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Research Article

On Sharp Triangle Inequalities in Banach Spaces II

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Sharp triangle inequality and its reverse in Banach spaces were recently showed by Mitani et al. (2007). In this paper, we present equality attainedness for these inequalities in strictly convex Banach spaces.

1. Introduction

In recent years, the triangle inequality and its reverse inequality have been treated in [1–5] (see also [6, 7]).Kato et al. [8] presented the following sharp triangle inequality and its reverse inequality with n elements in a Banach space X.

Theorem 1.1 (see [8]). *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X,*

$$\begin{aligned} \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \le j \le n} \|x_j\| &\le \sum_{j=1}^{n} \|x_j\| \\ &\le \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \le j \le n} \|x_j\|. \end{aligned} \tag{1.1}$$

These inequalities are useful to treat geometrical structure of Banach spaces, such as uniform non-ℓ_1^n-ness (see [8]). Moreover,Hsu et al. [9] presented these inequalities for strongly integrable functions with values in a Banach space.

Mitani et al. [10] showed the following inequalities which are sharper than Inequality (1.1) in Theorem 1.1.

Theorem 1.2 (see [10]). *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X with $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|$, $n \geq 2$,*

$$\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j}{\|x_j\|}\right\|\right)\left(\|x_k\| - \|x_{k+1}\|\right) \leq \sum_{j=1}^{n}\|x_j\| \tag{1.2}$$

$$\leq \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\left(\|x_{n-k}\| - \|x_{n-(k-1)}\|\right), \tag{1.3}$$

where $x_0 = x_{n+1} = 0$.

In this paper we first present a simpler proof of Theorem 1.2. To do this we consider the case $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$, as follows.

Theorem 1.3. *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X with $\|x_1\| > \|x_2\| > \cdots > \|x_n\|, n \geq 2$,*

$$\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j}{\|x_j\|}\right\|\right)\left(\|x_k\| - \|x_{k+1}\|\right) \leq \sum_{j=1}^{n}\|x_j\| \tag{1.4}$$

$$\leq \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\left(\|x_{n-k}\| - \|x_{n-(k-1)}\|\right), \tag{1.5}$$

where $x_0 = x_{n+1} = 0$.

From this result we can easily obtain Theorem 1.2.

Moreover we consider equality attainedness for sharp triangle inequality and its reverse inequality in strictly convex Banach spaces. Namely, we characterize equality attainedness of Inequalities (1.4) and (1.5) in Theorem 1.3.

2. Simple Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.3. According to Theorem 1.1 Inequalities (1.4) and (1.5) hold for the case $n = 2$ (cf. [3]). Therefore let $n \geq 3$. We first prove (1.4) by the induction. Assume that (1.4) holds true for all $n-1$ elements in X. Let $x_1, x_2, \ldots, x_n$ be any n elements in X with $\|x_1\| > \|x_2\| > \cdots > \|x_n\| > 0$. Let

$$u_j = \left(\|x_j\| - \|x_n\|\right)\frac{x_j}{\|x_j\|}, \tag{2.1}$$

for all positive numbers j with $1 \le j \le n$. Then

$$\sum_{j=1}^{n} x_j = \|x_n\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} + \sum_{j=1}^{n-1} u_j \tag{2.2}$$

and $\|u_1\| > \|u_2\| > \cdots > \|u_{n-1}\| > 0$. By assumption,

$$\left\|\sum_{j=1}^{n-1} u_j\right\| \le \sum_{j=1}^{n-1}\|u_j\| - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{u_j}{\|u_j\|}\right\|\right)(\|u_k\| - \|u_{k+1}\|) \tag{2.3}$$

holds, where $u_n = 0$. Since $\|u_k\| - \|u_{k+1}\| = \|x_k\| - \|x_{k+1}\|$, from (2.2) and (2.3),

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &= \left\|\|x_n\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} + \sum_{j=1}^{n-1} u_j\right\| \\
&\le \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \left\|\sum_{j=1}^{n-1} u_j\right\| \\
&\le \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \sum_{j=1}^{n-1}\|u_j\| - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{u_j}{\|u_j\|}\right\|\right)(\|u_k\| - \|u_{k+1}\|) \\
&= \|x_n\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| + \sum_{j=1}^{n-1}\left(\|x_j\| - \|x_n\|\right) \\
&\quad - \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k}\frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|) \\
&= \sum_{j=1}^{n}\|x_j\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k}\frac{x_j}{\|x_j\|}\right\|\right)(\|x_k\| - \|x_{k+1}\|)
\end{aligned} \tag{2.4}$$

and hence (1.4). Thus (1.4) holds true for all finite elements in X.

