z(\phi * \widehat{f})' \in \Sigma_m^s(g,h)$. Hence $\phi * \widehat{f} \in \Sigma_m^k(g,h)$. □

Remark 3.5. (1) The above theorem can be written in the following equivalent forms:

$$\Sigma_m^s(g,h) \subset \Sigma_m^s(\phi * g, h), \qquad \Sigma_m^k(g,h) \subset \Sigma_m^k(\phi * g, h). \tag{3.22}$$

(2) When $m = 1$, various known results are easily obtained as special cases of Theorem 3.4. For instance, if $g(z) = p_\mu * q_{\beta,\lambda}$ is defined by (3.7), then [9, Theorem 6, page 1265] follows from Theorem 3.4(1).

Corollary 3.6. *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$. *Define*

$$F_k(z) = \frac{\gamma + 1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} f_k(t)\,dt \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1, \ldots, m). \tag{3.23}$$

If $\widehat{f} \in \Sigma_m^s(g,h)$, then $\widehat{F} = \langle F_1, \ldots, F_m \rangle \in \Sigma_m^s(g,h)$. Similarly, if $\widehat{f} \in \Sigma_m^k(g,h)$, then $\widehat{F} \in \Sigma_m^k(g,h)$.

Proof. Define the function ϕ by

$$\phi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\gamma+1}{\gamma+2+n} z^n. \tag{3.24}$$

For $\operatorname{Re}\gamma \geq 0$, the function $z^2\phi(z)$ is a convex function [11], and hence $z^2\phi(z) \in \mathcal{R}_\alpha$ ([4, Theorem 2.1, page 49]). It is clear from the definition of F_k that

$$F_k(z) = f_k(z) * \left(\frac{1}{z} + \sum_{n=0}^{\infty} \frac{\gamma+1}{\gamma+2+n} z^n \right) = (f_k * \phi)(z), \tag{3.25}$$

so that $\widehat{F} = \widehat{f} * \phi$. By Theorem 3.4, it follows that $\widehat{F} = \widehat{f} * \phi \in \Sigma_m^s(g,h)$.

The second result is established analogously. □

Remark 3.7. Again we take note of how our results extend various earlier works. If $g(z) = p_\mu * q_{\beta,\lambda}$ is defined by (3.7), then [7, Proposition 2, page 512] follows from Corollary 3.6. If $g(z) = k_n(z)$ is defined by (3.8), then Corollary 3.6 yields [5, Theorem 2.2, page 4]. If $g(z) = k_a(z)$ is defined by (3.9), then Corollary 3.6 reduces to [6, Theorem 2, page 11].

Theorem 3.8. *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2-\alpha$, $0 \leq \alpha < 1$, *and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{R}_\alpha$.*

(1) *If $\widehat{f} \in \Sigma_m^c(g,h)$ with respect to $\widehat{\varphi} \in \Sigma_m^s(g,h)$, then $\widehat{f} * \phi \in \Sigma_m^c(g,h)$ with respect to $\widehat{\varphi} * \phi \in \Sigma_m^s(g,h)$.*

(2) *If $\widehat{f} \in \Sigma_m^q(g,h)$ with respect to $\widehat{\varphi} \in \Sigma_m^k(g,h)$, then $\widehat{f} * \phi \in \Sigma_m^q(g,h)$ with respect to $\widehat{\varphi} * \phi \in \Sigma_m^k(g,h)$.*

Proof. (1) In view of the fact that

$$\widehat{f} \in \Sigma_m^c(g,h) \Longleftrightarrow \widehat{f} * g \in \Sigma_m^c(h), \tag{3.26}$$

it is sufficient to prove that $\widehat{f} * \phi \in \Sigma_m^c(h)$ when $\widehat{f} \in \Sigma_m^c(h)$. Let $\widehat{f} \in \Sigma_m^c(h)$. For $k = 1,2,\ldots,m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} \varphi_j(z), \qquad H_k(z) = -\frac{z f_k'(z)}{F(z)}. \tag{3.27}$$

Inequality (3.18) shows that $z^2 F \in \mathcal{S}^*(\alpha)$.

