Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 351419, 13 pages
doi:10.1155/2010/351419

Research Article

On Multiple Interpolation Functions of the q-Genocchi Polynomials

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Received 14 December 2009; Accepted 29 March 2010

Academic Editor: Wing-Sum Cheung

Recently, many mathematicians have studied various kinds of the q-analogue of Genocchi numbers and polynomials. In the work (New approach to q-Euler, Genocchi numbers and their interpolation functions, "Advanced Studies in Contemporary Mathematics, vol. 18, no. 2, pp. 105–112, 2009.", Kim defined new generating functions of q-Genocchi, q-Euler polynomials, and their interpolation functions. In this paper, we give another definition of the multiple Hurwitz type q-zeta function. This function interpolates q-Genocchi polynomials at negative integers. Finally, we also give some identities related to these polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ (see [1]).

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$. In this paper, we use the following notation:

$$[x] = [x : q] = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

(see [2, 3]). Hence $\lim_{q\to 1}[x] = x$ for all $x \in \mathbb{Z}_p$.

We say that $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and we write $f \in UD(\mathbb{Z}_p)$ if the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_f(x,y) = \frac{f(x) - f(y)}{x - y} \tag{1.2}$$

have a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the q-deformed fermionic p-adic integral is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.3}$$

(see [4–6]). Note that

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \tag{1.4}$$

(see [7–9]). Let $f_1(x)$ be the translation with $f_1(x) = f(x+1)$. Then we have the following integral equation:

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.5}$$

(see [10–12]).

The ordinary Genocchi numbers and polynomials are defined by the generating functions as, respectively,

$$\begin{gathered} F(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi, \\ F(t,x) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \end{gathered} \tag{1.6}$$

Observe that $G_n(0) = G_n$ (see [10, 11, 13]).

These numbers and polynomials are interpolated by the Genocchi zeta function and Hurwitz-type Genocchi zeta function, respectively,

$$\begin{gathered} \zeta_G(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad s \in \mathbb{C}, \\ \zeta_G(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \quad s \in \mathbb{C},\ 0 < x \leq 1. \end{gathered} \tag{1.7}$$

Thus we note that Genocchi zeta functions are entire functions in the whole complex s-plane (see [14–16]).

Various kinds of the q-analogue of the Genocchi numbers and polynomials, recently, have been studied by many mathematicians. In this paper, we use Kim's [14–16] methods. By using p-adic q-Vokenborn integral [6], Kim [2, 7–9, 14–18] constructed many kind of generating functions of the q-Euler numbers and polynomials and their interpolation functions. He also gave many applications of these numbers and functions. He [14] defined q-extension Genocchi polynomials of higher order. He gave many applications and interesting identities. We give some of them in what follows.

Let $q \in \mathbb{C}$ with $|q| < 1$. The q-Genocchi numbers $G_{n,q}$ and polynomials $G_{n,q}(x)$ are defined by Kim of the generating functions as, respectively,

$$\begin{gathered} t\int_{\mathbb{Z}_p} e^{[x]t} d\mu_{-q}(x) = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!}, \quad |t| < \pi \\ t\int_{\mathbb{Z}_p} e^{[x+y]t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}, \quad |t| < \pi \end{gathered} \tag{1.8}$$

(see [1, 8–11, 13, 14, 17]). By using the Taylor expansion of $e^{[x]t}$,

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} [x]^n d\mu_{-q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^{n-1}}{n!} = G_{0,q} + \sum_{n=0}^{\infty}\frac{G_{n+1,q}}{n+1}\frac{t^n}{n!}. \tag{1.9}$$

By comparing the coefficient of both sides of $t^n/n!$ in the above,

$$\begin{gathered} G_{0,q} = 0, \\ \frac{G_{n+1,q}}{n+1} = \int_{\mathbb{Z}_p} [x]^n d\mu_{-q}(x) = \frac{[2]}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{l+1}}. \end{gathered} \tag{1.10}$$