Next we show Inequality (1.5). Let

$$v_j = \left(\|x_1\| - \|x_{n-j+1}\|\right)\frac{x_{n-j+1}}{\|x_{n-j+1}\|}, \quad 1 \le j \le n-1. \tag{2.5}$$

Then

$$\sum_{j=1}^{n} x_j = \|x_1\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} - \sum_{j=1}^{n-1} v_j \tag{2.6}$$

and $\|v_1\| > \cdots > \|v_{n-1}\| > 0$. Applying Inequality (1.4) to $v_1, \ldots, v_{n-1}$,

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &\geq \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \left\|\sum_{j=1}^{n-1} v_j\right\| \\
&\geq \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \sum_{j=1}^{n-1} \|v_j\| + \sum_{k=2}^{n-1} \left(k - \left\|\sum_{j=1}^{k} \frac{v_j}{\|v_j\|}\right\| \right) (\|v_k\| - \|v_{k+1}\|) \\
&= \|x_1\| \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| - \sum_{j=1}^{n-1} (\|x_1\| - \|x_{n-j+1}\|) \\
&\quad + \sum_{k=2}^{n-1} \left(k - \left\|\sum_{j=1}^{k} \frac{x_{n-j+1}}{\|x_{n-j+1}\|}\right\| \right) (\|x_{n-k+1}\| - \|x_{n-k}\|) \\
&= \sum_{j=1}^{n} \|x_j\| - \sum_{k=2}^{n} \left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|}\right\| \right) (\|x_{n-k+1}\| - \|x_{n-k}\|),
\end{aligned} \tag{2.7}$$

where $v_n = 0$. Thus we obtain (1.5). This completes the proof. □

Proof of Theorem 1.2. Let $x_1, x_2, \ldots, x_n$ be any nonzero elements in X with $\|x_1\| \geq \cdots \geq \|x_n\|$. For all positive numbers m with $m > n$ let

$$x_{k,m} = \left(1 - \frac{k}{m}\right) x_k, \quad k = 1, 2, \ldots, n. \tag{2.8}$$

Then $\|x_{1,m}\| > \|x_{2,m}\| > \cdots > \|x_{n,m}\| > 0$. Applying Theorem 1.3 to $x_{1,m}, \ldots, x_{n,m}$,

$$\begin{aligned}
&\left\|\sum_{j=1}^{n} x_{j,m}\right\| + \sum_{k=2}^{n} \left(k - \left\|\sum_{j=1}^{k} \frac{x_{j,m}}{\|x_{j,m}\|}\right\| \right) (\|x_{k,m}\| - \|x_{k+1,m}\|) \\
&\quad \leq \sum_{j=1}^{n} \|x_{j,m}\| \\
&\quad \leq \left\|\sum_{j=1}^{n} x_{j,m}\right\| - \sum_{k=2}^{n} \left(k - \left\|\sum_{j=n-(k-1)}^{n} \frac{x_{j,m}}{\|x_{j,m}\|}\right\| \right) (\|x_{n-k,m}\| - \|x_{n-(k-1),m}\|),
\end{aligned} \tag{2.9}$$

where $x_{0,m} = x_{n+1,m} = 0$ for all positive numbers m with $m > n$. As $m \to +\infty$, we have Inequalities (1.4) and (1.5). □

3. Equality Attainedness in a Strictly Convex Banach Space

In this section we consider equality attainedness for sharp triangle inequality and its reverse inequality in a strictly convex Banach space. Kato et al. in [8] showed the following.

Theorem 3.1 (see [8]). *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \le j \le n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \le j \le n\}$. Let $J_0 = \{j : \|x_j\| = \|x_{j_0}\|, 1 \le j \le n\}$. Then*

$$\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\min_{1\le j\le n}\|x_k\| = \sum_{j=1}^{n}\|x_k\| \tag{3.1}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $x_j/\|x_j\| = x_{j_1}/\|x_{j_1}\|$ *for all* $j \in J_0^c$ *and* $\sum_{j=1}^{n}(x_j/\|x_j\|) = \|\sum_{j=1}^{n}(x_j/\|x_j\|)\|(x_{j_1}/\|x_{j_1}\|)$.

Theorem 3.2 (see [8]). *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \le j \le n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \le j \le n\}$. Let $J_1 = \{j : \|x_j\| = \|x_{j_1}\|, 1 \le j \le n\}$. Then*

$$\sum_{j=1}^{n}\|x_j\| = \left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\max_{1\le j\le n}\|x_j\| \tag{3.2}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $x_j/\|x_j\| = x_{j_0}/\|x_{j_0}\|$ *for all* $j \in J_1^c$ *and* $\sum_{j=1}^{n} x_j = \|\sum_{j=1}^{n} x_j\|(x_{j_0}/\|x_{j_0}\|)$.

We present equality attainedness for (1.4) and (1.5) in Theorem 1.2. The following lemma given in [8] is quite powerful.

Lemma 3.3 (see [8]). *Let X be a strictly convex Banach space. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in X. Then the following are equivalent.*

(i) $\|\sum_{j=1}^{n}\alpha_j x_j\| = \sum_{j=1}^{n}\alpha_j\|x_j\|$ *holds for any positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(ii) $\|\sum_{j=1}^{n}\alpha_j x_j\| = \sum_{j=1}^{n}\alpha_j\|x_j\|$ *holds for some positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(iii) $x_1/\|x_1\| = x_2/\|x_2\| = \cdots = x_n/\|x_n\|$.

