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Research Article

Potential Operators in Variable Exponent Lebesgue Spaces: Two-Weight Estimates

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Two-weighted norm estimates with general weights for Hardy-type transforms and potentials in variable exponent Lebesgue spaces defined on quasimetric measure spaces (X,d,μ) are established. In particular, we derive integral-type easily verifiable sufficient conditions governing two-weight inequalities for these operators. If exponents of Lebesgue spaces are constants, then most of the derived conditions are simultaneously necessary and sufficient for corresponding inequalities. Appropriate examples of weights are also given.

1. Introduction

We study the two-weight problem for Hardy-type and potential operators in Lebesgue spaces with nonstandard growth defined on quasimetric measure spaces (X,d,μ). In particular, our aim is to derive easily verifiable sufficient conditions for the boundedness of the operators

$$(T_{\alpha(\cdot)}f)(x)=\int_X \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\alpha(x)}}d\mu(y), \qquad (I_{\alpha(\cdot)}f)(x)=\int_X \frac{f(y)}{d(x,y)^{1-\alpha(x)}}d\mu(y) \tag{1.1}$$

in weighted $L^{p(\cdot)}(X)$ spaces which enable us to effectively construct examples of appropriate weights. The conditions are simultaneously necessary and sufficient for corresponding inequalities when the weights are of special type and the exponent p of the space is constant. We assume that the exponent p satisfies the local log-Hölder continuity condition, and if the diameter of X is infinite, then we suppose that p is constant outside some ball. In the framework of variable exponent analysis such a condition first appeared in the paper [1], where the author established the boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$. As far as we know, unfortunately, an analog of the log-Hölder decay condition (at infinity) for $p : X \to [1,\infty)$ is not known even in the unweighted case, which is well-known and natural for the Euclidean spaces (see [2–5]). Local log-Hölder continuity condition for the exponent p, together with the log-Hölder decay condition, guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see, e.g., [6]). The technique developed here enables us to expect that results similar to those of this paper can be obtained also for other integral operators, for instance, for maximal and Calderón-Zygmund singular operators defined on X.

Considerable interest of researchers is focused on the study of mapping properties of integral operators defined on (quasi)metric measure spaces. Such spaces with doubling measure and all their generalities naturally arise when studying boundary value problems for partial differential equations with variable coefficients, for instance, when the quasimetric might be induced by a differential operator or tailored to fit kernels of integral operators. The problem of the boundedness of integral operators naturally arises also in the Lebesgue spaces with nonstandard growth. Historically the boundedness of the maximal and fractional integral operators in $L^{p(\cdot)}(X)$ spaces was derived in the papers [7–14]. Weighted inequalities for classical operators in $L^{p(\cdot)}_w$ spaces, where w is a power-type weight, were established in the papers [10–12, 15–19], while the same problems with general weights for Hardy, maximal, and fractional integral operators were studied in [10, 20–25]. Moreover, in the latter paper, a complete solution of the one-weight problem for maximal functions defined on Euclidean spaces is given in terms of Muckenhoupt-type conditions.

It should be emphasized that in the classical Lebesgue spaces the two-weight problem for fractional integrals is already solved (see [26, 27]), but it is often useful to construct concrete examples of weights from transparent and easily verifiable conditions.

To derive two-weight estimates for potential operators, we use the appropriate inequalities for Hardy-type transforms on X (which are also derived in this paper) and Hardy-Littlewood-Sobolev-type inequalities for $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ in $L^{p(\cdot)}(X)$ spaces.

The paper is organized as follows: in Section 1, we give some definitions and prove auxiliary results regarding quasimetric measure spaces and the variable exponent Lebesgue spaces; Section 2 is devoted to the sufficient governing two-weight inequalities for Hardy-type operators defined on quasimetric measure spaces, while in Section 3 we study the two-weight problem for potentials defined on X.

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c or C. The symbol $f(x) \approx g(x)$ means that there are positive constants c_1 and c_2 independent of x such that the inequality $f(x) \le c_1 g(x) \le c_2 f(x)$ holds. Throughout the paper is denoted the function $p(x)/(p(x)-1)$ by the symbol $p'(x)$.

2. Preliminaries

Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a nonnegative real-valued function (quasimetric) d on $X \times X$ satisfying the conditions:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) there exists a constant $a_1 > 0$, such that $d(x,y) \le a_1(d(x,z) + d(z,y))$ for all $x,y,z \in X$;

(iii) there exists a constant $a_0 > 0$, such that $d(x,y) \le a_0 d(y,x)$ for all $x,y,\in X$.

We assume that the balls $B(x,r) := \{y \in X : d(x,y) < r\}$ are measurable and $0 \le \mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$; for every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x,r) \subset V$. Throughout the paper we also suppose that $\mu\{x\} = 0$ and that

$$B(x,R) \setminus B(x,r) \neq \emptyset, \tag{2.1}$$

for all $x \in X$, positive r and R with $0 < r < R < L$, where

$$L := \operatorname{diam}(X) = \sup\{d(x,y) : x,y \in X\}. \tag{2.2}$$

We call the triple (X,d,μ) a quasimetric measure space. If μ satisfies the doubling condition $\mu(B(x,2r)) \le c\mu(B(x,r))$, where the positive constant c does not depend on $x \in X$ and $r > 0$, then (X,d,μ) is called a space of homogeneous type (SHT). For the definition, examples, and some properties of an SHT see, for example, monographs [28–30].

A quasimetric measure space, where the doubling condition is not assumed, is called a nonhomogeneous space.

Notice that the condition $L < \infty$ implies that $\mu(X) < \infty$ because we assumed that every ball in X has a finite measure.

We say that the measure μ is upper Ahlfors Q-regular if there is a positive constant c_1 such that $\mu B(x,r) \le c_1 r^Q$ for for all $x \in X$ and $r > 0$. Further, μ is lower Ahlfors Q-regular if there is a positive constant c_2 such that $\mu B(x,r) \ge c_2 r^q$ for all $x \in X$ and $r > 0$. It is easy to check that if (X,d,μ) is a quasimetric measure space and $L < \infty$, then μ is lower Ahlfors regular (see also, e.g., [8] for the case when d is a metric).

For the boundedness of potential operators in weighted Lebesgue spaces with constant exponents on nonhomogeneous spaces we refer, for example, to the monograph [31, Chapter 6] and references cited therein.

Let p be a nonnegative μ-measurable function on X. Suppose that E is a μ-measurable set in X. We use the following notation:

$$\begin{gathered} p_-(E) := \inf_E p; \qquad p_+(E) := \sup_E p; \qquad p_- := p_-(X); \qquad p_+ := p_+(X); \\ \overline{B}(x,r) := \{y \in X : d(x,y) \le r\}, \qquad kB(x,r) := B(x,kr); \qquad B_{xy} := B(x,d(x,y)); \\ \overline{B}_{xy} := \overline{B}(x,d(x,y)); \qquad g_B := \frac{1}{\mu(B)} \int_B |g(x)| d\mu(x). \end{gathered} \tag{2.3}$$

Assume that $1 \leq p_- \leq p_+ < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(X)$ (sometimes it is denoted by $L^{p(x)}(X)$) is the class of all μ-measurable functions f on X for which $S_p(f) := \int_X |f(x)|^{p(x)} d\mu(x) < \infty$. The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\lambda > 0 : S_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{2.4}$$

It is known (see, e.g., [8, 15, 32, 33]) that $L^{p(\cdot)}$ is a Banach space. For other properties of $L^{p(\cdot)}$ spaces we refer, for example, to [32–34].

We need some definitions for the exponent p which will be useful to derive the main results of the paper.

Definition 2.1. Let (X, d, μ) be a quasimetric measure space and let $N \geq 1$ be a constant. Suppose that p satisfies the condition $0 < p_- \leq p_+ < \infty$. We say that p belongs to the class $\mathcal{P}(N, x)$, where $x \in X$, if there are positive constants b and c (which might be depended on x) such that

$$\mu(B(x, Nr))^{p_-(B(x,r)) - p_+(B(x,r))} \leq c \tag{2.5}$$

holds for all r, $0 < r \leq b$. Further, $p \in \mathcal{P}(N)$ if there are positive constants b and c such that (2.5) holds for all $x \in X$ and all r satisfying the condition $0 < r \leq b$.

Definition 2.2. Let (X, d, μ) be an SHT. Suppose that $0 < p_- \leq p_+ < \infty$. We say that $p \in \mathrm{LH}(X, x)$ (p satisfies the log-Hölder-type condition at a point $x \in X$) if there are positive constants b and c (which might be depended on x) such that

$$|p(x) - p(y)| \leq \frac{c}{-\ln(\mu(B_{xy}))} \tag{2.6}$$

holds for all y satisfying the condition $d(x, y) \leq b$. Further, $p \in \mathrm{LH}(X)$ (p satisfies the log-Hölder type condition on X) if there are positive constants b and c such that (2.6) holds for all x, y with $d(x, y) \leq b$.

We will also need another form of the log-Hölder continuity condition given by the following definition.

Definition 2.3. Let (X, d, μ) be a quasimetric measure space, and let $0 < p_- \leq p_+ < \infty$. We say that $p \in \overline{\mathrm{LH}}(X, x)$ if there are positive constants b and c (which might be depended on x) such that

$$|p(x) - p(y)| \leq \frac{c}{-\ln d(x, y)} \tag{2.7}$$

for all y with $d(x, y) \leq b$. Further, $p \in \overline{\mathrm{LH}}(X)$ if (2.7) holds for all x, y with $d(x, y) \leq b$.

It is easy to see that if a measure μ is upper Ahlfors Q-regular and $p \in \mathrm{LH}(X)$ (resp., $p \in \mathrm{LH}(X,x)$), then $p \in \overline{\mathrm{LH}}(X)$ (resp., $p \in \overline{\mathrm{LH}}(X,x)$. Further, if μ is lower Ahlfors Q-regular and $p \in \overline{\mathrm{LH}}(X)$ (resp., $p \in \overline{\mathrm{LH}}(X,x)$), then $p \in \mathrm{LH}(X)$ (resp., $p \in \mathrm{LH}(X,x)$).