From the above, we can easily derive that

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(t\int_{\mathbb{Z}_p} [x]^n d\mu_{-q}(x)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(t\frac{[2]}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{l+1}}\right)\frac{t^n}{n!} \\ &= [2]t\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]t}. \end{aligned} \tag{1.11}$$

Thus we have, following that,

$$F_q(t) = [2]t\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]t} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!}. \tag{1.12}$$

Using similar method to the above, we can find that

$$G_{0,q}(x)=0,$$
$$\frac{G_{n+1,q}(x)}{n+1}=\int_{\mathbb{Z}_p}[x+y]^n d\mu_{-q}(y)=\frac{[2]}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{1}{1+q^{l+1}}. \tag{1.13}$$

Thus we can easily derive that

$$F_q(t,x)=[2]t\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]t}=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}. \tag{1.14}$$

Observe that $F_q(t)=F_q(t,0)$. Hence we have $G_{n,q}(0)=G_{n,q}$. If $q\to 1$ into (1.14), then we easily obtain $F(t,x)$ in (1.6).

Let $q\in\mathbb{C}$ with $|q|<1$, $r\in\mathbb{N}$, and $n\geq 0$. We now define as the generating functions of higher order q-extension Genocchi numbers $G_{n,q}^{(r)}$ and polynomials $G_{n,q}^{(r)}(x)$, respectively,

$$F_q^{(r)}(t)=t^r\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}e^{[x_1+\cdots+x_r]t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)=\sum_{n=0}^{\infty}G_{n,q}^{(r)}\frac{t^n}{n!},$$
$$F_q^{(r)}(t,x)=t^r\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}e^{[x+x_1+\cdots+x_r]t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)=\sum_{n=0}^{\infty}G_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{1.15}$$

Then we have

$$\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}G_{n,q}^{(r)}\frac{t^{n-r}}{n!}=\sum_{n=0}^{r-1}G_{n,q}^{(r)}\frac{t^{n-r}}{n!}+\sum_{n=0}^{\infty}\frac{G_{n+r,q}^{(r)}}{\binom{n+r}{r}r!}\frac{t^n}{n!}, \tag{1.16}$$

where $\binom{n+r}{r}=(n+r)!/n!r!$.

By comparing the coefficient of both sides of $t^n/n!$ in the above, we can derive that

$$G_{0,q}^{(r)}=G_{1,q}^{(r)}=\cdots=G_{r-1,q}^{(r)}=0,$$
$$\frac{G_{n+r,q}^{(r)}}{\binom{n+r}{r}r!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$
$$=\frac{[2]^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{(1+q^{l+1})^r}. \tag{1.17}$$

Therefore we obtain

$$F_q^{(r)}(t) = [2]^r t^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} e^{[m]t} = \sum_{n=0}^{\infty} G_{n,q}^{(r)} \frac{t^n}{n!}. \tag{1.18}$$

Using similar method to the above, we can also derive that

$$\begin{gathered} G_{0,q}^{(r)}(x) = G_{1,q}^{(r)}(x) = \cdots = G_{r-1,q}^{(r)}(x) = 0, \\ \frac{G_{n,q}^{(r)}(x)}{\binom{n+r}{r} r!} = \frac{[2]^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{(1+q^{l+1})^r}. \end{gathered} \tag{1.19}$$

Thus we can easily obtain the following theorem.

Theorem 1.1. *For $r \in \mathbb{N}$ and $n \geq 0$, one has*

$$F_q^{(r)}(t,x) = [2]^r t^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} e^{[m+x]t} = \sum_{n=0}^{\infty} G_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{1.20}$$

It is noted that if $r = 1$, then (1.20) reduces to (1.14).

