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Research Article

Some Comparison Inequalities for Generalized Muirhead and Identric Means

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For $x, y > 0$, $a, b \in \mathbb{R}$, with $a + b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ with parameters a and b and the identric mean $I(x, y)$ are defined by $M(a, b; x, y) = ((x^a y^b + x^b y^a)/2)^{1/(a+b)}$ and $I(x, y) = (1/e)(y^y/x^x)^{1/(y-x)}$, $x \neq y$, $I(x, y) = x$, $x = y$, respectively. In this paper, the following results are established: (1) $M(a, b; x, y) > I(x, y)$ for all $x, y > 0$ with $x \neq y$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a + b > 0,\ ab \leq 0,\ 2(a-b)^2 - 3(a+b) + 1 \geq 0,\ 3(a-b)^2 - 2(a+b) \geq 0\}$; (2) $M(a, b; x, y) < I(x, y)$ for all $x, y > 0$ with $x \neq y$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a \geq 0,\ b \geq 0,\ 3(a-b)^2 - 2(a+b) \leq 0\} \cup \{(a, b) \in \mathbb{R}^2 : a + b < 0\}$; (3) if $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a > 0,\ b > 0,\ 3(a-b)^2 - 2(a+b) > 0\} \cup \{(a, b) \in \mathbb{R}^2 : ab < 0,\ 3(a-b)^2 - 2(a+b) < 0\}$, then there exist $x_1, y_1, x_2, y_2 > 0$ such that $M(a, b; x_1, y_1) > I(x_1, y_1)$ and $M(a, b; x_2, y_2) < I(x_2, y_2)$.

1. Introduction

For $x, y > 0$, $a, b \in \mathbb{R}$, with $a + b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ with parameters a and b and the identric mean $I(x, y)$ are defined by

$$M(a, b; x, y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{1/(a+b)}, \tag{1.1}$$

$$I(x, y) = \begin{cases} \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{1/(y-x)}, & x \neq y, \\ x, & x = y, \end{cases} \tag{1.2}$$

respectively.

The generalized Muirhead mean was introduced by Trif [1], the monotonicity of $M(a,b;x,y)$ with respect to a or b was discussed, and a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $M(a,b;x,y)$ were discussed.

It is easy to see that the generalized Muirhead mean $M(a,b;x,y)$ is continuous on the domain $\{(a,b;x,y) : a+b \neq 0;\ x,y>0\}$ and differentiable with respect to $(x,y) \in (0,+\infty)\times(0,+\infty)$ for fixed $a,b \in \mathbb{R}$ with $a+b\neq 0$. It is symmetric in a and b and in x and y. Many means are special cases of the generalized Muirhead mean, for example,

$$\begin{aligned} M(p,0;x,y) &\quad \text{is the power or Hölder mean,} \\ M(0,1;x,y) &\quad \text{is the arithmetic mean,} \\ M(a,a;x,y) &\quad \text{is the geometric mean,} \\ M(0,-1;x,y) &\quad \text{is the harmonic mean.} \end{aligned} \tag{1.3}$$

The well-known Muirhead inequality [2] implies that if $x,y>0$ are fixed, then $M(a,b;x,y)$ is Schur convex on the domain $\{(a,b)\in\mathbb{R}^2 : a+b>0\}$ and Schur concave on the domain $\{(a,b)\in\mathbb{R}^2 : a+b<0\}$. Chu and Xia [3] discussed the Schur convexity and Schur concavity of $M(a,b;x,y)$ with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for fixed $a,b\in\mathbb{R}$ with $a+b\neq 0$.

Recently, the identric mean $I(x,y)$ has been the subject of intensive research. In particular, many remarkable inequalities for the identric mean $I(x,y)$ can be found in the literature [4–13].