It is evident that

$$-\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * \varphi_j)(z)} = \frac{(\phi * (-z f_k'))(z)}{\left(\phi * (1/m)\sum_{j=1}^{m} \varphi_j\right)(z)} = \frac{(\phi * H_k F)(z)}{(\phi * F)(z)}. \tag{3.28}$$

Since $z^2\phi \in \mathcal{R}_\alpha$ and $z^2F \in \mathcal{S}^*(\alpha)$, Theorem 3.3 yields

$$\frac{(\phi * H_k F)(z)}{(\phi * F)(z)} \in \overline{\text{co}}(H_k(\mathcal{U})), \tag{3.29}$$

and because $H_k(z) \prec h(z)$, it follows that

$$-\frac{z(\phi * f_k)'(z)}{(1/m)\sum_{j=1}^{m}(\phi * \psi_j)(z)} \prec h(z), \quad k = 1, \ldots, m. \tag{3.30}$$

Thus $\widehat{f} * \phi \in \Sigma_m^c(h)$.

(2) The function $\widehat{f}$ is in $\Sigma_m^q(g,h)$ if and only if $-z\widehat{f}'$ is in $\Sigma_m^c(g,h)$ and from the first part above, it follows that $\phi * (-z\widehat{f}') = -z(\phi * \widehat{f})' \in \Sigma_m^c(g,h)$. Hence $\phi * \widehat{f} \in \Sigma_m^q(g,h)$. □

Corollary 3.9. *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2-\alpha$, $0 \leq \alpha < 1$. *Let F_k be defined by* (3.23). *If $\widehat{f} \in \Sigma_m^c(g,h)$, then $\widehat{F} = \langle F_1, \ldots, F_m \rangle \in \Sigma_m^c(g,h)$.*

The proof is analogous to Corollary 2.3 and is omitted.

Remark 3.10. If $g(z) = k_n(z)$ is defined by (3.8), then Corollary 3.9 yields [5, Theorem 3.1, page 9]. If $g(z) = k_a(z)$ is defined by (3.9), then Corollary 3.9 reduces to [6, Theorem 4, page 14].

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References

[1] T. N. Shanmugam, "Convolution and differential subordination," *International Journal of Mathematics and Mathematical Sciences*, vol. 12, no. 2, pp. 333–340, 1989.

[2] K. S. Padmanabhan and R. Parvatham, "Some applications of differential subordination," *Bulletin of the Australian Mathematical Society*, vol. 32, no. 3, pp. 321–330, 1985.

[3] K. S. Padmanabhan and R. Manjini, "Certain applications of differential subordination," *Publications de l'Institut Mathématique*, vol. 39, no. 53, pp. 107–118, 1986.

[4] St. Ruscheweyh, *Convolutions in Geometric Function Theory*, vol. 83 of *Séminaire de Mathématiques Supérieures*, Presses de l'Université de Montréal, Montreal, Canada, 1982.

[5] F. M. Al-Oboudi and H. A. Al-Zkeri, "Applications of Briot-Bouquet differential subordination to some classes of meromorphic functions," *Arab Journal of Mathematical Sciences*, vol. 12, no. 1, pp. 17–30, 2006.

[6] R. Bharati and R. Rajagopal, "Meromorphic functions and differential subordination," in *New Trends in Geometric Function Theory and Applications (Madras, 1990)*, pp. 10–17, World Scientific, Hackensack, NJ, USA, 1991.

[7] N. E. Cho, O. S. Kwon, and H. M. Srivastava, "Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 505–520, 2004.

[8] M. Haji Mohd, R. M. Ali, L. S. Keong, and V. Ravichandran, “Subclasses of meromorphic functions associated with convolution,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 190291, 10 pages, 2009.

[9] K. Piejko and J. Sokół, “Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator,” *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1261–1266, 2008.

[10] V. Ravichandran, “Functions starlike with respect to n-ply symmetric, conjugate and symmetric conjugate points,” *Indian Academy of Mathematics Journal*, vol. 26, no. 1, pp. 35–45, 2004.

[11] S. Ruscheweyh, “New criteria for univalent functions,” *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.