Remark 2.4. It can be checked easily that if (X,d,μ) is an SHT, then $\mu B_{x_0 x} \approx \mu B_{x x_0}$.

Remark 2.5. Let (X,d,μ) be an SHT with $L < \infty$. It is known (see, e.g., [8, 35]) that if $p \in \overline{\mathrm{LH}}(X)$, then $p \in \mathcal{P}(1)$. Further, if μ is upper Ahlfors Q-regular, then the condition $p \in \mathcal{P}(1)$ implies that $p \in \overline{\mathrm{LH}}(X)$.

Proposition 2.6. *Let c be positive and let $1 < p_-(X) \le p_+(X) < \infty$ and $p \in \mathrm{LH}(X)$ (resp., $p \in \overline{\mathrm{LH}}(X)$), then the functions $cp(\cdot)$, $1/p(\cdot)$, and $p'(\cdot)$ belong to $\mathrm{LH}(X)$ (resp., $\overline{\mathrm{LH}}(X)$). Further if $p \in \mathrm{LH}(X,x)$ (resp., $p \in \overline{\mathrm{LH}}(X,x)$) then $cp(\cdot)$, $1/p(\cdot)$, and $p'(\cdot)$ belong to $\mathrm{LH}(X,x)$ (resp., $p \in \overline{\mathrm{LH}}(X,x)$).*

The proof of the latter statement can be checked immediately using the definitions of the classes $\mathrm{LH}(X,x)$, $\mathrm{LH}(X)$, $\overline{\mathrm{LH}}(X,x)$, and $\overline{\mathrm{LH}}(X)$.

Proposition 2.7. *Let (X,d,μ) be an SHT and let $p \in \mathcal{P}(1)$. Then $(\mu B_{xy})^{p(x)} \le c(\mu B_{yx})^{p(y)}$ for all $x,y \in X$ with $\mu(B(x,d(x,y))) \le b$, where b is a small constant, and the constant c does not depend on $x,y \in X$.*

Proof. Due to the doubling condition for μ, Remark 1.1, the condition $p \in \mathcal{P}(1)$ and the fact that $x \in B(y, a_1(a_0+1)d(y,x))$ we have the following estimates: $\mu(B_{xy})^{p(x)} \le \mu(B(y,a_1(a_0+1)d(x,y)))^{p(x)} \le c\mu B(y,a_1(a_0+1)d(x,y))^{p(y)} \le c(\mu B_{yx})^{p(y)}$, which proves the statement. □

The proof of the next statement is trivial and follows directly from the definition of the classes $\mathcal{P}(N,x)$ and $\mathcal{P}(N)$. Details are omitted.

Proposition 2.8. *Let (X,d,μ) be a quasimetric measure space and let $x_0 \in X$. Suppose that $N \ge 1$ be a constant. Then the following statements hold:*

(i) *if $p \in \mathcal{P}(N,x_0)$ (resp., $p \in \mathcal{P}(N)$), then there are positive constants r_0, c_1, and c_2 such that for all $0 < r \le r_0$ and all $y \in B(x_0,r)$ (resp., for all x_0, y with $d(x_0,y) < r \le r_0$), one has that $\mu(B(x_0,Nr))^{p(x_0)} \le c_1\mu(B(x_0,Nr))^{p(y)} \le c_2\mu(B(x_0,Nr))^{p(x_0)}$.*

(ii) *Let $p \in \mathcal{P}(N,x_0)$, then there are positive constants r_0, c_1, and c_2 (in general, depending on x_0) such that for all r ($r \le r_0$) and all $x,y \in B(x_0,r)$ one has $\mu(B(x_0,Nr))^{p(x)} \le c_1\mu(B(x_0,Nr))^{p(y)} \le c_2\mu(B(x_0,Nr))^{p(x)}$.*

(iii) *Let $p \in \mathcal{P}(N)$, then there are positive constants r_0, c_1, and c_2 such that for all balls B with radius r ($r \le r_0$) and all $x,y \in B$, one has that $\mu(NB)^{p(x)} \le c_1\mu(NB)^{p(y)} \le c_2\mu(NB)^{p(x)}$.*

It is known that (see, e.g., [32, 33]) if f is a measurable function on X and E is a measurable subset of X, then the following inequalities hold:

$$\begin{aligned} &\|f\|_{L^{p(\cdot)}(E)}^{p_+(E)} \le S_p(f\chi_E) \le \|f\|_{L^{p(\cdot)}(E)}^{p_-(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \le 1; \\ &\|f\|_{L^{p(\cdot)}(E)}^{p_-(E)} \le S_p(f\chi_E) \le \|f\|_{L^{p(\cdot)}(E)}^{p_+(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} > 1. \end{aligned} \tag{2.8}$$

Further, Hölder's inequality in the variable exponent Lebesgue spaces has the following form:

$$\int_E fg\,d\mu \le \left(\frac{1}{p_-(E)} + \frac{1}{(p')_-(E)}\right)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p'(\cdot)}(E)}. \tag{2.9}$$

Lemma 2.9. *Let* (X, d, μ) *be an SHT.*

(i) *If* β *is a measurable function on* X *such that* $\beta_+ < -1$ *and if* r *is a small positive number, then there exists a positive constant* c *independent of* r *and* x *such that*

$$\int_{X\setminus B(x_0,r)} \left(\mu B_{x_0 y}\right)^{\beta(x)} d\mu(y) \le c\frac{\beta(x)+1}{\beta(x)}\mu(B(x_0,r))^{\beta(x)+1}. \tag{2.10}$$

(ii) *Suppose that* p *and* α *are measurable functions on* X *satisfying the conditions* $1 < p_- \le p_+ < \infty$ *and* $\alpha_- > 1/p_-$. *Then there exists a positive constant* c *such that for all* $x \in X$ *the inequality*

$$\int_{\overline{B}(x_0,2d(x_0,x))} \left(\mu B(x, d(x,y))\right)^{(\alpha(x)-1)p'(x)} d\mu(y) \le c\left(\mu B(x_0, d(x_0,x))\right)^{(\alpha(x)-1)p'(x)+1} \tag{2.11}$$

holds.

Proof. Part (i) was proved in [35] (see also [31, page 372], for constant β). The proof of Part (ii) is given in [31, (Lemma 6.5.2, page 348)] for constant α and p, but repeating those arguments we can see that it is also true for variable α and p. Details are omitted. □

Lemma 2.10. *Let* (X, d, μ) *be an SHT. Suppose that* $0 < p_- \le p_+ < \infty$, *then* p *satisfies the condition* $p \in \mathcal{P}(1)$ *(resp.,* $p \in \mathcal{P}(1, x)$*) if and only if* $p \in \mathrm{LH}(X)$ *(resp.,* $p \in \mathrm{LH}(X, x)$*).*

Proof. We follow [1].

Necessity. Let $p \in \mathcal{P}(1)$, and let $x, y \in X$ with $d(x,y) < c_0$ for some positive constant c_0. Observe that $x, y \in B$, where $B := B(x, 2d(x,y))$. By the doubling condition for μ, we have that $(\mu B_{xy})^{-|p(x)-p(y)|} \le c(\mu B)^{-|p(x)-p(y)|} \le c(\mu B)^{p_-(B)-p_+(B)} \le C$, where C is a positive constant which is greater than 1. Taking now the logarithm in the last inequality, we have that $p \in \mathrm{LH}(X)$. If $p \in \mathcal{P}(1, x)$, then by the same arguments we find that $p \in \mathrm{LH}(X, x)$.

Sufficiency. Let $B := B(x_0, r)$. First observe that If $x, y \in B$, then $\mu B_{xy} \le c\mu B(x_0, r)$. Consequently, this inequality and the condition $p \in \mathrm{LH}(X)$ yield $|p_-(B) - p_+(B)| \le C/ - \ln(c_0\mu B(x_0, r))$. Further, there exists r_0 such that $0 < r_0 < 1/2$ and $c_1 \le (\ln(\mu(B)))/(-\ln(c_0\mu(B))) \le c_2$, $0 < r \le r_0$, where c_1 and c_2 are positive constants. Hence $(\mu(B))^{p_-(B)-p_+(B)} \le (\mu(B))^{C/\ln(c_0\mu(B))} = \exp(C\ln(\mu(B))/\ln(c_0\mu(B))) \le C$.

Let, now, $p \in \mathrm{LH}(X, x)$ and let $B_x := B(x, r)$ where r is a small number. We have that $p_+(B_x) - p(x) \le (c / -\ln(c_0 \mu B(x, r)))$ and $p(x) - p_-(B_x) \le (c / -\ln(c_0 \mu B(x, r)))$ for some positive constant c_0. Consequently,

$$\left(\mu(B_x)\right)^{p_-(B_x) - p_+(B_x)} = \left(\mu(B_x)\right)^{p(x) - p_+(B_x)} \left(\mu(B_x)\right)^{p_-(B_x) - p(x)} \le c\left(\mu(B_x)\right)^{(-2c/-\ln(c_0 \mu B_x))} \le C. \tag{2.12}$$

□

Definition 2.11. A measure μ on X is said to satisfy the reverse doubling condition ($\mu \in \mathrm{RDC}(X)$) if there exist constants $A > 1$ and $B > 1$ such that the inequality $\mu(B(a, Ar)) \ge B\mu(B(a, r))$ holds.

Remark 2.12. It is known that if all annulus in X are not empty (i.e., condition (2.1) holds), then $\mu \in \mathrm{DC}(X)$ implies that $\mu \in \mathrm{RDC}(X)$ (see, e.g., [28, page 11, Lemma 20]).