Theorem 3.4. *Let X be a strictly convex Banach space and x, y nonzero elements in X with $\|x\| > \|y\|$. Then*

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\| = \|x\| + \|y\| \tag{3.3}$$

if and only if there exists a real number α with $0 < \alpha < 1$ satisfying $y = \pm\alpha x$.

Proof. Assume that (3.3) is true. By Theorem 3.1, the equality (3.3) is equivalent to Equality

$$\frac{x}{\|x\|}+\frac{y}{\|y\|}=\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\frac{x}{\|x\|}. \tag{3.4}$$

Put

$$\alpha=\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|-1\right)\frac{\|y\|}{\|x\|}. \tag{3.5}$$

Then $y=\alpha x$. Since $\|x\|>\|y\|$, we obtain $0<|\alpha|<1$. Conversely, if $y=\alpha x$ where $0<|\alpha|<1$, then

$$\frac{x}{\|x\|}+\frac{y}{\|y\|}=\left(1+\frac{\alpha}{|\alpha|}\right)\frac{x}{\|x\|}. \tag{3.6}$$

By $1+(\alpha/|\alpha|)\geq 0$, we have (3.4). Thus we get (3.3). □

Next we consider the case $n=3$.

Theorem 3.5. *Let X be a strictly convex Banach space and x, y, z nonzero elements in X with $\|x\|>\|y\|>\|z\|$. Then*

$$\begin{aligned}\|x+y+z\|&+\left(3-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right\|\right)\|z\|+\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)(\|y\|-\|z\|)\\&=\|x\|+\|y\|+\|z\|\end{aligned} \tag{3.7}$$

if and only if there exist α, β with $0<\beta<\alpha<1$ satisfying one of the following conditions:

(a) $y=\alpha x, z=\pm\beta x$,

(b) $y=-\alpha x, z=\beta x$.

Proof. Assume that (3.7) is true. Put

$$u=(\|x\|-\|z\|)\frac{x}{\|x\|},\quad v=(\|y\|-\|z\|)\frac{y}{\|y\|}. \tag{3.8}$$

Then $\|u\|>\|v\|>0$ and

$$x+y+z=\|z\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right)+u+v. \tag{3.9}$$

Note that $u+v\neq 0$. As in the proof of Theorem 1.2 given in [10], we have (3.7) if and only if we have the equalities

$$\left\|\|z\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right)+u+v\right\|=\|z\|\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right\|+\|u+v\|, \tag{3.10}$$

$$\|u+v\|=\|u\|+\|v\|-\left(2-\left\|\frac{u}{\|u\|}+\frac{v}{\|v\|}\right\|\right)\|v\|. \tag{3.11}$$

By Theorem 3.4, Equality (3.11) implies that

$$v = \pm\gamma u \tag{3.12}$$

for some γ with $0 < \gamma < 1$. By (3.8) we have

$$y = \pm\alpha x \tag{3.13}$$

for some $0 < \alpha < 1$. On the other hand, by Lemma 3.3, Equality (3.10) implies

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \frac{u+v}{\|u+v\|}. \tag{3.14}$$

Hence, by using (3.8), (3.12), and (3.13) we have $z = \beta x$ for some real number β. Since $\|x\| > \|y\| > \|z\| > 0$, we have $0 < |\beta| < \alpha < 1$. We consider the following two cases

Case 1. $y = \alpha x$.

Equality (3.14) implies

$$\frac{x}{\|x\|} + \frac{x}{\|x\|} + \frac{\beta x}{\|\beta x\|} = \left\| \frac{x}{\|x\|} + \frac{x}{\|x\|} + \frac{\beta x}{\|\beta x\|} \right\| \frac{x + \alpha x - 2|\beta| x}{\|x + \alpha x - 2|\beta| x\|}. \tag{3.15}$$

Hence we have

$$2 + \frac{\beta}{|\beta|} = \left| 2 + \frac{\beta}{|\beta|} \right| \frac{1 + \alpha - 2|\beta|}{|1 + \alpha - 2|\beta||}. \tag{3.16}$$

By $2 + (\beta/|\beta|) \geq 0$ and $1 + \alpha - 2|\beta| = 1 - |\beta| + \alpha - |\beta| \geq 0$, Equality (3.16) is valid for all real numbers β with $\beta \neq 0$.

Case 2. $y = -\alpha x$.

Equality (3.14) implies

$$\frac{x}{\|x\|} - \frac{x}{\|x\|} + \frac{\beta x}{\|\beta x\|} = \left\| \frac{x}{\|x\|} - \frac{x}{\|x\|} + \frac{\beta x}{\|\beta x\|} \right\| \frac{x - \alpha x}{\|x - \alpha x\|}. \tag{3.17}$$

So we have

$$\frac{\beta}{|\beta|} = \left| \frac{\beta}{|\beta|} \right| \frac{1 - \alpha}{|1 - \alpha|} = \left| \frac{\beta}{|\beta|} \right|. \tag{3.18}$$

Hence $\beta > 0$. Thus ($\Rightarrow$) holds.

