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Research Article

Differences of Weighted Mixed Symmetric Means and Related Results

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Some improvements of classical Jensen's inequality are used to define the weighted mixed symmetric means. Exponential convexity and mean value theorems are proved for the differences of these improved inequalities. Related Cauchy means are also defined, and their monotonicity is established as an application.

1. Introduction and Preliminary Results

For $n \in \mathbb{N}$, let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. We define power means of order $r \in \mathbb{R}$, as follows:

$$M_r(\mathbf{x}, \mathbf{p}) = M_r(x_1, \ldots, x_n; p_1, \ldots, p_n) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} p_i x_i^r \right)^{1/r}, & r \neq 0, \\ \left(\Pi_{i=1}^{n} x_i^{p_i} \right), & r = 0. \end{cases} \tag{1.1}$$

We introduce the mixed symmetric means with positive weights as follows:

$$M_{s,t}^{1}(\mathbf{x},\mathbf{p};k) = \begin{cases} \left(\dfrac{1}{C_{k-1}^{n-1}} \displaystyle\sum_{1\le i_1<\cdots<i_k\le n} \left(\sum_{j=1}^{k} p_{i_j}\right) M_t^s\left(x_{i_1},\dots x_{i_k}; p_{i_1},\dots p_{i_k}\right)\right)^{1/s}, & s \neq 0, \\ \left(\Pi_{1\le i_1<\cdots<i_k\le n}\left(M_t\left(x_{i_1},\dots,x_{i_k}; p_{i_1},\dots,p_{i_k}\right)\right)^{\left(\sum_{j=1}^{k} p_{i_j}\right)}\right)^{1/C_{k-1}^{n-1}}, & s = 0. \end{cases} \tag{1.2}$$

We obtain the monotonicity of these means as a consequence of the following improvement of Jensen's inequality [1].

Theorem 1.1. *Let $I \subseteq \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_n) \in I^n$, $\mathbf{p} = (p_1,\dots,p_n)$ be a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$. Also let $f : I \to \mathbb{R}$ be a convex function and*

$$f_{k,n}^{1}(\mathbf{x},\mathbf{p}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1\le i_1<\cdots<i_k\le n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right), \tag{1.3}$$

then

$$f_{k+1,n}^{1}(\mathbf{x},\mathbf{p}) \le f_{k,n}^{1}(\mathbf{x},\mathbf{p}), \quad k = 1,2,\dots,n-1, \tag{1.4}$$

that is

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = f_{n,n}^{1}(\mathbf{x},\mathbf{p}) \le \cdots \le f_{k,n}^{1}(\mathbf{x},\mathbf{p}) \le \cdots \le f_{1,n}^{1}(\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i). \tag{1.5}$$

If f is a concave function, then the inequality (1.4) *is reversed.*

Corollary 1.2. *Let $s,t \in \mathbb{R}$ such that $s \le t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$, then, we have*

$$M_t^1 = M_{t,s}^{1}(\mathbf{x},\mathbf{p};1) \ge \cdots \ge M_{t,s}^{1}(\mathbf{x},\mathbf{p};k) \ge \cdots \ge M_{t,s}^{1}(\mathbf{x},\mathbf{p};n) = M_s^1, \tag{1.6}$$

$$M_s^1 = M_{s,t}^{1}(\mathbf{x},\mathbf{p};1) \le \cdots \le M_{s,t}^{1}(\mathbf{x},\mathbf{p};k) \le \cdots \le M_{s,t}^{1}(\mathbf{x},\mathbf{p};n) = M_t^1. \tag{1.7}$$

Proof. Let $s,t \in \mathbb{R}$ such that $s \le t$, if $s,t \neq 0$, then we set $f(x) = x^{t/s}$, $x_{i_j} = x_{i_j}^{s}$ in (1.4) and raising the power $1/t$, we get (1.6). Similarly we set $f(x) = x^{s/t}$, $x_{i_j} = x_{i_j}^{t}$ in (1.4) and raising the power $1/s$, we get (1.7).

When $s = 0$ or $t = 0$, we get the required results by taking limit. □

Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{x}$, $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Also let $h, g : I \to \mathbb{R}$ be continuous and strictly monotonic functions. We define the quasiarithmetic means with respect to (1.3) as follows:

$$M_{h,g}^{1}(\mathbf{x},\mathbf{p};\mathbf{k}) = h^{-1}\left(\frac{1}{C_{k-1}^{n-1}}\sum_{1\le i_1<\cdots<i_k\le n}\left(\sum_{j=1}^{k}p_{i_j}\right)h\circ g^{-1}\left(\frac{\sum_{j=1}^{k}p_{i_j}g\left(x_{i_j}\right)}{\sum_{j=1}^{k}p_{i_j}}\right)\right), \tag{1.8}$$

where $h \circ g^{-1}$ is the convex function.

We obtain generalized means by setting $f = h \circ g^{-1}$, $x_{i_j} = g(x_{i_j})$ and applying h^{-1} to (1.3).