The power mean of order r of the positive real numbers x and y is defined by

$$M_r(x,y) = \begin{cases} \left(\dfrac{x^r+y^r}{2}\right)^{1/r}, & r\neq 0, \\ \sqrt{xy}, & r=0. \end{cases} \tag{1.4}$$

The main properties of the power mean $M_r(x,y)$ are given in [14]. In particular, $M_r(x,y)$ is continuous and increasing with respect to $r\in\mathbb{R}$ for fixed $x,y>0$. Let $A(x,y) = (1/2)(x+y)$,

$$L(x,y) = \begin{cases} \dfrac{y-x}{\log y - \log x}, & x\neq y, \\ x, & x=y, \end{cases} \tag{1.5}$$

$G(x,y)=\sqrt{xy}$, and $H(x,y)=2xy/(x+y)$ be the arithmetic, logarithmic, geometric, and harmonic means of two positive numbers x and y. Then it is well known that

$$\begin{aligned} \min\{x,y\} &< H(x,y) = M(0,-1;x,y) = M_{-1}(x,y) \\ &< G(x,y) = M(a,a;x,y) = M_0(x,y) < L(x,y) < I(x,y) \\ &< A(x,y) = M(0,1;x,y) = M_1(x,y) < \max\{x,y\} \end{aligned} \tag{1.6}$$

for all $x,y>0$ with $x\neq y$.

The following sharp inequality is due to Carlson [15]:

$$L(x,y) < \frac{1}{3}M(0,1;x,y) + \frac{2}{3}M(a,a;x,y) \tag{1.7}$$

for all $x, y > 0$ with $x \neq y$.

Pittenger [16] proved that

$$M\left(\frac{2}{3},0;x,y\right) = M_{2/3}(x,y) < I(x,y) < M_{\log 2}(x,y) = M(\log 2,0;x,y) \tag{1.8}$$

for all $x, y > 0$ with $x \neq y$, and $M_{\log 2}(x,y)$ and $M_{2/3}(x,y)$ are the optimal upper and lower power mean bounds for the identric mean $I(x,y)$.

In [8, 9], Sándor established that

$$I(x,y) > \frac{2}{3}M(0,1;x,y) + \frac{1}{3}M(a,a;x,y) \tag{1.9}$$

for all $x, y > 0$ with $x \neq y$.

Alzer and Qiu [5] proved the inequalities

$$\alpha M(0,1;x,y) + (1-\alpha)M(a,a;x,y) < I(x,y) < \beta M(0,1;x,y) + (1-\beta)M(a,a;x,y) \tag{1.10}$$

for all $x, y > 0$ with $x \neq y$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e$.

In [3], Chu and Xia proved that

$$M(a,b;x,y) \geq A(x,y) \tag{1.11}$$

for all $x, y > 0$ and $(a,b) \in \{(a,b) \in \mathbb{R}^2 : (a-b)^2 \geq a+b >, ab \leq 0\}$, and

$$M(a,b;x,y) \leq A(x,y) \tag{1.12}$$

for all $x, y > 0$ and $(a,b) \in \{(a,b) \in \mathbb{R}^2 : (a-b)^2 \leq a+b, a^2+b^2 \neq 0\} \cup \{(a,b) \in \mathbb{R}^2 : a+b < 0\}$.

Our purpose in what follows is to compare the generalized Muirhead mean $M(a,b;x,y)$ with the identric mean $I(x,y)$. Our main result is Theorem 1.1 which follows.

Theorem 1.1. *Suppose that* $E_1 = \{(a,b) \in \mathbb{R}^2 : a+b > 0, ab \leq 0, 2(a-b)^2 - 3(a+b) + 1 \geq 0, 3(a-b)^2 - 2(a+b) \geq 0\}$, $E_2 = \{(a,b) \in \mathbb{R}^2 : a \geq 0, b \geq 0, a^2+b^2 \neq 0, 3(a-b)^2 - 2(a+b) \leq 0\} \cup \{(a,b) \in \mathbb{R}^2 : a+b < 0\}$, *and* $E_3 = \{(a,b) \in \mathbb{R}^2 : a > 0, b > 0, 3(a-b)^2 - 2(a+b) > 0\} \cup \{(a,b) \in \mathbb{R}^2 : ab < 0, 3(a-b)^2 - 2(a+b) < 0\}$. *The following statements hold,*

(1) *If* $(a,b) \in E_1$, *then* $M(a,b;x,y) > I(x,y)$ *for all* $x, y > 0$ *with* $x \neq y$.