Remark 1.2. In (1.20), we easily see that

$$\begin{aligned} \lim_{q \to 1} F_q^{(r)}(t,x) &= 2^r t^r \sum_{m=0}^{\infty} (-1)^m \binom{m+r-1}{m} e^{(m+x)t} \\ &= 2^r t^r e^{tx} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-e^t)^m \\ &= \frac{2^r t^r e^{tx}}{(1+e^t)^r} \\ &= F^{(r)}(t,x). \end{aligned} \tag{1.21}$$

From the above, we obtain generating function of the Genocchi numbers of higher order. That is

$$F^{(r)}(t,x) = \frac{2^r t^r e^{tx}}{(1+e^t)^r} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}. \tag{1.22}$$

Thus we have

$$\lim_{q \to 1} G_{n,q}^{(r)}(x) = G_n^{(r)}(x). \tag{1.23}$$

Hence we have

$$
\begin{aligned}
F^{(r)}(t,x) &= \underbrace{\left(\frac{2t}{e^t+1}\right)\left(\frac{2t}{e^t+1}\right)\cdots\left(\frac{2t}{e^t+1}\right)}_{r\text{times}}e^{tx} \\
&= 2^r t^r e^{tx}\sum_{n_1=0}^{\infty}(-1)^{n_1}e^{n_1 t}\sum_{n_2=0}^{\infty}(-1)^{n_2}e^{n_2 t}\cdots\sum_{n_r=0}^{\infty}(-1)^{n_r}e^{n_r t} \\
&= 2^r t^r e^{tx}\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}(-1)^{n_1+n_2+\cdots+n_r}e^{(n_1+n_2+\cdots+n_r)t} \\
&= \sum_{n=0}^{\infty}G_n^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}
\tag{1.24}
$$

In [14], Kim defined new generating functions of q-Genocchi, q-Euler polynomials and their interpolation functions. In this paper, we give another definition of the multiple Hurwitz type q-zeta function. This function interpolates q-Genocchi polynomials at negative integers. Finally, we also give some identities related to these polynomials.

2. Modified Generating Functions of Higher Order q-Genocchi Polynomials and Numbers

In this section, we study modified generating functions of the higher order q-Genocchi numbers and polynomials. We obtain some relations related to these numbers and polynomials. Therefore we define generating function of modified higher order q-Genocchi polynomials and numbers, which are denoted by $G_{n,q}^{(r)}(x)$ and $G_{n,q}^{(r)}$, respectively, in (1.15). We give relations between these numbers and polynomials.

We modify (1.20) as follows:

$$
\mathfrak{F}_q^{(r)}(t,x) = F_q^{(r)}(q^{-x}t,x),
\tag{2.1}
$$

where $F_q^{(r)}(t,x)$ is defined in (1.20). From the above we find that

$$
\mathfrak{F}_q^{(r)}(t,x) = \sum_{n=0}^{\infty}q^{-(n+r)x}G_{n,q}^{(r)}(x)\frac{t^n}{n!}.
\tag{2.2}
$$

After some elementary calculations, we obtain

$$
\mathfrak{F}_q^{(r)}(t,x) = q^{-rx}\exp([x]q^{-x}t)F_q^{(r)}(t),
\tag{2.3}
$$

where

$$
F_q^{(r)}(t) = [2]^r t^r\sum_{m=0}^{\infty}(-1)^m q^m\binom{r+m-1}{m}e^{[m]t} = \sum_{n=0}^{\infty}G_{n,q}^{(r)}\frac{t^n}{n!}.
\tag{2.4}
$$

From the above, we can define the modified higher order q-Genocchi polynomials $\varepsilon_{n,q}^{(r)}(x)$ as follows

$$\mathfrak{F}_q^{(r)}(t,x) = \sum_{n=0}^{\infty} \varepsilon_{n,q}^{(r)}(x) \frac{t^n}{n!} \tag{2.5}$$

Then we have

$$\varepsilon_{n,q}^{(r)}(x) = q^{-(n+r)x} G_{n,q}^{(r)}(x). \tag{2.6}$$

By using Cauchy product in (2.3), we arrive at following theorem.

Theorem 2.1. *For $r \in \mathbb{N}$ and $n \geq 0$, one has*

$$\varepsilon_{n,q}^{(r)}(x) = q^{-(n+r)x} \sum_{j=0}^{n} \binom{n}{j} q^{jx} [x]^{n-j} G_{j,q}^{(r)}. \tag{2.7}$$

By using (2.7), we easily obtain the following result.