Corollary 1.3. *By similar setting in* (1.4)*, one gets the monotonicity of generalized means as follows:*

$$M_{h}^{1}(\mathbf{x},\mathbf{p}) = M_{h,g}^{1}(\mathbf{x},\mathbf{p};1) \ge \cdots \ge M_{h,g}^{1}(\mathbf{x},\mathbf{p};k) \ge \cdots \ge M_{h,g}^{1}(\mathbf{x},\mathbf{p};n) = M_{g}^{1}(\mathbf{x},\mathbf{p}), \tag{1.9}$$

where $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M_{g}^{1}(\mathbf{x},\mathbf{p}) = M_{g,h}^{1}(\mathbf{x},\mathbf{p};1) \le \cdots \le M_{g,h}^{1}(\mathbf{x},\mathbf{p};k) \le \cdots \le M_{g,h}^{1}(\mathbf{x},\mathbf{p};n) = M_{h}^{1}(\mathbf{x},\mathbf{p}), \tag{1.10}$$

where $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

Remark 1.4. In fact Corollaries 1.2 and 1.3 are weighted versions of results in [2].

The inequality of Popoviciu as given by Vasić and Stanković in [3] (see also [4, page 173]) can be written in the following form:

Theorem 1.5. *Let the conditions of Theorem 1.1 be satisfied for $k \in \mathbb{N}$, $2 \le k \le n-1$, $n \ge 3$. Then*

$$f_{k,n}^{1}(\mathbf{x},\mathbf{p}) \le \frac{n-k}{n-1}f_{1,n}^{1}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}f_{n,n}^{1}(\mathbf{x},\mathbf{p}), \tag{1.11}$$

where $f_{k,n}^{1}(\mathbf{x},\mathbf{p})$ is given by (1.3) *for convex function f.*

By inequality (1.11), we write

$$\Omega^{4}(\mathbf{x},\mathbf{p};f) = \frac{n-k}{n-1}f_{1,n}^{1}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}f_{n,n}^{1}(\mathbf{x},\mathbf{p}) - f_{k,n}^{1}(\mathbf{x},\mathbf{p}) \ge 0. \tag{1.12}$$

Corollary 1.6. *Let $s, t \in \mathbb{R}$ such that $s \le t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Then, we have*

$$M_{t,s}^{t}(\mathbf{x},\mathbf{p};k) \le \frac{n-k}{n-1}M_{t}^{t}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}M_{s}^{t}(\mathbf{x},\mathbf{p}), \tag{1.13}$$

$$M_{s,t}^{s}(\mathbf{x},\mathbf{p};k) \ge \frac{n-k}{n-1}M_{s}^{s}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1}M_{t}^{s}(\mathbf{x},\mathbf{p}). \tag{1.14}$$

Proof. Let $s, t \in \mathbb{R}$ such that $s \leq t$, if $s, t \neq 0$, then we set $f(x) = x^{t/s}$, $x_{i_j} = x_{i_j}^s$ in (1.11) to obtain (1.13) and we set $f(x) = x^{s/t}$, $x_{i_j} = x_{i_j}^t$ in (1.11) to obtain (1.14).

When $s = 0$ or $t = 0$, we get the required results by taking limit. □

Corollary 1.7. *We set $x_{i_j} = g(x_{i_j})$ and the convex function $f = h \circ g^{-1}$ in* (1.11) *to get*

$$h\left(M_{h,g}(\mathbf{x}, \mathbf{p}; k)\right) \leq \frac{n-k}{n-1} h\left(M_h(\mathbf{x}, \mathbf{p})\right) + \frac{k-1}{n-1} h\left(M_g(\mathbf{x}, \mathbf{p})\right). \tag{1.15}$$

The following result is valid [5, page 8].

Theorem 1.8. *Let f be a convex function defined on an interval $I \subseteq \mathbb{R}$, $\mathbf{x}$, $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^n p_i = 1$ and $x_1, \ldots, x_n \in I$. Then*

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \cdots \leq f_{k+1,n}^2(\mathbf{x}, \mathbf{p}) \leq f_{k,n}^2(\mathbf{x}, \mathbf{p}) \leq \cdots \leq f_{1,n}^2(\mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i), \tag{1.16}$$

where

$$f_{k,n}^2(\mathbf{x}, \mathbf{p}; \mathbf{k}) = \frac{1}{C_{k-1}^{n+k-1}} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left(\sum_{j=1}^k p_{i_j}\right) f\left(\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}}\right). \tag{1.17}$$

If f is a concave function then the inequality (1.16) *is reversed.*

We introduce the mixed symmetric means with positive weights related to (1.17) as follows:

$$M_{s,t}^2(\mathbf{x}, \mathbf{p}; k) = \begin{cases} \left(\dfrac{1}{C_{k-1}^{n+k-1}} \displaystyle\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left(\sum_{j=1}^k p_{i_j}\right) M_t^s(x_{i_1}, \ldots, x_{i_k}; p_{i_1}, \ldots p_{i_k})\right)^{1/s}, & s \neq 0; \\ \left(\Pi_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left(M_t(x_{i_1}, \ldots, x_{i_k}; p_{i_1}, \ldots p_{i_k})\right)^{(\sum_{j=1}^k p_{i_j})}\right)^{1/C_{k-1}^{n+k-1}}, & s = 0. \end{cases} \tag{1.18}$$