Conversely, assume that there exist α, β with $0 < \beta < \alpha < 1$ satisfying one of the conditions (a) and (b). Then it is clear that (3.7) holds. Thus we obtain ($\Leftarrow$). □

Moreover we consider general cases. For each m with $1 \le m \le n$, we put $I_m = \{1, 2, \ldots, m\}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ and $1 \le m \le n$, we define

$$I_m^+(\alpha) = \{k \in I_m : \alpha_k > 0\},$$
$$I_m^-(\alpha) = \{k \in I_m : \alpha_k < 0\}. \tag{3.19}$$

For a finite set A, the cardinal number of A is denoted by $|A|$.

Lemma 3.6. *Let $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_n| > 0$. If $|I_m^+(\alpha)| \ge |I_m^-(\alpha)|$ for all m with $1 \le m \le n$, then*

$$\sum_{j=1}^{n} \alpha_j > 0. \tag{3.20}$$

Proof. Let

$$I_n^+(\alpha) = \{m_1, m_2, \ldots, m_\ell\} \left(= I_{m_\ell}^+(\alpha)\right),$$
$$I_n^-(\alpha) = \{n_1, n_2, \ldots, n_k\} \left(= I_{n_k}^+(\alpha)\right), \tag{3.21}$$

where $m_1 < m_2 < \cdots < m_\ell$ and $n_1 < n_2 < \cdots < n_k$. By the assumption, we have $\ell > k$. We first show $m_j < n_j$ for all j with $1 \le j \le k$. It is clear that $m_1 < n_1$. Assume that $m_i < n_i$ for all i with $1 \le i \le j$. We will show $m_{j+1} < n_{j+1}$. Suppose that $m_{j+1} > n_{j+1}$. By $m_j < n_j < n_{j+1} < m_{j+1}$, we have

$$I_{n_{j+1}}^+(\alpha) = I_{m_j}^+(\alpha) = \left\{m_1, m_2, \ldots, m_j\right\},$$
$$I_{n_{j+1}}^-(\alpha) = \left\{n_1, n_2, \ldots, n_{j+1}\right\}. \tag{3.22}$$

Hence we have $|I_{n_{j+1}}^+(\alpha)| < |I_{n_{j+1}}^-(\alpha)|$, which is a contradiction. Therefore we have $m_{j+1} < n_{j+1}$. Namely, $m_j < n_j$ for all j with $1 \le j \le k$. From this result, we obtain $\alpha_{m_j} + \alpha_{n_j} = |\alpha_{m_j}| - |\alpha_{n_j}| > 0$. Hence

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{\ell} \alpha_{m_j} + \sum_{j=1}^{k} \alpha_{n_j}$$
$$= \sum_{j=1}^{k} \left(\alpha_{m_j} + \alpha_{n_j}\right) + \sum_{j=k+1}^{\ell} \alpha_{m_j} > 0. \tag{3.23}$$

□

Theorem 3.7. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X with $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$. Then*

$$\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j}{\|x_j\|} \right\| \right) \left(\|x_k\| - \|x_{k+1}\|\right) = \sum_{j=1}^{n} \|x_j\| \tag{3.24}$$

if and only if there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *with* $1 = \alpha_1 > |\alpha_2| > |\alpha_3| > \cdots > |\alpha_n|$ *such that*

$$x_m = \alpha_m x_1, \tag{3.25}$$

$$|I_m^+(\alpha)| \geq \left|I_m^-(\alpha)\right| \tag{3.26}$$

for every m *with* $1 \leq m \leq n$.

Proof. ($\Rightarrow$): According to Theorems 3.4 and 3.5, Theorem 3.7 is valid for the cases $n = 2, 3$. Therefore let $n \geq 4$. We will prove Theorem 3.7 by the induction. Assume that this theorem holds true for all nonzero elements in X less than n. Let $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$ and assume that Equality (3.24) holds. Let

$$u_j = \left(\left\|x_j\right\| - \|x_n\|\right) \frac{x_j}{\left\|x_j\right\|} \tag{3.27}$$

for positive integer j with $1 \leq j \leq n-1$. As in the proof of Theorem 1.3, Equality (3.24) holds if and only if

$$\left\| \|x_n\| \sum_{j=1}^{n} \frac{x_j}{\left\|x_j\right\|} + \sum_{j=1}^{n-1} u_j \right\| = \|x_n\| \left\| \sum_{j=1}^{n} \frac{x_j}{\left\|x_j\right\|} \right\| + \left\| \sum_{j=1}^{n-1} u_j \right\|, \tag{3.28}$$

$$\left\| \sum_{j=1}^{n-1} u_i \right\| = \sum_{j=1}^{n-1} \left\|u_j\right\| - \sum_{k=2}^{n-1} \left(k - \left\| \sum_{j=1}^{k} \frac{u_j}{\left\|u_j\right\|} \right\| \right) \left(\|u_k\| - \|u_{k+1}\|\right) \tag{3.29}$$

hold, where $u_n = 0$. Hence, by assumption, there exists $\beta = (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{R}^{n-1}$ with $1 = \beta_1 > |\beta_2| > \cdots > |\beta_{n-1}| > 0$ such that

$$\begin{gathered} u_j = \beta_j u_1, \quad (1 \leq j \leq n-1), \\ \left|I_m^+(\beta)\right| \geq \left|I_m^-(\beta)\right|, \quad (1 \leq m \leq n-1). \end{gathered} \tag{3.30}$$