Corollary 2.2. *For $r \in \mathbb{N}$, and $n \geq 0$, one has*

$$\varepsilon_{n,q}^{(r)}(x) = q^{-(n+r)x} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j, l, n-j-l} \binom{n-j+m-1}{m} (-1)^l q^{m+x(j+l)} G_{j,q}^{(r)}. \tag{2.8}$$

We now give some identity related to the Genocchi polynomials and numbers of higher order.

Substituting $x = 0$ into (1.24), we find that

$$G_n^{(r)} = 2^r t^r \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} \sum_{\substack{j_1, j_2, \ldots, j_r = 0 \\ j_1 + j_2 + \cdots + j_r = n}}^{\infty} \binom{n}{j_1, j_2, \ldots, j_r} (-1)^{n_1 + n_2 + \cdots + n_r} \prod_{k=0}^{r} n_k^{j_k}. \tag{2.9}$$

By (1.24) and (2.8), we arrive at the following theorem.

Theorem 2.3. *For $r \in \mathbb{N}$ and $n \geq 0$, one has*

$$G_n^{(r)} = \sum_{j=0}^{n} \binom{n}{j} (-x)^{n-j} G_j^{(r)}(x). \tag{2.10}$$

By using (1.24), we easily arrive at the following result.

Corollary 2.4. *For $r, v \in \mathbb{N}$ and $n \geq 0$, one has*

$$\left(G^{(r)}(x) + G^{(v)}(y)\right)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} G_j^{(r+v)}(y), \tag{2.11}$$

where $(G^{(r)}(x))^n$ is replace by $G_n^{(r)}(x)$.

3. Interpolation Function of Higher Order q-Genocchi Polynomials

Recently, higher order Bernoulli polynomials, Euler polynomials, and Genocchi polynomials have been studied by many mathematicians. Especially, in this paper, we study higher order Genocchi polynomials which constructed by Kim [15] and see also the references cited in each of the these earlier works.

In [14], by using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, the set of p-adic integers, Kim gave a new construction of q-Genocchi numbers, Euler numbers of higher order. By using q-Genocchi, Euler numbers of higher order, he investigated the interesting relationship between w-q-Euler polynomials and w-q-Genocchi polynomials. He also defined the multiple w-q-zeta functions which interpolate q-Genocchi, Euler numbers of higher order.

By using similar method to that in the papers given by Kim [14], in this section, we give interpolation function of the generating functions of higher order q-Genocchi polynomials. From (1.20), we easily see that

$$\sum_{k=0}^{\infty} G_{k,q}^{(r)}(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty}[2]^r r!\binom{k+r}{r}\sum_{m=0}^{\infty}(-1)^m q^m\binom{m+r-1}{m}[m+x]^k\frac{t^{k+r}}{(k+r)!}. \tag{3.1}$$

From the above we have

$$\begin{gathered} G_{k+r,q}^{(r)}(x) = [2]^r r!\binom{k+r}{r}\sum_{m=0}^{\infty}(-1)^m q^m\binom{m+r-1}{m}[m+x]^k, \\ G_{0,q}^{(r)}(x) = G_{1,q}^{(r)}(x) = \cdots = G_{r-1,q}^{(r)}(x) = 0. \end{gathered} \tag{3.2}$$

Hence we have obtain the following theorem.

Theorem 3.1. *Let $r, k \in \mathbb{Z}_+$. Then one has*

$$G_{k+r,q}^{(r)}(x) = [2]^r r!\binom{k+r}{r}\sum_{m=0}^{\infty}(-1)^m q^m\binom{m+r-1}{m}[m+x]^k. \tag{3.3}$$

Let us define interpolation function of the $G_{k+r,q}^{(r)}(x)$ as follows.

Definition 3.2. Let $q, s \in \mathbb{C}$ with $|q| < 1$ and $0 < x \leq 1$. Then we define

$$\zeta_q^{(r)}(s,x) = [2]^r\sum_{n=0}^{\infty}\binom{n+r-1}{n}\frac{(-1)^n q^n}{[n+x]^s}. \tag{3.4}$$

We call $\zeta_q^{(r)}(s,x)$ are the multiple Hurwitz type q-zeta function.