(2) *If* $(a,b) \in E_2$, *then* $M(a,b;x,y) < I(x,y)$ *for all* $x, y > 0$ *with* $x \neq y$.

(3) *If* $(a,b) \in E_3$, *then there exist* $x_1, y_1, x_2, y_2 > 0$ *such that* $M(a,b;x_1,y_1) > I(x_1,y_1)$ *and* $M(a,b;x_2,y_2) < I(x_2,y_2)$.

2. Lemma

In order to prove Theorem 1.1 we need Lemma 2.1 that follows.

Lemma 2.1. *Let a and b be two real numbers such that $a > b$ and $a+b \neq 0$. Let one define the function $f : [1, +\infty) \to \mathbb{R}$ as follows:*

$$f(t) = \frac{1}{a+b}\left[-bt^{a-b+1} + at^{a-b} - at^{b-a+1} + bt^{b-a} + \left(a^2 + b^2 - 2ab - a - b\right)(t-1)\right], \tag{2.1}$$

then the following statements hold.

(1) *If $b > 0$ and $3(a-b)^2 - 2(a+b) \leq 0$, then $f(t) < 0$ for $t > 1$.*

(2) *If $b < 0$, $a+b > 0$, $2(a-b)^2 - 3(a+b) + 1 \geq 0$, and $3(a-b)^2 - 2(a+b) \geq 0$, then $f(t) > 0$ for $t > 1$.*

(3) *If $a + b < 0$, then $f(t) < 0$ for $t > 1$.*

Proof. Simple computations lead to

$$f(1) = 0, \tag{2.2}$$

$$\begin{aligned} f'(t) = \frac{1}{a+b}\big[&-b(a-b+1)t^{a-b} + a(a-b)t^{a-b-1} + a(a-b-1)t^{b-a} \\ &-b(a-b)t^{b-a-1} + a^2 + b^2 - 2ab - a - b\big], \end{aligned} \tag{2.3}$$

$$f'(1) = \frac{3(a-b)^2 - 2(a+b)}{a+b}, \tag{2.4}$$

$$f''(t) = (a-b)t^{b-a-2} f_1(t), \tag{2.5}$$

where

$$\begin{aligned} f_1(t) = \frac{1}{a+b}\big[&-b(a-b+1)t^{2a-2b+1} + a(a-b-1)t^{2a-2b} \\ &-a(a-b-1)t + b(a-b+1)\big], \end{aligned} \tag{2.6}$$

$$f_1(1) = 0, \tag{2.7}$$

$$\begin{aligned} f_1'(t) = \frac{1}{a+b}\big[&-b(a-b+1)(2a-2b+1)t^{2a-2b} \\ &+2a(a-b)(a-b-1)t^{2a-2b-1} - a(a-b-1)\big], \end{aligned} \tag{2.8}$$

$$f_1'(1) = \frac{a-b}{a+b}\left[2(a-b)^2 - 3(a+b) + 1\right], \tag{2.9}$$

$$f_1''(t) = 2(a-b)t^{2a-2b-2} f_2(t), \tag{2.10}$$

where

$$f_2(t) = \frac{1}{a+b}[-b(a-b+1)(2a-2b+1)t + a(a-b-1)(2a-2b-1)], \tag{2.11}$$

$$f_2(1) = \frac{a-b}{a+b}\left[2(a-b)^2 - 3(a+b) + 1\right], \tag{2.12}$$

$$f_2'(t) = -\frac{b(a-b+1)(2a-2b+1)}{a+b}. \tag{2.13}$$

(1) We divide the proof of Lemma 2.1(1) into two cases.