Lemma 2.13. *Let (X, d, μ) be an SHT. Suppose that there is a point $x_0 \in X$ such that $p \in \mathrm{LH}(X, x_0)$. Let A be the constant defined in Definition 2.11. Then there exist positive constants r_0 and C (which might be depended on x_0) such that for all $r, 0 < r \le r_0$, the inequality*

$$(\mu B_A)^{p_-(B_A) - p_+(B_A)} \le C \tag{2.13}$$

holds, where $B_A := B(x_0, Ar) \setminus B(x_0, r)$ and the constant C is independent of r.

Proof. Taking into account condition (2.1) and Remark 2.12, we have that $\mu \in \mathrm{RDC}(X)$. Let $B := B(x_0, r)$. By the doubling and reverse doubling conditions, we have that $\mu B_A = \mu B(x_0, Ar) - \mu B(x_0, r) \ge (B-1)\mu B(x_0, r) \ge c\mu(AB)$. Suppose that $0 < r < c_0$, where c_0 is a sufficiently small constant. Then by using Lemma 2.10 we find that $(\mu B_A)^{p_-(B_A) - p_+(B_A)} \le c(\mu(AB))^{p_-(B_A) - p_+(B_A)} \le c(\mu(AB))^{p_-(AB) - p_+(AB)} \le c$. □

In the sequel we will use the notation:

$$I_{1,k} := \begin{cases} B\left(x_0, \dfrac{A^{k-1} L}{a_1}\right) & \text{if } L < \infty, \\ B\left(x_0, \dfrac{A^{k-1}}{a_1}\right) & \text{if } L = \infty, \end{cases}$$

$$I_{2,k} := \begin{cases} \overline{B}\left(x_0, \mathrm{A}^{k+2} a_1 L\right) \setminus B\left(x_0, \dfrac{A^{k-1} L}{a_1}\right) & \text{if } L < \infty, \\ \overline{B}\left(x_0, A^{k+2} a_1\right) \setminus B\left(x_0, \dfrac{A^{k-1}}{a_1}\right) & \text{if } L = \infty, \end{cases}$$

$$I_{3,k} := \begin{cases} X \setminus B\left(x_0, A^{k+2} L a_1\right) & \text{if } L < \infty, \\ X \setminus B\left(x_0, A^{k+2} a_1\right) & \text{if } L = \infty, \end{cases}$$

$$E_k := \begin{cases} \overline{B}(x_0, A^{k+1}L) \setminus B(x_0, A^k L) & \text{if } L < \infty, \\ \overline{B}(x_0, A^{k+1}) \setminus B(x_0, A^k) & \text{if } L = \infty, \end{cases} \tag{2.14}$$

where the constants A and a_1 are taken, respectively, from Definition 2.11 and the triangle inequality for the quasimetric d, and L is a diameter of X.

Lemma 2.14. *Let (X, d, μ) be an SHT and let $1 < p_-(x) \le p(x) \le q(x) \le q_+(X) < \infty$. Suppose that there is a point $x_0 \in X$ such that $p, q \in \mathrm{LH}(X, x_0)$. Assume that if $L = \infty$, then $p(x) \equiv p_c \equiv$ const and $q(x) \equiv q_c \equiv$ const outside some ball $B(x_0, a)$. Then there exists a positive constant C such that*

$$\sum_k \left\| f\chi_{I_{2,k}} \right\|_{L^{p(\cdot)}(X)} \left\| g\chi_{I_{2,k}} \right\|_{L^{q'(\cdot)}(X)} \le C \|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{q'(\cdot)}(X)}, \tag{2.15}$$

for all $f \in L^{p(\cdot)}(X)$ and $g \in L^{q'(\cdot)}(X)$.

Proof. Suppose that $L = \infty$. To prove the lemma, first observe that $\mu(E_k) \approx \mu B(x_0, A^k)$ and $\mu(I_{2,k}) \approx \mu B(x_0, A^{k-1})$. This holds because μ satisfies the reverse doubling condition and, consequently,

$$\begin{aligned} \mu E_k &= \mu\left(\overline{B}\left(x_0, A^{k+1}\right) \setminus B\left(x_0, A^k\right)\right) = \mu\overline{B}\left(x_0, A^{k+1}\right) - \mu B\left(x_0, A^k\right) \\ &= \mu\overline{B}\left(x_0, AA^k\right) - \mu B\left(x_0, A^k\right) \ge B\mu B\left(x_0, A^k\right) - \mu B\left(x_0, A^k\right) = (B-1)\mu B\left(x_0, A^k\right). \end{aligned} \tag{2.16}$$

Moreover, the doubling condition yields $\mu E_k \le \mu B(x_0, AA^k) \le c\mu B(x_0, A^k)$, where $c > 1$. Hence, $\mu E_k \approx \mu B(x_0, A^k)$.

Further, since we can assume that $a_1 \ge 1$, we find that

$$\begin{aligned} \mu I_{2,k} &= \mu\left(\overline{B}\left(x_0, A^{k+2}a_1\right) \setminus B\left(x_0, \frac{A^{k-1}}{a_1}\right)\right) = \mu\overline{B}\left(x_0, A^{k+2}a_1\right) - \mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) \\ &= \mu\overline{B}\left(x_0, AA^{k+1}a_1\right) - \mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) \ge B\mu B\left(x_0, A^{k+1}a_1\right) - \mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) \\ &\ge B^2\mu B\left(x_0, \frac{A^k}{a_1}\right) - \mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) \ge B^3\mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) - \mu B\left(x_0, \frac{A^{k-1}}{a_1}\right) \\ &= \left(B^3 - 1\right)\mu B\left(x_0, \frac{A^{k-1}}{a_1}\right). \end{aligned} \tag{2.17}$$

Moreover, using the doubling condition for μ we have that $\mu I_{2,k} \le \mu\overline{B}(x_0, A^{k+2}r) \le c\mu B(x_0, A^{k+1}r) \le c^2\mu B(x_0, A^k/a_1) \le c^3\mu B(x_0, A^{k-1}/a_1)$. This gives the estimates $(B^3 - 1)\mu B(x_0, A^{k-1}/a_1) \le \mu(I_{2,k}) \le c^3\mu B(x_0, A^{k-1}/a_1)$.

For simplicity, assume that $a = 1$. Suppose that m_0 is an integer such that $A^{m_0-1}/a_1 > 1$. Let us split the sum as follows:

$$\sum_i \|f\chi_{I_{2,i}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{q'(\cdot)}(X)} = \sum_{i\le m_0}(\cdots) + \sum_{i>m_0}(\cdots) =: J_1 + J_2. \tag{2.18}$$

Since $p(x) \equiv p_c = \text{const}$, $q(x) = q_c = \text{const}$ outside the ball $B(x_0, 1)$, by using Hölder's inequality and the fact that $p_c \le q_c$, we have

$$J_2 = \sum_{i>m_0} \|f\chi_{I_{2,i}}\|_{L^{p_c}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{(q_c)'}(X)} \le c\|f\|_{L^{p(\cdot)}(X)} \cdot \|g\|_{L^{q'(\cdot)}(X)}. \tag{2.19}$$

Let us estimate J_1. Suppose that $\|f\|_{L^{p(\cdot)}(X)} \le 1$ and $\|g\|_{L^{q'(\cdot)}(X)} \le 1$. Also, by Proposition 2.6, we have that $1/q' \in \text{LH}(X, x_0)$. Therefore, by Lemma 2.13 and the fact that $1/q' \in \text{LH}(X, x_0)$, we obtain that $\mu(I_{2,k})^{1/q_+(I_{2,k})} \approx \|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \approx \mu(I_{2,k})^{1/q_-(I_{2,k})}$ and $\mu(I_{2,k})^{1/q'_+(I_{2,k})} \approx \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)} \approx \mu(I_{2,k})^{1/q'_-(I_k)}$, where $k \le m_0$. Further, observe that these estimates and Hölder's inequality yield the following chain of inequalities:

$$\begin{aligned}
J_1 &\le c\sum_{k\le m_0}\int_{\overline{B}(x_0,A^{m_0+1})} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}\chi_{E_k}(x)\,d\mu(x) \\
&= c\int_{\overline{B}(x_0,A^{m_0+1})} \sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}\chi_{E_k}(x)d\mu(x) \\
&\le c\left\|\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)}}\chi_{E_k}(x)\right\|_{L^{q(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} \\
&\quad\times\left\|\sum_{k\le m_0} \frac{\|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}\chi_{E_k}(x)\right\|_{L^{q'(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} =: cS_1(f) \cdot S_2(g).
\end{aligned} \tag{2.20}$$

Now we claim that $S_1(f) \le cI(f)$, where

$$I(f) := \left\|\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}}\chi_{E_k}(\cdot)\right\|_{L^{p(\cdot)}(\overline{B}(x_0,A^{m_0+1}))}, \tag{2.21}$$

and the positive constant c does not depend on f. Indeed, suppose that $I(f) \le 1$. Then taking into account Lemma 2.13 we have that

$$\begin{aligned}
&\sum_{k\le m_0}\frac{1}{\mu(I_{2,k})}\int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)}d\mu(x) \\
&\quad\le c\int_{\overline{B}(x_0,A^{m_0+1})}\left(\sum_{k\le m_0}\frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}\chi_{E_k(x)}\right)^{p(x)} d\mu(x) \le c.
\end{aligned} \tag{2.22}$$

Consequently, since $p(x) \le q(x)$, $E_k \subseteq I_{2,k}$ and $\|f\|_{L^{p(\cdot)}(X)} \le 1$, we find that

$$\sum_{k\le m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{q(x)}_{L^{p(\cdot)}(X)} d\mu(x) \le \sum_{k\le m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)} d\mu(x) \le c. \tag{2.23}$$