Remark 3.3. It holds that

$$\lim_{q\to 1}\zeta_q^{(r)}(s,x) = 2^r\sum_{n=0}^{\infty}\binom{n+r-1}{n}\frac{(-1)^n}{(n+x)^s}. \tag{3.5}$$

From (1.24), we easily see that

$$\zeta^{(r)}(s,x) = 2^r\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\frac{(-1)^{n_1+n_2+\cdots+n_r}}{\left(\sum_{j=1}^{r} n_j + x\right)^s}, \tag{3.6}$$

where $s\in\mathbb{C}$.

The functions in (3.5) and (3.6) interpolate the same numbers at negative integers. That is, these functions interpolate higher order q-Genocchi numbers at negative integers. So, by (3.5), we modify (3.6) in sense of q-analogue.

In (3.5) and (3.6), setting $r=1$, we have

$$\zeta^{(1)}(s,x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^s} = \zeta_G(s,x), \tag{3.7}$$

where $\zeta_G(s,x)$ denotes Hurwitz type Genocchi zeta function, which interpolates classical Genocchi polynomials at negative integers.

Substituting $s=-k$, $k\in\mathbb{Z}^+$ into (3.4). Then we have

$$\zeta_q^{(r)}(-k,x) = [2]^r\sum_{n=0}^{\infty}\binom{r+n-1}{n}(-1)^n q^n [n+x]^k. \tag{3.8}$$

Setting (3.3) into the above, we easily arrive at the following result.

Theorem 3.4. *Let $r,k\in\mathbb{Z}_+$. Then one has*

$$\zeta_q^{(r)}(-n,x) = \frac{G_{n+r,q}^{(r)}(x)}{r!\binom{n+r}{r}}. \tag{3.9}$$

4. Some Relations Related to Higher Order q-Genocchi Polynomials

In this section, by using generating function of the higher order q-Genocchi polynomials, which is defined by (1.20), we obtain the following identities.

By using (1.20), we find that

$$\begin{aligned}
\frac{G_{k+r,q}^{(r)}(x)}{r!\binom{k+r}{r}} &= [2]^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \left([m] + q^m [x]\right)^k \\
&= [2]^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \sum_{j=0}^{k} \binom{k}{j} [m]^j q^{m(k-j)} [x]^{k-j} \\
&= [2]^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \sum_{j=0}^{k} \binom{k}{j} \frac{(1-q^m)^j}{(1-q)^j} q^{m(k-j)} [x]^{k-j} \\
&= [2]^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \sum_{j=0}^{k} \binom{k}{j} \sum_{a=0}^{j} \binom{j}{a} \frac{(-1)^a q^{m(a+k-j)} [x]^{k-j}}{(1-q)^j} \\
&= [2]^r \sum_{j=0}^{k} \sum_{a=0}^{j} \binom{k}{j} \binom{j}{a} \frac{(-1)^a [x]^{k-j}}{(1-q)^j} \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} q^{m(a+k-j)} \\
&= [2]^r \sum_{j=0}^{k} \sum_{a=0}^{j} \binom{k}{j} \binom{j}{a} \frac{(-1)^a [x]^{k-j}}{(1-q)^j \left(1 + q^{a+k-j+1}\right)^r} \\
&= [2]^r \sum_{j=0}^{k} \sum_{a=0}^{j} (-1)^a \binom{k}{a, j-a, k-j} \frac{[x]^{k-j}}{(1-q)^j \left(1 + q^{a+k-j+1}\right)^r}.
\end{aligned} \tag{4.1}$$

Thus we have the following theorem.