Case 1. $b > 0$, $3(a-b)^2 - 2(a+b) \le 0$, and $2(a-b)^2 - 3(a+b) + 1 \le 0$. From (2.13), (2.12), (2.9), and (2.4), we clearly see that

$$\begin{aligned} f_2'(t) &< 0, \qquad f_2(1) \le 0, \\ f_1'(1) &\le 0, \qquad f'(1) \le 0. \end{aligned} \tag{2.14}$$

Therefore, $f(t) < 0$ for $t \in (1, +\infty)$ easily follows from (2.2), (2.5), (2.7), (2.10), and (2.14).

Case 2. $b > 0$, $3(a-b)^2 - 2(a+b) \le 0$, and $2(a-b)^2 - 3(a+b) + 1 > 0$; we conclude that

$$a < \frac{1}{2}. \tag{2.15}$$

In fact, we clearly see that $2(a-b)^2 - 3(a+b) + 1 = (2a^2 - 3a + 1) - (4ab - 2b^2 + 3b) < 2a^2 - 3a + 1 = (2a-1)(a-1) \le 0$ for $1/2 \le a < 1$, and $2(a-b)^2 - 3(a+b) + 1 \le -(5/3)(a+b) + 1 < -2/3 < 0$ for $a \ge 1$ and $3(a-b)^2 - 2(a+b) \le 0$.

Equation (2.15) and $3(a-b)^2 - 2(a+b) \le 0$ imply that

$$\begin{aligned} &2a - 2b - 1 < 0, \\ &a^2 + b^2 - 2ab - a - b = (a-b)^2 - (a+b) < 0. \end{aligned} \tag{2.16}$$

Therefore, $f(t) < 0$ for $t > 1$ follows from (2.16) together with that $f(t)$ can be rewritten as

$$\begin{aligned} f(t) = \frac{1}{a+b}\Big[&a t^{b-a+1}\left(t^{2a-2b-1} - 1\right) - b t^{b-a}\left(t^{2a-2b+1} - 1\right) \\ &+ \left(a^2 + b^2 - 2ab - a - b\right)(t-1)\Big]. \end{aligned} \tag{2.17}$$

(2) If $b < 0$, $a + b > 0$, $2(a - b)^2 - 3(a + b) + 1 \geq 0$ and $3(a - b)^2 - 2(a + b) \geq 0$, then from (2.13), (2.12), (2.9), and (2.4) we get

$$f_2'(t) > 0, \qquad f_2(1) \geq 0,$$
$$f_1'(1) \geq 0, \qquad f'(1) \geq 0. \tag{2.18}$$

Therefore, $f(t) > 0$ for $t \in (1, +\infty)$ easily follows from (2.2), (2.5), (2.7), and (2.10) together with (2.18).

(3) If $a + b < 0$, then we clearly see that inequalities (2.14) again hold, and $f(t) < 0$ for $t > 1$ follows from (2.2), (2.5), (2.7), and (2.10) together with (2.14). □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For convenience, we introduce the following classified regions in $\mathbb{R}^2$:

$$E_{11} = \left\{(a,b) \in \mathbb{R}^2 : a + b > 0, a > 0, b < 0, 2(a-b)^2 - 3(a+b) + 1 \geq 0, 3(a-b)^2 - 2(a+b) \geq 0\right\},$$
$$E_{12} = \left\{(a,b) \in \mathbb{R}^2 : a + b > 0, a < 0, b > 0, 2(a-b)^2 - 3(a+b) + 1 \geq 0, 3(a-b)^2 - 2(a+b) \geq 0\right\},$$
$$E_{13} = \left\{(a,b) \in \mathbb{R}^2 : a = 0, b \geq 1\right\},$$
$$E_{14} = \left\{(a,b) \in \mathbb{R}^2 : b = 0, a \geq 1\right\},$$
$$E_{21} = \left\{(a,b) \in \mathbb{R}^2 : a > b > 0, 3(a-b)^2 - 2(a+b) \leq 0\right\},$$
$$E_{22} = \left\{(a,b) \in \mathbb{R}^2 : b > a > 0, 3(a-b)^2 - 2(a+b) \leq 0\right\},$$
$$E_{23} = \left\{(a,b) \in \mathbb{R}^2 : a = 0, 0 < b \leq \frac{2}{3}\right\},$$
$$E_{24} = \left\{(a,b) \in \mathbb{R}^2 : b = 0, 0 < a \leq \frac{2}{3}\right\},$$
$$E_{25} = \left\{(a,b) \in \mathbb{R}^2 : a > b, a + b < 0\right\},$$
$$E_{26} = \left\{(a,b) \in \mathbb{R}^2 : b > a, a + b < 0\right\},$$
$$E_{27} = \left\{(a,b) \in \mathbb{R}^2 : a = b \neq 0\right\},$$