This implies that $S_1(f) \le c$. Thus, the desired inequality is proved. Further, let us introduce the following function:

$$\mathbb{P}(y) := \sum_{k\le 2} p_+(I_{2,k})\chi_{E_k(y)}. \tag{2.24}$$

It is clear that $p(y) \le \mathbb{P}(y)$ because $E_k \subset I_{2,k}$. Hence

$$I(f) \le c \left\| \sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(\cdot)} \right\|_{L^{\mathbb{P}(\cdot)}(\overline{B}(x_0, A^{m_0+1}))} \tag{2.25}$$

for some positive constant c. Then, by using this inequality, the definition of the function $\mathbb{P}$, the condition $p \in \mathrm{LH}(X)$, and the obvious estimate $\|\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)} \ge c\mu(I_{2,k})$, we find that

$$\begin{aligned}
&\int_{\overline{B}(x_0, A^{m_0+1})} \left(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \right)^{\mathbb{P}(x)} d\mu(x) \\
&\quad = \int_{\overline{B}(x_0, A^{m_0+1})} \left(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \right) d\mu(x) \\
&\quad \le c \int_{\overline{B}(x_0, A^{m_0+1})} \left(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}}{\mu(I_{2,k})} \chi_{E_k(x)} \right) d\mu(x) \le c \sum_{k\le m_0} \|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)} \\
&\quad \le c \sum_{k\le m_0} \int_{I_{2,k}} |f(x)|^{p(x)} d\mu(x) \le c \int_X |f(x)|^{p(x)} d\mu(x) \le c.
\end{aligned} \tag{2.26}$$

Consequently, $I(f) \le c\|f\|_{L^{p(\cdot)}(X)}$. Hence, $S_1(f) \le c\|f\|_{L^{p(\cdot)}(X)}$. Analogously taking into account the fact that $q' \in \mathrm{DL}(X)$ and arguing as above, we find that $S_2(g) \le c\|g\|_{L^{q'(\cdot)}(X)}$. Thus, summarizing these estimates we conclude that

$$\sum_{i\le m_0} \|f\chi_{I_i}\|_{L^{p(\cdot)}(X)} \|g\chi_{I_i}\|_{L^{q'(\cdot)}(X)} \le c\|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{q'(\cdot)}(X)}. \tag{2.27}$$

□

Lemma 2.14 for $L^{p(\cdot)}([0,1])$ spaces defined with respect to the Lebesgue measure was derived in [24] (see also [22] for $X = \mathbb{R}^n$, $d(x,y) = |x-y|$, and $d\mu(x) = dx$).

3. Hardy-Type Transforms

In this section, we derive two-weight estimates for the operators:

$$T_{v,w}f(x) = v(x)\int_{B_{x_0x}} f(y)w(y)\,d\mu(y), \qquad T'_{v,w}f(x) = v(x)\int_{X\setminus\overline{B}_{x_0x}} f(y)w(y)d\mu(y). \tag{3.1}$$

Let a be a positive constant, and let p be a measurable function defined on X. Let us introduce the notation:

$$p_0(x) := p_-\left(\overline{B}_{x_0x}\right); \qquad \widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } d(x_0,x) \le a; \\ p_c = \text{const} & \text{if } d(x_0,x) > a. \end{cases}$$
$$p_1(x) := p_-\left(\overline{B}(x_0,a) \setminus B_{x_0x}\right); \qquad \widetilde{p}_1(x) := \begin{cases} p_1(x) & \text{if } d(x_0,x) \le a; \\ p_c = \text{const} & \text{if } d(x_0,x) > a. \end{cases} \tag{3.2}$$

Remark 3.1. If we deal with a quasimetric measure space with $L < \infty$, then we will assume that $a = L$. Obviously, $\widetilde{p}_0 \equiv p_0$ and $\widetilde{p}_1 \equiv p_1$ in this case.

Theorem 3.2. *Let (X, d, μ) be a quasimetric measure space. Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \le \widetilde{p}_0(x) \le q(x) \le q_+ < \infty$. In the case when $L = \infty$, suppose that $p \equiv p_c \equiv$ const, $q \equiv q_c \equiv$ const, outside some ball $\overline{B}(x_0, a)$. If the condition*

$$A_1 := \sup_{0\le t\le L} \int_{t<d(x_0,x)\le L} (v(x))^{q(x)} \left(\int_{d(x_0,x)\le t} w^{(\widetilde{p}_0)'(x)}(y)d\mu(y)\right)^{q(x)/(\widetilde{p}_0)'(x)} d\mu(x) < \infty, \tag{3.3}$$

holds, then $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Proof. Here we use the arguments of the proofs of Theorem 1.1.4 in [31, (see page 7)] and of Theorem 2.1 in [21]. First, we notice that $p_- \le p_0(x) \le p(x)$ for all $x \in X$. Let $f \ge 0$ and let $S_p(f) \le 1$. First, assume that $L < \infty$. We denote

$$I(s) := \int_{d(x_0,y)<s} f(y)w(y)d\mu(y) \quad \text{for } s \in [0, L]. \tag{3.4}$$

Suppose that $I(L) < \infty$, then $I(L) \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. Let us denote $s_j := \sup\{s : I(s) \le 2^j\}$, $j \le m$, and $s_{m+1} := L$. Then $\{s_j\}_{j=-\infty}^{m+1}$ is a nondecreasing sequence. It is easy to check that $I(s_j) \le 2^j$, $I(s) > 2^j$ for $s > s_j$, and $2^j \le \int_{s_j \le d(x_0,y) \le s_{j+1}} f(y)w(y)d\mu(y)$. If $\beta := \lim_{j\to-\infty} s_j$, then $d(x_0,x) < L$ if and only if $d(x_0,x) \in [0,\beta] \cup \bigcup_{j=-\infty}^{m}(s_j,s_{j+1}]$. If $I(L) = \infty$, then we take $m = \infty$. Since $0 \le I(\beta) \le I(s_j) \le 2^j$ for every j, we have that $I(\beta) = 0$. It is obvious that $X = \bigcup_{j\le m}\{x : s_j < d(x_0,x) \le s_{j+1}\}$. Further, we have that

$$\begin{aligned} S_q(T_{v,w}f) &= \int_X (T_{v,w}f(x))^{q(x)} d\mu(x) = \int_X \left(v(x) \int_{B(x_0,d(x_0,x))} f(y)w(y)d\mu(y) \right)^{q(x)} d\mu(x) \\ &= \int_X (v(x))^{q(x)} \left(\int_{B(x_0,d(x_0,x))} f(y)w(y)d\mu(y) \right)^{q(x)} d\mu(x) \\ &\le \sum_{j=-\infty}^{m} \int_{s_j<d(x_0,x)\le s_{j+1}} (v(x))^{q(x)} \left(\int_{d(x_0,y)<s_{j+1}} f(y)w(y)d\mu(y) \right)^{q(x)} d\mu(x). \end{aligned} \tag{3.5}$$

Let us denote

$$B_j(x_0) := \{x \in X : s_{j-1} \le d(x_0,x) \le s_j\}. \tag{3.6}$$

Notice that $I(s_{j+1}) \le 2^{j+1} \le 4\int_{B_j(x_0)} w(y)f(y)d\mu(y)$. Consequently, by this estimate and Hölder's inequality with respect to the exponent $p_0(x)$ we find that

$$\begin{aligned} S_q(T_{v,w}f) &\le c \sum_{j=-\infty}^{m} \int_{s_j<d(x_0,x)\le s_{j+1}} (v(x))^{q(x)} \left(\int_{B_j(x_0)} f(y)w(y)d\mu(y) \right)^{q(x)} d\mu(x) \\ &\le c \sum_{j=-\infty}^{m} \int_{s_j<d(x_0,x)\le s_{j+1}} (v(x))^{q(x)} J_k(x)d\mu(x), \end{aligned} \tag{3.7}$$

where

$$J_k(x) := \left(\int_{B_j(x_0)} f(y)^{p_0(x)} d\mu(y) \right)^{q(x)/p_0(x)} \left(\int_{B_j(x_0)} w(y)^{(p_0)'(x)} d\mu(y) \right)^{q(x)/(p_0)'(x)}. \tag{3.8}$$

Observe now that $q(x) \geq p_0(x)$. Hence, this fact and the condition $S_p(f) \leq 1$ imply that

$$
\begin{aligned}
J_k(x) &\leq c\left(\int_{B_j(x_0)\cap\{y:f(y)\leq 1\}} f(y)^{p_0(x)}d\mu(y) + \int_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y)\right)^{q(x)/p_0(x)} \\
&\quad \times \int_{B_j(x_0)} w\big((y)^{(p_0)'(x)}d\mu(y)\big)^{q(x)/(p_0)'(x)} \\
&\leq c\left(\mu(B_j(x_0)) + \int_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y)\right) \\
&\quad \times \left(\int_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y)\right)^{q(x)/(p_0)'(x)}.
\end{aligned}
\tag{3.9}
$$

It follows now that

$$
\begin{aligned}
S_q(T_{v,w}f) \leq c\Bigg(&\sum_{j=-\infty}^{m} \mu(B_j(x_0)) \int_{s_j<d(x_0,x)\leq s_{j+1}} v(x)^{q(x)} \\
&\times \left(\int_{B_j(x_0)} w(y)^{(p_0')(x)}d\mu(y)\right)^{q(x)/(p_0)'(x)} d\mu(x) \\
&+ \sum_{j=-\infty}^{m}\left(\int_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y)\right)\int_{s_j<d(x_0,x)\leq s_{j+1}} v(x)^{q(x)} \\
&\times\left(\int_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y)\right)^{q(x)/(p_0)'(x)} d\mu(x)\Bigg) := c(N_1+N_2).
\end{aligned}
\tag{3.10}
$$

Since $L < \infty$, it is obvious that

$$
\begin{aligned}
N_1 &\leq A_1\sum_{j=-\infty}^{m+1}\mu(B_j(x_0)) \leq CA_1, \\
N_2 &\leq A_1\sum_{j=-\infty}^{m+1}\int_{B_j(x_0)} f(y)^{p(y)}d\mu(y) \leq C\int_X (f(y))^{p(y)}d\mu(y) = A_1 S_p(f) \leq A_1.
\end{aligned}
\tag{3.11}
$$

Finally, $S_q(T_{v,w}f) \leq c(CA_1 + A_1) < \infty$. Thus, $T_{v,w}$ is bounded if $A_1 < \infty$.