Since $\sum_{j=1}^{n-1} \beta_j > 0$ by Lemma 3.6, we have

$$\sum_{j=1}^{n-1} u_j = \sum_{j=1}^{n-1} \beta_j u_1 \neq 0. \tag{3.31}$$

Since

$$\left(1 - \frac{\|x_n\|}{\left\|x_j\right\|}\right) x_j = \beta_j \left(1 - \frac{\|x_n\|}{\|x_1\|}\right) x_1 \tag{3.32}$$

by the definition of u_j, we have

$$x_j = \alpha_j x_1, \quad (1 \leq j \leq n-1), \tag{3.33}$$

where

$$\alpha_j = \beta_j \frac{\|x_1\| - \|x_n\|}{\|x_j\| - \|x_n\|} \cdot \frac{\|x_j\|}{\|x_1\|}. \tag{3.34}$$

Since

$$I_m^+(\alpha) = I_m^+(\beta), \quad I_m^-(\alpha) = I_m^-(\beta), \tag{3.35}$$

we have

$$|I_m^+(\alpha)| \geq \left|I_m^-(\alpha)\right| \tag{3.36}$$

for all m with $1 \leq m \leq n-1$. By Lemma 3.3, Equality (3.28) implies

$$\left\|\sum_{j=1}^{n-1} u_j\right\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| \sum_{j=1}^{n-1} u_j. \tag{3.37}$$

Hence there exists $\alpha_n \in \mathbb{R}$ such that $x_n = \alpha_n x_1$. Also,

$$\begin{aligned}\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} &= \sum_{j=1}^{n} \frac{\alpha_j x_1}{|\alpha_j| \|x_1\|} \\ &= \left(\left|I_{n-1}^+(\alpha)\right| - \left|I_{n-1}^-(\alpha)\right| + \frac{\alpha_n}{|\alpha_n|}\right) \frac{x_1}{\|x_1\|},\end{aligned} \tag{3.38}$$

$$\sum_{j=1}^{n-1} u_j = \left(\sum_{j=1}^{n-1} \beta_j\right) u_1. \tag{3.39}$$

Since $\sum_{j=1}^{n-1} \beta_j > 0$, we have from (3.37) and (3.38),

$$\left|I_{n-1}^+(\alpha)\right| - \left|I_{n-1}^-(\alpha)\right| + \frac{\alpha_n}{|\alpha_n|} = \left|\left|I_{n-1}^+(\alpha)\right| - \left|I_{n-1}^-(\alpha)\right| + \frac{\alpha_n}{|\alpha_n|}\right|, \tag{3.40}$$

which implies

$$\left|I_{n-1}^+(\alpha)\right| - \left|I_{n-1}^-(\alpha)\right| + \frac{\alpha_n}{|\alpha_n|} \geq 0. \tag{3.41}$$

If $|I_{n-1}^+(\alpha)| > |I_{n-1}^-(\alpha)|$, then it is clear that $|I_n^+(\alpha)| \geq |I_n^-(\alpha)|$.

If $|I_{n-1}^+(\alpha)| = |I_{n-1}^-(\alpha)|$, then, by (3.41), we have $\alpha_n/|\alpha_n| \geq 0$. Hence $\alpha_n > 0$. Thus we obtain $|I_n^+(\alpha)| > |I_n^-(\alpha)|$.

(⇐): Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ with $1 = \alpha_1 > |\alpha_2| > \cdots > |\alpha_n| > 0$ satisfying (3.25) and (3.26), and let $\alpha_{n+1} = 0$. From (3.25) we have

$$\begin{aligned} &\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j}{\|x_j\|} \right\| \right) (\|x_k\| - \|x_{k+1}\|) \\ &\quad = \left\| \sum_{j=1}^{n} \alpha_j x_1 \right\| + \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{n} \frac{\alpha_j x_1}{\|\alpha_j x_1\|} \right\| \right) (\|\alpha_k x_1\| - \|\alpha_{k+1} x_1\|) \\ &\quad = \left| \sum_{j=1}^{n} \alpha_j \right| + \sum_{k=2}^{n} \left(k - \left| \sum_{j=1}^{k} \frac{\alpha_j}{\|\alpha_j\|} \right| \right) (|\alpha_k| - |\alpha_{k+1}|) \times \|x_1\|. \end{aligned} \tag{3.42}$$

By Lemma 3.6,

$$\left| \sum_{j=1}^{n} \alpha_j \right| = \sum_{j=1}^{n} \alpha_j = \sum_{j \in I_n^+(\alpha)} |\alpha_j| - \sum_{j \in I_n^-(\alpha)} |\alpha_j|. \tag{3.43}$$