Theorem 4.1. *Let $q \in \mathbb{C}$ with $|q| < 1$. Let r be a positive integer. Then one has*

$$\frac{G_{k+r,q}^{(r)}(x)}{r!\binom{k+r}{r}} = [2]^r \sum_{j=0}^{k} \sum_{a=0}^{j} (-1)^a \binom{k}{a, j-a, k-j} \frac{[x]^{k-j}}{(1-q)^j \left(1 + q^{a+k-j+1}\right)^r}. \tag{4.2}$$

By using (1.20), we have

$$\begin{aligned}
F_q^{(r)}(t,x) &= [2]^r t^r \sum_{m=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} e^{[m+x]t} \\
&= [2]^r t^r \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \left(\frac{1-q^{m+x}}{1-q}\right)^n \frac{t^n}{n!} \\
&= [2]^r t^r \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m q^m \binom{m+r-1}{m} \frac{1}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} \left(-q^{m+x}\right)^j \frac{t^n}{n!} \\
&= [2]^r t^r \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j q^{jx}}{(1-q)^n} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^{(j+1)m} \frac{t^n}{n!}.
\end{aligned} \tag{4.3}$$

Thus we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(r)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[2]^r t^r \sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{jx}\left(1+q^{j+1}\right)^{-r}(1-q)^{-n}\frac{t^n}{n!}. \tag{4.4}$$

By comparing the coefficients $t^n/n!$ of both sides in the above, we arrive at the following theorem.

Theorem 4.2. *Let $q \in \mathbb{C}$ with $|q| < 1$. Let r be a positive integer. Then one has*

$$\frac{G_{n+r,q}^{(r)}(x)}{r!\binom{n+r}{r}} = [2]^r \sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{jx}\left(1+q^{j+1}\right)^{-r}(1-q)^{-n}. \tag{4.5}$$

By using (1.20), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,q}^{(r)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} G_{n,q}^{(y)}(x)\frac{t^n}{n!} \\ = [2]^{r+y}t^{r+y}\sum_{n=0}^{\infty}(-1)^n q^n\binom{n+r-1}{n}e^{[n+x]t}\sum_{n=0}^{\infty}(-1)^n q^n\binom{n+y-1}{n}e^{[n+x]t}.\end{aligned} \tag{4.6}$$

By using Cauchy product into the above, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}G_{j,q}^{(r)}(x)G_{n-j,q}^{(r)}(y)\frac{t^n}{n!} \\ &\quad= [2]^{r+y}t^{r+y}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^j\binom{j+r-1}{j}(-1)^{n-j}q^{n-j}\binom{n-j+y-1}{n-j}e^{[j+x]t}e^{[n-j+x]t}.\end{aligned} \tag{4.7}$$

From the above, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}G_{j,q}^{(r)}(x)G_{m-j,q}^{(r)}(y)\right)\frac{t^m}{m!} \\ &\quad= \sum_{m=0}^{\infty}\left([2]^{r+y}t^{r+y}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^n q^n\binom{j+r-1}{j}\binom{n-j+y-1}{n-j}([j+x]+[n-j+x])^m\right)\frac{t^m}{m!}.\end{aligned} \tag{4.8}$$

By comparing the coefficients of both sides of $t^m/m!$ in the above, we have the following theorem.

Theorem 4.3. *Let $r, y \in \mathbb{Z}^+$. Then one has*

$$\frac{\sum_{j=0}^{k+r+y}\binom{k+r+y}{j}G_{j,q}^{(r)}(x)G_{k+r+y-j,q}^{(y)}(x)}{(r+y)!\binom{k+r+y}{k}} = [2]^{r+y}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^n q^n\binom{j+r-1}{j}\binom{n-j+y-1}{n-j}\left([j+x]+[n-j+x]\right)^k. \tag{4.9}$$

Remark 4.4. In (4.9) setting $y = 1$, we have

$$\frac{\sum_{j=0}^{k+r+1}\binom{k+r+1}{j}G_{j,q}^{(r)}(x)G_{k+r+1-j,q}^{(1)}(x)}{(r+1)!\binom{k+r+1}{k}} = [2]^{r+1}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^n q^n\binom{j+r-1}{j}\left([j+x]+[n-j+x]\right)^k. \tag{4.10}$$

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