Let us now suppose that $L = \infty$. We have

$$T_{v,w}f(x) = \chi_{B(x_0,a)}(x)v(x)\int_{B_{x_0x}} f(y)w(y)d\mu(y) + \chi_{X\setminus B(x_0,a)}(x)v(x)\int_{B_{x_0x}} f(y)w(y)d\mu(y) =: T^{(1)}_{v,w}f(x) + T^{(2)}_{v,w}f(x). \tag{3.12}$$

By using the already proved result for $L < \infty$ and the fact that $\operatorname{diam}(B(x_0, a)) < \infty$, we find that $\|T^{(1)}_{v,w}f\|_{L^{q(\cdot)}(B(x_0,a))} \leq c\|f\|_{L^{p(\cdot)}(B(x_0,a))} \leq c$ because

$$A_1^{(a)} := \sup_{0\leq t\leq a}\int_{t<d(x_0,x)\leq a}(v(x))^{q(x)}\left(\int_{d(x_0,x)\leq t} w^{(p_0)'(x)}(y)d\mu(y)\right)^{q(x)/(p_0)'(x)} d\mu(x) \leq A_1 < \infty. \tag{3.13}$$

Further, observe that

$$T^{(2)}_{v,w}f(x) = \chi_{X\setminus B(x_0,a)}(x)v(x)\int_{B_{x_0x}} f(y)w(y)d\mu(y) = \chi_{X\setminus B(x_0,a)}(x)v(x) \times\int_{d(x_0,y)\leq a} f(y)w(y)d\mu(y) + \chi_{X\setminus B(x_0,a)}(x)v(x)\int_{a\leq d(x_0,y)\leq d(x_0,x)} f(y)w(y)d\mu(y) =: T^{(2,1)}_{v,w}f(x) + T^{(2,2)}_{v,w}f(x). \tag{3.14}$$

It is easy to see that (see also [31, Theorems 1.1.3 or 1.1.4]) the condition

$$\overline{A}_1^{(a)} := \sup_{t\geq a}\left(\int_{d(x_0,x)\geq t}(v(x))^{q_c}d\mu(x)\right)^{1/q_c}\left(\int_{a\leq d(x_0,y)\leq t} w(y)^{(p_c)'}d\mu(y)\right)^{1/(p_c)'} < \infty \tag{3.15}$$

guarantees the boundedness of the operator

$$T_{v,w}f(x) = v(x)\int_{a\leq d(x_0,y)<d(x_0,x)} f(y)w(y)d\mu(y) \tag{3.16}$$

from $L^{p_c}(X \setminus B(x_0,a))$ to $L^{q_c}(X \setminus B(x_0,a))$. Thus, $T_{v,w}^{(2,2)}$ is bounded. It remains to prove that $T_{v,w}^{(2,1)}$ is bounded. We have

$$\begin{aligned}\left\|T_{v,w}^{(2,1)} f\right\|_{L^{p(\cdot)}(X)} &= \left(\int_{(B(x_0,a))^c} v(x)^{q_c} d\mu(x)\right)^{1/q_c} \left(\int_{\overline{B}(x_0,a)} f(y) w(y) d\mu(y)\right) \\ &\le \left(\int_{(B(x_0,a))^c} v(x)^{q_c} d\mu(x)\right)^{1/q_c} \|f\|_{L^{p(\cdot)}(\overline{B}(x_0,a))} \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))}.\end{aligned} \tag{3.17}$$

Observe, now, that the condition $A_1 < \infty$ guarantees that the integral

$$\int_{(B(x_0,a))^c} v(x)^{q_c} d\mu(x) \tag{3.18}$$

is finite. Moreover, $N := \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))} < \infty$. Indeed, we have that

$$N \le \begin{cases} \left(\int_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y)\right)^{1/(p_-(\overline{B}(x_0,a)))'} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))} \le 1, \\ \left(\int_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y)\right)^{1/(p_+(\overline{B}(x_0,a)))'} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))} > 1. \end{cases} \tag{3.19}$$

Further,

$$\begin{aligned}&\int_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y) \\ &\quad = \int_{\overline{B}(x_0,a)\cap\{w\le 1\}} w(y)^{p'(y)} d\mu(y) + \int_{\overline{B}(x_0,a)\cap\{w>1\}} w(y)^{p'(y)} d\mu(y) := I_1 + I_2.\end{aligned} \tag{3.20}$$

For I_1, we have that $I_1 \le \mu(\overline{B}(x_0,a)) < \infty$. Since $L = \infty$ and condition (2.1) holds, there exists a point $y_0 \in X$ such that $a < d(x_0,y_0) < 2a$. Consequently, $\overline{B}(x_0,a) \subset \overline{B}(x_0,d(x_0,y_0))$ and $p(y) \ge p_-(\overline{B}(x_0,d(x_0,y_0))) = p_0(y_0)$, where $y \in \overline{B}(x_0,a)$. Consequently, the condition $A_1 < \infty$ yields $I_2 \le \int_{\overline{B}(x_0,a)} w(y)^{(p_0)'(y_0)} dy < \infty$. Finally, we have that $\|T_{v,w}^{(2,1)}\|f\|_{L^{p(\cdot)}(X)} \le C$. Hence, $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. □

The proof of the following statement is similar to that of Theorem 3.2; therefore, we omit it (see also the proofs of Theorem 1.1.3 in [31] and Theorems 2.6 and 2.7 in [21] for similar arguments).

Theorem 3.3. *Let (X, d, μ) be a quasimetric measure space. Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \le \tilde{p}_1(x) \le q(x) \le q_+ < \infty$. If $L = \infty$, then, one assumes that $p \equiv p_c \equiv const$, $q \equiv q_c \equiv const$ outside some ball $B(x_0, a)$. If*

$$B_1 = \sup_{0 \le t \le L} \int_{d(x_0,x) \le t} (v(x))^{q(x)} \left(\int_{t \le d(x_0,x) \le L} w^{(\tilde{p}_1)'(x)}(y) d\mu(y) \right)^{q(x)/(\tilde{p}_1)'(x)} d\mu(x) < \infty, \quad (3.21)$$

then $T'_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Remark 3.4. If $p \equiv$ const, then the condition $A_1 < \infty$ in Theorem 3.2 (resp., $B_1 < \infty$ in Theorem 3.3) is also necessary for the boundedness of $T_{v,w}$ (resp., $T'_{v,w}$) from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. See [31, pages 4-5] for the details.

4. Potentials

In this section, we discuss two-weight estimates for the potential operators $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ on quasimetric measure spaces, where $0 < \alpha_- \le \alpha_+ < 1$. If $\alpha \equiv$ const , then we denote $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ by T_α and I_α, respectively.

The boundedness of Riesz potential operators in $L^{p(\cdot)}(\Omega)$ spaces, where Ω is a domain in $\mathbb{R}^n$ was established in [5, 6, 36, 37].

For the following statement we refer to [11].

Theorem A. *Let (X, d, μ) be an SHT . Suppose that $1 < p_- \le p_+ < \infty$ and $p \in \mathcal{P}(1)$. Assume that if $L = \infty$, then $p \equiv const$ outside some ball. Let α be a constant satisfying the condition $0 < \alpha < 1/p_+$. One sets $q(x) = p(x)/(1 - \alpha p(x))$. Then, T_α is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Theorem B (see [9]). *Let (X, d, μ) be a nonhomogeneous space with $L < \infty$ and let N be a constant defined by $N = a_1(1 + 2a_0)$, where the constants a_0 and a_1 are taken from the definition of the quasimetric d. Suppose that $1 < p_- < p_+ < \infty$, $p, \alpha \in \mathcal{P}(N)$ and that μ is upper Ahlfors 1-regular. One defines $q(x) = p(x)/(1 - \alpha(x)p(x))$, where $0 < \alpha_- \le \alpha_+ < 1/p_+$. Then $I_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

For the statements and their proofs of this section, we keep the notation of the previous sections and, in addition, introduce the new notation:

$$\begin{aligned}
&v_\alpha^{(1)}(x) := v(x)\left(\mu B_{x_0 x}\right)^{\alpha-1}, \quad w_\alpha^{(1)}(x) := w^{-1}(x); \quad v_\alpha^{(2)}(x) := v(x); \\
&w_\alpha^{(2)}(x) := w^{-1}(x)\left(\mu B_{x_0 x}\right)^{\alpha-1}; \\
&F_x := \begin{cases} \left\{ y \in X : \dfrac{d(x_0,y)L}{A^2 a_1} \le d(x_0,y) \le A^2 L a_1 d(x_0,x) \right\}, & \text{if } L < \infty, \\ \left\{ y \in X : \dfrac{d(x_0,y)}{A^2 a_1} \le d(x_0,y) \le A^2 a_1 d(x_0,x) \right\}, & \text{if } L = \infty, \end{cases}
\end{aligned} \quad (4.1)$$

where A and a_1 are constants defined in Definition 2.11 and the triangle inequality for d, respectively. We begin this section with the following general-type statement.