Let $I_n^-(\alpha) = \{k_1, k_2, \dots, k_m\}$, where $1 < k_1 < k_2 < \cdots < k_m < n$. From (3.26) and $|I_k^+(\alpha)| + |I_k^-(\alpha)| = k$ we have

$$\begin{aligned} &\sum_{k=2}^{n} \left(k - \left| \sum_{j=1}^{k} \frac{\alpha_j}{|\alpha_j|} \right| \right) (|\alpha_k| - |\alpha_{k+1}|) \\ &\quad = \sum_{k=2}^{n} \left(k - \left| I_k^+(\alpha) \right| + \left| I_k^-(\alpha) \right| \right) (|\alpha_k| - |\alpha_{k+1}|) \\ &\quad = 2 \sum_{k=2}^{n} \left| I_k^-(\alpha) \right| (|\alpha_k| - |\alpha_{k+1}|) \\ &\quad = 2 \left(\sum_{k=2}^{k_1 - 1} \left| I_k^-(\alpha) \right| (|\alpha_k| - |\alpha_{k+1}|) + \sum_{k=k_1}^{k_2 - 1} \left| I_k^-(\alpha) \right| (|\alpha_k| - |\alpha_{k+1}|) + \cdots + \sum_{k=k_m}^{n} \left| I_k^-(\alpha) \right| (|\alpha_k| - |\alpha_{k+1}|) \right) \\ &\quad = 2 \left(\sum_{k=k_1}^{k_2 - 1} (|\alpha_k| - |\alpha_{k+1}|) + \sum_{k=k_2}^{k_3 - 1} 2(|\alpha_k| - |\alpha_{k+1}|) + \cdots + \sum_{k=k_m}^{n} m(|\alpha_k| - |\alpha_{k+1}|) \right) \\ &\quad = (|\alpha_{k_1}| - |\alpha_{k_2}|) + 2(|\alpha_{k_2}| - |\alpha_{k_3}|) \\ &\qquad + \cdots + (m-1)(|\alpha_{k_{m-1}}| - |\alpha_{k_m}|) + m(|\alpha_{k_m}| - |\alpha_{k_{n+1}}|) \\ &\quad = 2 \sum_{j \in I_n^-(\alpha)} |\alpha_j|. \end{aligned} \tag{3.44}$$

Thus we obtain (3.24). This completes the proof. □

In what follows, we characterize the equality condition of Inequality (1.3) in Theorem 1.2. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ and positive integer m with $2 \le m \le n-1$ we define $J_m = \{n-(m-1), \ldots, n-1, n\}$, $J_m^+(\alpha) = \{j \in J_m : \alpha_j > 0\}$, and $J_m^-(\alpha) = \{j \in J_m : \alpha_j < 0\}$.

Lemma 3.8. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *with* $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_{n-1}| > \alpha_n = 1$. *If*

$$|J_m^+(\alpha)| \ge \left|J_m^-(\alpha)\right| \tag{3.45}$$

for all positive integers m *with* $2 \le m \le n-1$, *then one has*

$$|\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \sum_{j=2}^{n} \alpha_j \ge 0. \tag{3.46}$$

Proof. Let $J_{n-1}^+(\alpha) = \{m_1, m_2, \ldots, m_\ell, m_{\ell+1}, \ldots, m_k\}$ and $J_{n-1}^-(\alpha) = \{n_1, n_2, \ldots, n_\ell\}$, where $n = m_1 > \cdots > m_k \ge 2$ and $n > n_1 > \cdots > n_\ell \ge 2$. As in the proof of Lemma 3.6, we have $m_j > n_j$ for all j. So $|\alpha_{m_j}| < |\alpha_{n_j}|$ for all j. Hence

$$\begin{aligned}
&|\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \sum_{j=2}^{n} \alpha_j \\
&\quad = |\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \left(\sum_{j \in J_{n-1}^+(\alpha)} \alpha_j + \sum_{j \in J_{n-1}^-(\alpha)} \alpha_j \right) \\
&\quad = |\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \left(\sum_{j \in J_{n-1}^+(\alpha)} \left|\alpha_j\right| - \sum_{j \in J_{n-1}^-(\alpha)} \left|\alpha_j\right| \right) \\
&\quad = \sum_{j \in J_{n-1}^+(\alpha)} \left(|\alpha_1| - \left|\alpha_j\right|\right) - \sum_{j \in J_{n-1}^-(\alpha)} \left(|\alpha_1| - \left|\alpha_j\right|\right) \\
&\quad = \sum_{j=1}^{k} \left(|\alpha_1| - \left|\alpha_{m_j}\right|\right) - \sum_{j=1}^{\ell} \left(|\alpha_1| - \left|\alpha_{n_j}\right|\right) \\
&\quad = \sum_{j=\ell+1}^{k} \left(|\alpha_1| - \left|\alpha_{m_j}\right|\right) + \sum_{j=1}^{\ell} \left(\left|\alpha_{n_j}\right| - \left|\alpha_{m_j}\right|\right) \\
&\quad \ge 0.
\end{aligned} \tag{3.47}$$

This completes the proof. □

Theorem 3.9. *Let X be a strictly convex Banach space and* $x_1, x_2, \ldots, x_n$ *nonzero elements in X with* $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$. *Then*

$$\sum_{j=1}^{n} \|x_j\| = \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{k=2}^{n} \left(k - \left\| \sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \left(\|x_{n-k}\| - \left\|x_{n-(k-1)}\right\| \right) \tag{3.48}$$

holds if and only if there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *with* $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_{n-1}| > \alpha_n = 1$, $x_j = \alpha_j x_n$ *for all positive integers* j *with* $1 \leq j \leq n$ *satisfying*

$$|J_m^+(\alpha)| \geq \left|J_m^-(\alpha)\right| \tag{3.49}$$

for all positive integers m *with* $2 \leq m \leq n-1$ *and* $\sum_{j=1}^n \alpha_j \geq 0$.