$$E_{31} = \left\{(a,b) \in \mathbb{R}^2 : a > b > 0, 3(a-b)^2 - 2(a+b) > 0\right\},$$
$$E_{32} = \left\{(a,b) \in \mathbb{R}^2 : b > a > 0, 3(a-b)^2 - 2(a+b) > 0\right\},$$
$$E_{33} = \left\{(a,b) \in \mathbb{R}^2 : a > 0, b < 0, 3(a-b)^2 - 2(a+b) < 0\right\},$$
$$E_{34} = \left\{(a,b) \in \mathbb{R}^2 : a < 0, b > 0, 3(a-b)^2 - 2(a+b) < 0\right\}. \tag{3.1}$$

Then we clearly see that $E_1 = \bigcup_{i=1}^{4} E_{1i}$, $E_2 = \bigcup_{i=1}^{7} E_{2i}$, and $E_3 = \bigcup_{i=1}^{4} E_{3i}$.

Without loss of generality, we assume that $y > x$. From the symmetry we clearly see that Theorem 1.1 is true if we prove that $M(a,b;x,y) - I(x,y)$ is positive, negative, and neither positive nor negative with respect to $(x,y) \in \{(x,y) \in \mathbb{R}^2 : y > x > 0\}$ for $(a,b) \in E_{11} \cup E_{13}$, $E_{21} \cup E_{23} \cup E_{25} \cup E_{27}$, and $E_{31} \cup E_{33}$.

Let $t = y/x > 1$, then (1.1) and (1.2) lead to

$$\log M(a,b;x,y) - \log I(x,y) = \frac{1}{a+b} \log \frac{t^a + t^b}{2} - \frac{t}{t-1} \log t + 1. \tag{3.2}$$

Let

$$g(t) = \frac{1}{a+b} \log \frac{t^a + t^b}{2} - \frac{t}{t-1} \log t + 1. \tag{3.3}$$

Then simple computations yield

$$\lim_{t \to 1} g(t) = 0,$$
$$g'(t) = \frac{g_1(t)}{(t-1)^2}, \tag{3.4}$$

where

$$g_1(t) = \log t - \frac{(t-1)\left(bt^{b-1} + at^{a-1} + at^b + bt^a\right)}{(a+b)\left(t^a + t^b\right)}. \tag{3.5}$$

Note that

$$g_1(1) = 0, \tag{3.6}$$

$$g_1'(t) = \frac{(t-1)t^{a+b-2}}{\left(t^a + t^b\right)^2} f(t), \tag{3.7}$$

where $f(t)$ is defined as in Lemma 2.1.

We divide the proof into three cases.

Case 3. $(a,b) \in E_{11} \cup E_{13}$. We divide our discussion into two subcases.

Subcase 1. $(a,b) \in E_{11}$. From Lemma 2.1(2) we get

$$f(t) > 0 \tag{3.8}$$

for $t > 1$.

Equations (3.3)–(3.8) imply that

$$g(t) > 0 \tag{3.9}$$

for $t > 1$.

Therefore, $M(a,b;x,y) > I(x,y)$ follows from (3.2) and (3.9).

Subcase 2. $(a,b) \in E_{13}$. Then from (1.1), (1.4), and (1.6) together with the monotonicity of the power mean $M_r(x,y)$ with respect to $r \in \mathbb{R}$ for fixed $x,y > 0$, we get

$$M(a,b;x,y) = M(0,b;x,y) = M_b(x,y) \geq M_1(x,y) > I(x,y). \tag{3.10}$$

Case 4. $(a,b) \in E_{21} \cup E_{23} \cup E_{25} \cup E_{27}$. We divide our discussion into four subcases.