Theorem 4.1. *Let (X, d, μ) be an SHT without atoms. Suppose that $1 < p_- \le p_+ < \infty$ and α is a constant satisfying the condition $0 < \alpha < 1/p_+$. Let $p \in \mathcal{P}(1)$. One sets $q(x) = p(x)/(1 - \alpha p(x))$. Further, if $L = \infty$, then one assumes that $p \equiv p_c \equiv$ const outside some ball $B(x_0, a)$. Then the inequality*

$$\left\| v(T_\alpha f) \right\|_{L^{q(\cdot)}(X)} \le c \left\| w f \right\|_{L^{p(\cdot)}(X)} \tag{4.2}$$

holds if the following three conditions are satisfied:

(a) *$T_{v_\alpha^{(1)}, w_\alpha^{(1)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(b) *$T_{v_\alpha^{(2)}, w_\alpha^{(2)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(c) *there is a positive constant b such that one of the following inequalities hold: (1) $v_+(F_x) \le b w(x)$ for $\mu-$ a.e. $x \in X$; (2) $v(x) \le b w_-(F_x)$ for $\mu-$ a.e. $x \in X$.*

Proof. For simplicity, suppose that $L < \infty$. The proof for the case $L = \infty$ is similar to that of the previous case. Recall that the sets $I_{i,k}$, $i = 1, 2, 3$ and E_k are defined in Section 2. Let $f \ge 0$ and let $\|g\|_{L^{q'(\cdot)}(X)} \le 1$. We have

$$\begin{aligned}
&\int_X (T_\alpha f)(x) g(x) v(x) d\mu(x) \\
&\quad = \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f)(x) g(x) v(x) d\mu(x) \\
&\quad \le \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{1,k})(x) g(x) v(x) d\mu(x) + \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{2,k})(x) g(x) v(x) d\mu(x) \\
&\qquad + \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{3,k})(x) g(x) v(x) d\mu(x) =: S_1 + S_2 + S_3,
\end{aligned} \tag{4.3}$$

where $f_{1,k} = f \cdot \chi_{I_{1,k}}$, $f_{2,k} = f \cdot \chi_{I_{2,k}}$, $f_{3,k} = f \cdot \chi_{I_{3,k}}$.

Observe that if $x \in E_k$ and $y \in I_{1,k}$, then $d(x_0, y) \le d(x_0, x)/A a_1$. Consequently, the triangle inequality for d yields $d(x_0, x) \le A' a_1 a_0 d(x, y)$, where $A' = A/(A-1)$. Hence, by using Remark 2.4, we find that $\mu(B_{x_0 x}) \le c\mu(B_{xy})$. Applying condition (a) now, we have that

$$S_1 \le c \left\| (\mu B_{x_0 x})^{\alpha - 1} v(x) \int_{B_{x_0 x}} f(y) d\mu(y) \right\|_{L^{q(x)}(X)} \|g\|_{L^{q'(\cdot)}(X)} \le c \|f\|_{L^{p(\cdot)}(X)}. \tag{4.4}$$

Further, observe that if $x \in E_k$ and $y \in I_{3,k}$, then $\mu(B_{x_0 y}) \le c\mu(B_{xy})$. By condition (b), we find that $S_3 \le c\|f\|_{L^{p(\cdot)}(X)}$.

Now we estimate S_2. Suppose that $v_+(F_x) \le bw(x)$. Theorem A and Lemma 2.14 yield

$$\begin{aligned} S_2 &\le \sum_k \left\| (T_\alpha f_{2,k})(\cdot)\chi_{E_k}(\cdot)v(\cdot)\right\|_{L^{q(\cdot)}(X)} \left\| g\chi_{E_k}(\cdot)\right\|_{L^{q'(\cdot)}(X)} \\ &\le \sum_k (v_+(E_k)) \left\| (T_\alpha f_{2,k})(\cdot)\right\|_{L^{q(\cdot)}(X)} \left\| g(\cdot)\chi_{E_k}(\cdot)\right\|_{L^{q'(\cdot)}(X)} \\ &\le c\sum_k (v_+(E_k)) \left\| f_{2,k}\right\|_{L^{p(\cdot)}(X)} \left\| g(\cdot)\chi_{E_k}(\cdot)\right\|_{L^{q'(\cdot)}(X)} \\ &\le c\sum_k \left\| f_{2,k}(\cdot)w(\cdot)\chi_{I_{2,k}}(\cdot)\right\|_{L^{p(\cdot)}(X)} \left\| g(\cdot)\chi_{E_k}(\cdot)\right\|_{L^{q'(\cdot)}(X)} \\ &\le c\left\| f(\cdot)w(\cdot)\right\|_{L^{p(\cdot)}(X)} \left\| g(\cdot)\right\|_{L^{q'(\cdot)}(X)} \le c\left\| f(\cdot)w(\cdot)\right\|_{L^{p(\cdot)}(X)}. \end{aligned} \tag{4.5}$$

The estimate of S_2 for the case when $v(x) \le bw_-(F_x)$ is similar to that of the previous one. Details are omitted. □

Theorems 4.1, 3.2, and 3.3 imply the following statement.

Theorem 4.2. *Let (X, d, μ) be an SHT. Suppose that $1 < p_- \le p_+ < \infty$ and α is a constant satisfying the condition $0 < \alpha < 1/p_+$. Let $p \in \mathcal{P}(1)$. One sets $q(x) = p(x)/(1 - \alpha p(x))$. If $L = \infty$, then, one supposes that $p \equiv p_c \equiv$ const outside some ball $B(x_0, a)$. Then inequality (4.2) holds if the following three conditions are satisfied:*

(i)

$$\begin{aligned} P_1 := \sup_{0<t\le L} \int_{t<d(x_0,x)\le L} \left(\frac{v(x)}{(\mu(B_{x_0 x}))^{1-\alpha}} \right)^{q(x)} \\ \times \left(\int_{d(x_0,y)\le t} w^{-(\tilde{p}_0)'(x)}(y)\, d\mu(y) \right)^{q(x)/(\tilde{p}_0)'(x)} d\mu(x) < \infty; \end{aligned} \tag{4.6}$$

(ii)

$$\begin{aligned} P_2 := \sup_{0<t\le L} \int_{d(x_0,x)\le t} (v(x))^{q(x)} \\ \times \left(\int_{t<d(x_0,y)\le L} \left(w(y)(\mu B_{x_0 y})^{1-\alpha} \right)^{-(\tilde{p}_1)'(x)} d\mu(y) \right)^{q(x)/(\tilde{p}_1)'(x)} d\mu(x) < \infty, \end{aligned} \tag{4.7}$$

(iii) condition (c) of Theorem 4.1 holds.

Remark 4.3. If $p = p_c \equiv$ const on X, then the conditions $P_i < \infty$, $i = 1, 2$, are necessary for (4.2). Necessity of the condition $P_1 < \infty$ follows by taking the test function $f = w^{-(p_c)'}\chi_{B(x_0,t)}$ in (4.2) and observing that $\mu B_{xy} \le c\mu B_{x_0 x}$ for those x and y which satisfy the conditions $d(x_0, x) \ge t$ and $d(x_0, y) \le t$ (see also [31, Theorem 6.6.1, page 418] for the similar arguments) while necessity of the condition $P_2 < \infty$ can be derived by choosing the test function

$f(x) = w^{-(p_c)'}(x)\chi_{X\setminus B(x_0,t)}(x)(\mu B_{x_0x})^{(\alpha-1)((p_c)'-1)}$ and taking into account the estimate $\mu B_{xy} \le \mu B_{x_0y}$ for $d(x_0,x) \le t$ and $d(x_0,y) \ge t$.

The next statement follows in the same manner as the previous one. In this case, Theorem B is used instead of Theorem A. The proof is omitted.

Theorem 4.4. *Let (X,d,μ) be a nonhomogeneous space with $L < \infty$. Let N be a constant defined by $N = a_1(1+2a_0)$. Suppose that $1 < p_- \le p_+ < \infty$, $p, \alpha \in \mathcal{P}(N)$ and that μ is upper Ahlfors 1-regular. We define $q(x) = p(x)/(1-\alpha(x)p(x))$, where $0 < \alpha_- \le \alpha_+ < 1/p_+$. Then the inequality*

$$\left\|v(\cdot)(I_{\alpha(\cdot)}f)(\cdot)\right\|_{L^{q(\cdot)}(X)} \le c\left\|w(\cdot)f(\cdot)\right\|_{L^{p(\cdot)}(X)} \tag{4.8}$$

holds if

(i)

$$\sup_{0\le t\le L}\int_{t<d(x_0,x)\le L}\left(\frac{v(x)}{(d(x_0,x))^{1-\alpha(x)}}\right)^{q(x)}\left(\int_{\overline{B}(x_0,t)} w^{-(p_0)'(x)}(y)d\mu(y)\right)^{q(x)/(p_0)'(x)} d\mu(x) < \infty; \tag{4.9}$$

(ii)

$$\sup_{0\le t\le L}\int_{\overline{B}(x_0,t)}(v(x))^{q(x)}\left(\int_{t<d(x_0,y)\le L}\left(w(y)d(x_0,y)^{1-\alpha(y)}\right)^{-(p_1)'(x)}d\mu(y)\right)^{q(x)/(p_1)'(x)} d\mu(x)<\infty, \tag{4.10}$$

and (iii) condition (c) of Theorem 4.1 is satisfied.

Remark 4.5. It is easy to check that if p and α are constants, then conditions (i) and (ii) in Theorem 4.4 are also necessary for (4.8). This follows easily by choosing appropriate test functions in (4.8) (see also Remark 4.3).