Proof. ($\Rightarrow$): Let $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$ and

$$\sum_{j=1}^n \|x_j\| = \left\|\sum_{j=1}^n x_j\right\| - \sum_{k=2}^n \left(k - \left\|\sum_{j=n-(k-1)}^n \frac{x_j}{\|x_j\|}\right\|\right)\left(\|x_{n-k}\| - \|x_{n-(k-1)}\|\right). \tag{3.50}$$

For positive integers j with $2 \leq j \leq n$ we put

$$v_j = \left(\frac{\|x_1\|}{\|x_j\|} - 1\right) x_j. \tag{3.51}$$

Note that $0 < \|v_2\| < \cdots < \|v_n\|$ and

$$\sum_{j=1}^n x_j = \|x_1\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} - \sum_{j=2}^n v_j. \tag{3.52}$$

Then Equality (3.48) holds if and only if

$$\left\|\sum_{j=1}^n x_j + \sum_{j=2}^n v_j\right\| = \left\|\sum_{j=1}^n x_j\right\| + \left\|\sum_{j=2}^n v_j\right\|, \tag{3.53}$$

$$\begin{aligned} &\left\|\sum_{j=2}^n v_j\right\| + \left(n - 1 - \left\|\sum_{j=2}^n \frac{v_j}{\|v_j\|}\right\|\right)\|v_2\| \\ &\quad + \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=n-(k-1)}^n \frac{v_j}{\|v_j\|}\right\|\right)\left(\|v_{n-(k-1)}\| - \|v_{n-k}\|\right) = \sum_{j=2}^n \|v_j\|. \end{aligned} \tag{3.54}$$

Thus, by the equality condition of sharp triangle inequality with $n-1$ elements, there exists $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ such that

$$v_j = \beta_j v_n \tag{3.55}$$

for all positive integers j with $2 \leq j \leq n$,

$$\begin{gathered} 0 < |\beta_2| < |\beta_3| < \cdots < |\beta_{n-1}| < \beta_n = 1, \\ |J_m^+(\beta)| \geq |J_m^-(\beta)| \end{gathered} \tag{3.56}$$

for all positive integers m with $2 \leq m \leq n-1$. Since

$$\left(\frac{\|x_1\|}{\|x_j\|}-1\right)x_j = \beta_j\left(\frac{\|x_1\|}{\|x_n\|}-1\right)x_n \tag{3.57}$$

for each j with $2 \leq j \leq n$, we have $x_j = \alpha_j x_n$, where

$$\alpha_j = \beta_j \frac{\|x_1\| - \|x_n\|}{\|x_1\| - \|x_j\|} \cdot \frac{\|x_j\|}{\|x_n\|}. \tag{3.58}$$

Note that $|J_m^+(\alpha)| = |J_m^+(\beta)|$ and $|J_m^+(\alpha)| = |J_m^-(\beta)|$. By (3.53),

$$\left\|\sum_{j=1}^n x_j\right\| \sum_{j=2}^n v_j = \left\|\sum_{j=2}^n v_j\right\| \sum_{j=1}^n x_j. \tag{3.59}$$

Hence there exists $\alpha_1 \in \mathbb{R}$ such that $x_1 = \alpha_1 x_n$. We also have $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_{n-1}| > \alpha_n = 1$ and

$$\begin{aligned}
\sum_{j=2}^n v_j &= \sum_{j=2}^n \left(\frac{\|x_1\|}{\|x_j\|}-1\right)x_j \\
&= \left(|\alpha_1|\sum_{j=2}^n \frac{\alpha_j}{|\alpha_j|} - \sum_{j=2}^n \alpha_j\right)x_n \\
&= \left(|\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \sum_{j=2}^n \alpha_j\right)x_n.
\end{aligned} \tag{3.60}$$

Hence we have

$$\begin{aligned}
&\left|\sum_{j=1}^n \alpha_j\right| \left(|\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \sum_{j=2}^n \alpha_j\right) \\
&\quad = \left||\alpha_1|\left(\left|J_{n-1}^+(\alpha)\right| - \left|J_{n-1}^-(\alpha)\right|\right) - \sum_{j=2}^n \alpha_j\right| \sum_{j=1}^n \alpha_j.
\end{aligned} \tag{3.61}$$

From (3.3), we obtain $\sum_{j=1}^n \alpha_j \geq 0$. Thus we have $(\Rightarrow)$.