Subcase 3. $(a,b) \in E_{21}$. Then Lemma 2.1(1) leads to

$$f(t) < 0 \tag{3.11}$$

for $t > 1$.

Therefore, $M(a,b;x,y) < I(x,y)$ follows from (3.2)–(3.7) and (3.11).

Subcase 4. $(a,b) \in E_{23}$. Then from (1.1), (1.4), and (1.8) together with the monotonicity of the power mean $M_r(x,y)$ with respect to $r \in \mathbb{R}$ for fixed $x,y > 0$ we clearly see that

$$M(a,b;x,y) = M_b(x,y) \leq M_{2/3}(x,y) < I(x,y). \tag{3.12}$$

Subcase 5. $(a,b) \in E_{25}$. Then from Lemma 2.1(3) we know that (3.11) holds again; hence, $M(a,b;x,y) < I(x,y)$.

Subcase 6. $(a,b) \in E_{27}$. Then (1.6) leads to

$$M(a,b;x,y) = M(a,a;x,y) = G(x,y) < I(x,y). \tag{3.13}$$

Case 5. $(a,b) \in E_{31} \cup E_{33}$. We divide our discussion into two subcases.

Subcase 7. $(a,b) \in E_{31}$. Then (2.4) leads to

$$f'(1) > 0. \tag{3.14}$$

Inequality (3.14) and the continuity of $f'(t)$ imply that there exists $\delta_1 > 0$ such that

$$f'(t) > 0 \tag{3.15}$$

for $t \in [1, 1+\delta_1)$.

From (2.2) and (3.15) we clearly see that

$$f(t) > 0 \tag{3.16}$$

for $t \in (1, 1+\delta_1)$.

Therefore, $M(a,b;x,y) > I(x,y)$ for $(x,y) \in \{(x,y) \in \mathbb{R}^2 : y > x > 0, y < (1+\delta_1)x\}$ follows from (3.2)–(3.7) and (3.16).

On the other hand, from (3.3) we clearly see that

$$\lim_{t \to +\infty} g(t) = -\infty. \tag{3.17}$$

Equations (3.2) and (3.3) together with (3.17) imply that there exists sufficient large $\lambda_1 > 1$ such that $M(a,b;x,y) < I(x,y)$ for $(x,y) \in \{(x,y) \in \mathbb{R}^2 : y > \lambda_1 x > 0\}$.

Subcase 8. $(a,b) \in E_{33}$. Then (2.2) and (2.4) together with the continuity of $f'(t)$ imply that there exists $\delta_2 > 0$ such that

$$f(t) < 0 \tag{3.18}$$

for $t \in (1, 1+\delta_2)$.

Therefore, $M(a,b;x,y) < I(x,y)$ for $(x,y) \in \{(x,y) \in \mathbb{R}^2 : y > x > 0, y < (1+\delta_2)x\}$ follows from (3.2)–(3.7) and (3.18).

On the other hand, from (3.3) we clearly see that

$$\lim_{t \to +\infty} g(t) = +\infty. \tag{3.19}$$

Equations (3.2) and (3.3) together with (3.19) imply that there exists sufficient large $\lambda_2 > 1$ such that $M(a,b;x,y) > I(x,y)$ for $(x,y) \in \{(x,y) \in \mathbb{R}^2 : y > \lambda_2 x > 0\}$. □

Remark 3.1. Let $E_4 = \{(a,b) \in \mathbb{R}^2 : a+b \neq 0\} \setminus (E_1 \cup E_2 \cup E_3)$, then $E_4 = \{(a,b) \in \mathbb{R}^2 : ab < 0, 3(a-b)^2 - 2(a+b) > 0, 2(a-b)^2 - 3(a+b) + 1 < 0\}$. Unfortunately, in this paper we cannot discuss the case of $(a,b) \in E_4$; we leave it as an open problem to the readers.

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