Theorem 4.6. *Let (X,d,μ) be an SHT without atoms. Let $1 < p_- \le p_+ < \infty$ and let α be a constant with the condition $0 < \alpha < 1/p_+$. One sets $q(x) = p(x)/(1-\alpha p(x))$. Assume that p has a minimum at x_0 and that $p \in \mathrm{LH}(X)$. Suppose also that if $L = \infty$, then p is constant outside some ball $B(x_0,a)$. Let v and w be positive increasing functions on $(0,2L)$. Then the inequality*

$$\left\|v(d(x_0,\cdot))(T_\alpha f)(\cdot)\right\|_{L^{q(\cdot)}(X)} \le c\left\|w(d(x_0,\cdot))f(\cdot)\right\|_{L^{p(\cdot)}(X)} \tag{4.11}$$

holds if

$$\begin{aligned} I_1 := \sup_{0<t\le L} I_1(t) &:= \sup_{0<t\le L}\int_{t<d(x_0,x)\le L}\left(\frac{v(d(x_0,x))}{(\mu(B_{x_0x}))^{1-\alpha}}\right)^{q(x)} \\ &\times\left(\int_{d(x_0,y)\le t} w^{-(\tilde{p}_0)'(x)}(d(x_0,y))d\mu(y)\right)^{q(x)/(\tilde{p}_0)'(x)} d\mu(x) < \infty, \end{aligned} \tag{4.12}$$

for $L = \infty$;

$$J_1 := \sup_{0<t\le L} \int_{t<d(x_0,x)\le L} \left(\frac{v(d(x_0,x))}{(\mu(B_{x_0 x}))^{1-\alpha}} \right)^{q(x)} \times \left(\int_{d(x_0,y)\le t} w^{-p'(x_0)}(d(x_0,y))\,d\mu(y) \right)^{q(x)/p'(x_0)} d\mu(x) < \infty, \tag{4.13}$$

for $L < \infty$.

Proof. We prove the theorem for $L = \infty$. The proof for the case when $L < \infty$ is similar. Observe that by Lemma 2.10 the condition $p \in \mathrm{LH}(X)$ implies $p \in \mathcal{P}(1)$. We will show that the condition $I_1 < \infty$ implies the inequality $v(A^2 a_1 t)/w(t) \le C$ for all $t > 0$, where A and a_1 are constants defined in Definition 2.11 and the triangle inequality for d, respectively. Indeed, let us assume that $t \le b_1$, where b_1 is a small positive constant. Then, taking into account the monotonicity of v and w and the facts that $\widetilde{p}_0(x) = p_0(x)$ (for small $d(x_0,x)$) and $\mu \in \mathrm{RDC}(X)$, we have

$$\begin{aligned} I_1(t) &\ge \int_{A^2 a_1 t \le d(x_0,x) < A^3 a_1 t} \left(\frac{v(A^2 a_1 t)}{w(t)} \right)^{q(x)} (\mu B(x_0,t))^{(\alpha - 1/p_0(x))q(x)}\, d\mu(x) \\ &\ge \left(\frac{v(A^2 a_1 t)}{w(t)} \right)^{q_-} \int_{A^2 a_1 t \le d(x_0,x) < A^3 a_1 t} (\mu B(x_0,t))^{(\alpha - 1/p_0(x))q(x)} d\mu(x) \ge c \left(\frac{v(A^2 a_1 t)}{w(t)} \right)^{q_-}. \end{aligned} \tag{4.14}$$

Hence, $\overline{c} := \overline{\lim}_{t\to 0}(v(A^2 a_1 t)/w(t)) < \infty$. Further, if $t > b_2$, where b_2 is a large number, then since p and q are constants, for $d(x_0,x) > t$, we have that

$$\begin{aligned} I_1(t) &\ge \left(\int_{A^2 a_1 t \le d(x_0,x) < A^3 a_1 t} v(d(x_0,x))^{q_c} (\mu B(x_0,t))^{(\alpha-1)q_c} d\mu(x) \right) \\ &\quad \times \left(\int_{B(x_0,t)} w^{-(p_c)'}(x)\,d\mu(x) \right)^{q_c/(p_c)'} d\mu(x) \\ &\ge C \left(\frac{v(A^2 a_1 t)}{w(t)} \right)^{q_c} \int_{A^2 a_1 t \le d(x_0,x) < A^3 a_1 t} (\mu B(x_0,t))^{(\alpha - 1/p_c)\, q_c} d\mu(x) \ \ge c \left(\frac{v(A^2 a_1 t)}{w(t)} \right)^{q_c}. \end{aligned} \tag{4.15}$$

In the last inequality we used the fact that μ satisfies the reverse doubling condition.

Now we show that the condition $I_1 < \infty$ implies

$$\sup_{t>0} I_2(t) := \sup_{t>0} \int_{d(x_0,x)\le t} (v(d(x_0,x)))^{q(x)} \times \left(\int_{d(x_0,y)>t} w^{-(\tilde{p}_1)'(x)}(d(x_0,y))(\mu(B_{x_0y}))^{(\alpha-1)(\tilde{p}_1)'(x)} d\mu(y) \right)^{q(x)/(\tilde{p}_1)'(x)} d\mu(x) < \infty. \tag{4.16}$$

Due to monotonicity of functions v and w, the condition $p \in \mathrm{LH}(X)$, Proposition 2.6, Lemmas 2.9, and 2.10 and the assumption that p has a minimum at x_0, we find that for all $t > 0$,

$$\begin{aligned} I_2(t) &\le \int_{d(x_0,x)\le t} \left(\frac{v(t)}{w(t)}\right)^{q(x)} (\mu(B(x_0,t)))^{(\alpha-1/p(x_0))q(x)} d\mu(x) \\ &\le c \int_{d(x_0,x)\le t} \left(\frac{v(t)}{w(t)}\right)^{q(x)} (\mu(B(x_0,t)))^{(\alpha-1/p(x_0))q(x_0)} d\mu(x) \\ &\le c \left(\int_{d(x_0,x)\le t} \left(\frac{v(A^2 a_1 t)}{w(t)}\right)^{q(x)} d\mu(x) \right) (\mu(B(x_0,t)))^{-1} \le C. \end{aligned} \tag{4.17}$$

Now, Theorem 4.2 completes the proof. □

Theorem 4.7. *Let (X, d, μ) be an SHT with $L < \infty$. Suppose that p, q and α are measurable functions on X satisfying the conditions: $1 < p_- \le p(x) \le q(x) \le q_+ < \infty$ and $1/p_- < \alpha_- \le \alpha_+ < 1$. Assume that $\alpha \in \mathrm{LH}(X)$ and there is a point $x_0 \in X$ such that $p, q \in \mathrm{LH}(X, x_0)$. Suppose also that w is a positive increasing function on $(0, 2L)$. Then the inequality*

$$\left\| (T_{\alpha(\cdot)} f) v \right\|_{L^{q(\cdot)}(X)} \le c \left\| w(d(x_0,\cdot)) f(\cdot) \right\|_{L^{p(\cdot)}(X)} \tag{4.18}$$

holds if the following two conditions are satisfied:

$$\begin{aligned} \widetilde{I}_1 &:= \sup_{0<t\le L} \int_{t\le d(x_0,x)\le L} \left(\frac{v(x)}{(\mu B_{x_0x})^{1-\alpha(x)}} \right)^{q(x)} \times \left(\int_{d(x_0,x)\le t} w^{-(p_0)'(x)}(d(x_0,y)) d\mu(y) \right)^{q(x)/(p_0)'(x)} d\mu(x) < \infty; \\ \widetilde{I}_2 &:= \sup_{0<t\le L} \int_{d(x_0,x)\le t} (v(x))^{q(x)} \times \left(\int_{t\le d(x_0,x)\le L} \left(w(d(x_0,y)) \times (\mu B_{x_0y})^{1-\alpha(x)} \right)^{-(p_1)'(x)} d\mu(y) \right)^{q(x)/(p_1)'(x)} d\mu(x) < \infty. \end{aligned} \tag{4.19}$$

Proof. For simplicity, assume that $L = 1$. First observe that by Lemma 2.10 we have $p, q \in \mathcal{P}(1, x_0)$ and $\alpha \in \mathcal{P}(1)$. Suppose that $f \geq 0$ and $S_p(w(d(x_0,\cdot))f(\cdot)) \leq 1$. We will show that $S_q(v(T_{\alpha(\cdot)}f)) \leq C$.

We have

$$\begin{aligned} &S_q\left(vT_{\alpha(\cdot)}f\right) \\ &\quad\leq C_q\left[\int_X\left(v(x)\int_{d\left(x_0,y\right)\leq d(x_0,x)/(2a_1)} f(y)\left(\mu B_{xy}\right)^{\alpha(x)-1}d\mu(y)\right)^{q(x)}d\mu(x)\right. \\ &\qquad+\int_X\left(v(x)\int_{d(x_0,x)/(2a_1)\leq d\left(x_0,y\right)\leq 2a_1d(x_0,x)} f(y)\left(\mu B_{xy}\right)^{\alpha(x)-1}d\mu(y)\right)^{q(x)}d\mu(x) \\ &\qquad\left.+\int_X\left(v(x)\int_{d\left(x_0,y\right)\geq 2a_1d(x_0,x)} f(y)\left(\mu B_{xy}\right)^{\alpha(x)-1}d\mu(y)\right)^{q(x)}d\mu(x)\right] := C_q[I_1+I_2+I_3]. \end{aligned} \tag{4.20}$$

First, observe that by virtue of the doubling condition for μ, Remark 2.4, and simple calculation we find that $\mu(B_{x_0x}) \leq c\mu(B_{xy})$. Taking into account this estimate and Theorem 3.2 we have that

$$I_1 \leq c\int_X\left(\frac{v(x)}{\left(\mu B_{x_0x}\right)^{1-\alpha(x)}}\int_{d\left(x_0,y\right)<d(x_0,x)} f(y)d\mu(y)\right)^{q(x)}d\mu(x) \leq C. \tag{4.21}$$

Further, it is easy to see that if $d(x_0, y) \geq 2a_1d(x_0, x)$, then the triangle inequality for d and the doubling condition for μ yield that $\mu B_{x_0y} \leq c\mu B_{xy}$. Hence, due to Proposition 2.7, we see that $(\mu B_{x_0y})^{\alpha(x)-1} \geq c(\mu B_{xy})^{\alpha(y)-1}$ for such x and y. Therefore, Theorem 3.3 implies that $I_3 \leq C$.