$(\Leftarrow)$: Assume that there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_{n-1}| > \alpha_n = 1$, $x_j = \alpha_j x_n$ for all positive integers j with $1 \leq j \leq n$ satisfying (3.49) for all positive integers m with $2 \le m \le n-1$ and $\sum_{j=1}^{n} \alpha_j \ge 0$. By Theorem 3.7, we have (3.54). From the assumption,

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \alpha_j x_n, \quad \sum_{j=1}^{n} v_j = \left(|\alpha_1| \left(\left| J_{n-1}^{+}(\alpha) \right| - \left| J_{n-1}^{-}(\alpha) \right| \right) - \sum_{j=2}^{n} \alpha_j \right) x_n. \tag{3.62}$$

By $\sum_{j=1}^{n} \alpha_j \ge 0$ and Lemma 3.8, we obtain (3.53). Thus we have ($\Leftarrow$). This completes the proof. □

If $n = 2$, then we have the following corollary.

Corollary 3.10. *Let X be a strictly convex Banach space and x, y nonzero elements in X with $\|x\| > \|y\|$. Then*

$$\|x\| + \|y\| = \|x + y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| \tag{3.63}$$

if and only if there exists a real number α with $\alpha > 1$ such that $x = \alpha y$.

If $n = 3$, then we have the following corollary.

Corollary 3.11. *Let X be a strictly convex Banach space and x, y, z nonzero elements in X with $\|x\| > \|y\| > \|z\|$. Then*

$$\begin{aligned} \|x\| + \|y\| + \|z\| = \|x + y + z\| &+ \left(3 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \right) \|x\| \\ &- \left(2 - \left\| \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \right) (\|x\| - \|y\|) \end{aligned} \tag{3.64}$$

if and only if there exist α, β with $|\alpha| > |\beta| > 1$ such that $x = \alpha z, y = \beta z$ and $\alpha + \beta + 1 \ge 0$.

4. Remark

In this section we consider equality attainedness for sharp triangle inequality in a more general case, that is, the case without the assumption that $\|x_1\| > \|x_2\| > \cdots > \|x_n\|$. Let us consider the case $n = 3$.

Proposition 4.1. *Let X be a strictly convex Banach space, and x, y, z nonzero elements in X.*

(i) *If $\|x\| = \|y\| = \|z\|$, then the equality*

$$\begin{aligned} &\|x + y + z\| + \left(3 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \right) \|z\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) (\|y\| - \|z\|) \\ &\quad = \|x\| + \|y\| + \|z\| \end{aligned} \tag{4.1}$$

always holds.

(ii) *If* $\|x\| > \|y\| = \|z\|$, *then the equality* (4.1) *holds if and only if there exists a real number* α *satisfying* $\alpha \geq -\|y\|/\|x\|$ *and* $y + z = \alpha x$.

(iii) *If* $\|x\| = \|y\| > \|z\|$, *then the equality* (4.1) *holds if and only if there exists a real number* α *satisfying* $\alpha \geq -\|x\|/\|z\|$ *and* $x + y = \alpha z$.

Proof. (i) Is clear.

(ii) Assume that (4.1) holds. By $\|y\| = \|z\|$, (4.1) implies

$$\|x + y + z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| = \|x\| + \|y\| + \|z\|. \tag{4.2}$$

From Theorem 3.1, we have

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} = \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\frac{x}{\|x\|}. \tag{4.3}$$

Hence $y + z = \alpha x$ for some $\alpha \in \mathbb{R}$. The following

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} = \left(1 + \alpha\frac{\|x\|}{\|y\|}\right)\frac{x}{\|x\|}, \tag{4.4}$$

implies

$$\left|1 + \alpha\frac{\|x\|}{\|y\|}\right| = 1 + \alpha\frac{\|x\|}{\|y\|}. \tag{4.5}$$

Hence $1 + \alpha(\|x\|/\|y\|) \geq 0$ and so $\alpha \geq -\|y\|/\|x\|$. The converse is clear.

(iii) Assume that (4.1) holds. Put $u = (\|x\| - \|z\|)(x/\|x\|)$ and $v = (\|y\| - \|z\|)(y/\|y\|)$. As in the proof of Theorem 3.5, we have

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} = \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\left(\frac{u + v}{\|u + v\|}\right) \tag{4.6}$$

Since $\|x\| = \|y\|$, we have $x + y = \alpha z$ for some $\alpha \in \mathbb{R}$. The following

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} = \left(1 + \frac{\|z\|}{\|x\|}\alpha\right)\frac{z}{\|z\|} \tag{4.7}$$

implies

$$\left|1 + \frac{\|z\|}{\|x\|}\alpha\right| = 1 + \frac{\|z\|}{\|x\|}\alpha. \tag{4.8}$$

Hence $\alpha \geq -\|x\|/\|z\|$. The converse is clear. □

Conjecture 1. *What is the necessary and sufficient condition when Equality* (3.24) *(resp. Equality* (3.48)*) holds for n elements $x_1, x_2, \ldots, x_n$ with $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|$ in Theorem 3.7 (resp. Theorem 3.9)?*

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