It remains to estimate I_2. Let us denote:

$$E^{(1)}(x) := \overline{B}_{x_0x} \setminus B\left(x_0, \frac{d(x_0,x)}{2a_1}\right); \qquad E^{(2)}(x) := \overline{B}(x_0, 2a_1d(x_0,x)) \setminus B_{x_0x}. \tag{4.22}$$

Then we have that

$$\begin{aligned} I_2 &\leq C\left[\int_X\left[v(x)\int_{E^{(1)}(x)} f(y)\left(\mu B_{xy}\right)^{\alpha(x)-1}d\mu(y)\right]^{q(x)}d\mu(x)\right. \\ &\qquad\left.+\int_X\left[v(x)\int_{E^{(2)}(x)} f(y)\left(\mu B_{xy}\right)^{\alpha(x)-1}d\mu(y)\right]^{q(x)}d\mu(x)\right] =: c[I_{21}+I_{22}]. \end{aligned} \tag{4.23}$$

Using Hölder's inequality for the classical Lebesgue spaces we find that

$$I_{21} \le \int_X v^{q(x)}(x)\left(\int_{E^{(1)}(x)} w^{p_0(x)}(d(x_0,y))(f(y))^{p_0(x)}d\mu(y)\right)^{q(x)/p_0(x)} \times\left(\int_{E^{(1)}(x)} w^{-(p_0)'(x)}(d(x_0,y))(\mu B_{xy})^{(\alpha(x)-1)(p_0)'(x)}d\mu(y)\right)^{q(x)/(p_0)'(x)} d\mu(x) . \tag{4.24}$$

Denote the first inner integral by $J^{(1)}$ and the second one by $J^{(2)}$.

By using the fact that $p_0(x) \le p(y)$, where $y \in E^{(1)}(x)$, we see that $J^{(1)} \le \mu(B_{x_0x}) + \int_{E^{(1)}(x)}(f(y))^{p(y)}(w(d(x_0,y)))^{p(y)}d\mu(y)$, while by applying Lemma 2.9, for $J^{(2)}$, we have that

$$\begin{aligned} J^{(2)} &\le cw^{-(p_0)'(x)}\left(\frac{d(x_0,x)}{2a_1}\right)\int_{E^{(1)}(x)}(\mu B_{xy})^{(\alpha(x)-1)(p_0)'(x)}d\mu(y) \\ &\le cw^{-(p_0)'(x)}\left(\frac{d(x_0,x)}{2a_1}\right)(\mu B_{x_0x})^{(\alpha(x)-1)(p_0)'(x)+1}. \end{aligned} \tag{4.25}$$

Summarizing these estimates for $J^{(1)}$ and $J^{(2)}$ we conclude that

$$\begin{aligned} \mathrm{I}_{21} &\le \int_X v^{q(x)}(x)(\mu B_{x_0x})^{q(x)\alpha(x)}w^{-q(x)}\left(\frac{d(x_0,x)}{2a_1}\right)d\mu(x) + \int_X v^{q(x)}(x) \\ &\times\left(\int_{E^{(1)}(x)} w^{p(y)}(d(x_0,y))(f(y))^{p(y)}d\mu(y)\right)^{q(x)/p_0(x)}(\mu B_{x_0x})^{q(x)(\alpha(x)-1/p_0(x))} \\ &\times w^{-q(x)}\left(\frac{d(x_0,x)}{2a_1}\right)d\mu(x) =: I_{21}^{(1)} + I_{21}^{(2)}. \end{aligned} \tag{4.26}$$

By applying monotonicity of w, the reverse doubling property for μ with the constants A and B (see Remark 2.12), and the condition $\widetilde{I}_1 < \infty$ we have that

$$\begin{aligned} I_{21}^{(1)} &\le c\sum_{k=-\infty}^{0}\int_{\overline{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v(x)^{q(x)}\left(\int_{B(x_0,A^{k-1}/2a_1)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y)\right)^{q(x)/(p_0)'(x)} \\ &\times(\mu B_{x_0,x})^{q(x)/p_0(x)+(\alpha(x)-1)q(x)}d\mu(x) \le c\sum_{k=-\infty}^{0}\left(\mu\overline{B}\left(x_0,A^k\right)\right)^{q_-/p_+} \end{aligned}$$

$$\times \int_{\overline{B}(x_0,A^k)\backslash B(x_0,A^{k-1})} v(x)^{q(x)} \left(\int_{B(x_0,A^k)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \right)^{q(x)/(p_0)'(x)}$$

$$\times (\mu B_{x_0,x})^{q(x)(\alpha(x)-1)} d\mu(x) \le c \sum_{k=-\infty}^{0} \left(\mu\overline{B}\left(x_0, A^k\right) \setminus B\left(x_0, A^{k-1}\right)\right)^{q_-/p_+}$$

$$\le c \sum_{k=-\infty}^{0} \int_{\mu\overline{B}(x_0,A^k)\backslash B(x_0,A^{k-1})} (\mu B_{x_0,x})^{q_-/p_+-1} d\mu(y)$$

$$\le c \int_X (\mu B_{x_0,x})^{q_-/p_+-1} d\mu(y) < \infty. \tag{4.27}$$

Due to the facts that $q(x) \ge p_0(x)$, $S_p(w(d(x_0,\cdot)f(\cdot))) \le 1$, $\widetilde{I}_1 < \infty$ and w is increasing, for $I_{21}^{(2)}$, we find that

$$I_{21}^{(2)} \le c \sum_{k=-\infty}^{0} \left(\int_{\mu\overline{B}\left(x_0,A^{k+1}a_1\right)\setminus B\left(x_0,A^{k-2}\right)} w^{p(y)}(d(x_0,y))(f(y))^{p(y)} d\mu(y) \right)$$

$$\times \left(\int_{\mu\overline{B}\left(x_0,A^k\right)\backslash B\left(x_0,A^{k-1}\right)} v^{q(x)}(x) \left(\int_{B\left(x_0,A^{k-1}\right)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \right)^{q(x)/(p_0)'(x)} \right.$$

$$\left. \times (\mu B_{x_0,x})^{(\alpha(x)-1)q(x)} d\mu(x) \right) \le c S_p(f(\cdot)w(d(x_0,\cdot))) \le c. \tag{4.28}$$

Analogously, the estimate for I_{22} follows. In this case, we use the condition $\widetilde{I}_2 < \infty$ and the fact that $p_1(x) \le p(y)$ when $d(x_0,x) \le d(x_0,y) < 2a_1 d(x_0,x)$. The details are omitted. The theorem is proved. □

Taking into account the proof of Theorem 4.6, we can easily derive the following statement, proof of which is omitted.

Theorem 4.8. *Let (X,d,μ) be an SHT with $L < \infty$. Suppose that p, q and α are measurable functions on X satisfying the conditions $1 < p_- \le p(x) \le q(x) \le q_+ < \infty$ and $1/p_- < \alpha_- \le \alpha_+ < 1$. Assume that $\alpha \in \mathrm{LH}(X)$. Suppose also that there is a point x_0 such that $p, q \in \mathrm{LH}(X, x_0)$ and p has a minimum at x_0. Let v and w be a positive increasing function on $(0, 2L)$ satisfying the condition $J_1 < \infty$ (see Theorem 4.6). Then inequality (4.11) is fulfilled.*

Theorem 4.9. *Let (X,d,μ) be an SHT with $L < \infty$ and let μ be upper Ahlfors 1-regular. Suppose that $1 < p_- \le p_+ < \infty$ and that $p \in \overline{\mathrm{LH}}(X)$. Let p have a minimum at x_0. Assume that α is constant* *satisfying the condition* $\alpha < 1/p_+$. *We set* $q(x) = p(x)/(1-\alpha p(x))$. *If* v *and* w *are positive increasing functions on* $(0, 2L)$ *satisfying the condition*

$$E := \sup_{0\le t\le L} \int_{t<d(x_0,x)\le L} \left(\frac{v(d(x_0,x))}{(d(x_0,x))^{1-\alpha}} \right)^{q(x)} \times \left(\int_{d(x_0,x)\le t} w^{-(p_0)'(x)}(y)\, d\mu(y) \right)^{q(x)/(p_0)'(x)} d\mu(x) < \infty, \tag{4.29}$$

then the inequality

$$\left\| v(d(x_0,\cdot))(I_\alpha f)(\cdot) \right\|_{L^{q(\cdot)}(X)} \le c \left\| w(d(x_0,\cdot)) f(\cdot) \right\|_{L^{p(\cdot)}(X)} \tag{4.30}$$

holds.

Proof. The proof is similar to that of Theorem 4.6, we only discuss some details. First, observe that due to Remark 2.5 we have that $p \in \mathcal{P}(N)$, where $N = a_1(1 + 2a_0)$. It is easy to check that the condition $E < \infty$ implies that $v(A^2 a_1 t)/w(t) \le C$ for all t, where the constant A is defined in Definition 2.11 and a_1 is from the triangle inequality for d. Further, Lemmas 2.9 and 2.10, the fact that p has a minimum at x_0, and the inequality

$$\int_{d(x_0,y)>t} (d(x_0,y))^{(\alpha-1)(p_1)'(x)} d\mu(y) \le c t^{(\alpha-1)(p_1)'(x)+1}, \tag{4.31}$$

where the constant c does not depend on t and x, yield that

$$\sup_{0\le t\le L} \int_{d(x_0,x)\le t} (v(d(x_0,x)))^{q(x)} \times \left(\int_{d(x_0,y)>t} \left(\frac{w(d(x_0,y))}{(d(x_0,y))^{1-\alpha}} \right)^{-(p_1)'(x)} d\mu(y) \right)^{q(x)/(p_1)'(x)} d\mu(x) < \infty. \tag{4.32}$$

Theorem 4.4 completes the proof. □

Example 4.10. Let $v(t) = t^\gamma$ and $w(t) = t^\beta$, where γ and β are constants satisfying the condition $0 \le \beta < 1/(p_-)'$, $\gamma \ge \max\{0, 1 - \alpha - (1/q_+) - (q_-/q_+)(-\beta + (1/(p_-)'))\}$. Then (v, w) satisfies the conditions of Theorem 4.6.

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