Corollary 1.9. *Let $s, t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Then, we have*

$$M_t^2 = M_{t,s}^2(\mathbf{x}, \mathbf{p}; 1) \geq \cdots \geq M_{t,s}^2(\mathbf{x}, \mathbf{p}; k) \geq \cdots \geq M_s^2, \tag{1.19}$$

$$M_s^2 = M_{s,t}^2(\mathbf{x}, \mathbf{p}; 1) \leq \cdots \leq M_{s,t}^2(\mathbf{x}, \mathbf{p}; k) \leq \cdots \leq M_t^2. \tag{1.20}$$

Proof. Let $s, t \in \mathbb{R}$ such that $s \leq t$, if $s, t \neq 0$, then we set $f(x) = x^{t/s}$, $x_{i_j} = x_{i_j}^s$ in (1.16) and raising the power $1/t$, we get (1.19). Similarly we set $f(x) = x^{s/t}$, $x_{i_j} = x_{i_j}^t$ in (1.16) and raising the power $1/s$, we get (1.20).

When $s = 0$ or $t = 0$, we get the required results by taking limit. □

We define the quasiarithmetic means with respect to (1.17) as follows:

$$M_{h,g}^{2}(\mathbf{x},\mathbf{p};\mathbf{k}) = h^{-1}\left(\frac{1}{C_{k-1}^{n+k-1}}\sum_{1\le i_1\le\cdots\le i_k\le n}\left(\sum_{j=1}^{k}p_{i_j}\right)h\circ g^{-1}\left(\frac{\sum_{j=1}^{k}p_{i_j}g\left(x_{i_j}\right)}{\sum_{j=1}^{k}p_{i_j}}\right)\right), \tag{1.21}$$

where $h\circ g^{-1}$ is the convex function.

We obtain these generalized means by setting $f = h\circ g^{-1}$, $x_{i_j} = g(x_{i_j})$ and applying h^{-1} to (1.17).

Corollary 1.10. *By similar setting in* (1.16), *we get the monotonicity of these generalized means as follows:*

$$M_{h}^{2}(\mathbf{x},\mathbf{p}) = M_{h,g}^{2}(\mathbf{x},\mathbf{p};1) \ge \cdots \ge M_{h,g}^{2}(\mathbf{x},\mathbf{p};k) \ge \cdots \ge M_{g}^{2}(\mathbf{x},\mathbf{p}), \tag{1.22}$$

where $f = h\circ g^{-1}$ is convex and h is increasing, or $f = h\circ g^{-1}$ is concave and h is decreasing;

$$M_{g}^{2}(\mathbf{x},\mathbf{p}) = M_{g,h}^{2}(\mathbf{x},\mathbf{p};1) \le \cdots \le M_{g,h}^{2}(\mathbf{x},\mathbf{p};k) \le \cdots \le M_{h}^{2}(\mathbf{x},\mathbf{p}), \tag{1.23}$$

where $f = g\circ h^{-1}$ is convex and g is decreasing, or $f = g\circ h^{-1}$ is concave and g is increasing.

The following result is given in [4, page 90].

Theorem 1.11. *Let M be a real linear space, U a non empty convex set in M, $f : U \to \mathbb{R}$ a convex function, and also let $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$ and $x_1,\ldots,x_n \in U$. Then*

$$f\left(\sum_{i=1}^{n}p_i x_i\right) \le \cdots \le f_{k,n}^{3}(\mathbf{x},\mathbf{p}) \le \cdots \le f_{1,n}^{3}(\mathbf{x},\mathbf{p}), \tag{1.24}$$

where $1\le k\le n$ and for $I = \{1,\ldots,n\}$,

$$f_{k,n}^{3}(\mathbf{x},\mathbf{p}) = \sum_{i_1,\ldots,i_k\in I} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k}x_{i_j}\right). \tag{1.25}$$

The mixed symmetric means with positive weights related to (1.25) are

$$M^3_{s,t}(\mathbf{x},\mathbf{p};k) = \begin{cases} \left(\displaystyle\sum_{i_1,\ldots,i_k \in I} \left(\Pi_{j=1}^k p_{i_j}\right) M_t^s(x_{i_1},\ldots,x_{i_k}) \right)^{1/s}, & s \neq 0, \\ \Pi_{i_1,\ldots,i_k \in I} (M_t(x_{i_1},\ldots,x_{i_k}))^{(\Pi_{j=1}^k p_{i_j})}, & s = 0. \end{cases} \tag{1.26}$$

Corollary 1.12. *Let $s,t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Then, we have*

$$M^3_t = M^3_{t,s}(\mathbf{x},\mathbf{p};1) \geq \cdots \geq M^3_{t,s}(\mathbf{x},\mathbf{p};k) \geq \cdots \geq M^3_s, \tag{1.27}$$

$$M^3_s = M^3_{s,t}(\mathbf{x},\mathbf{p};1) \leq \cdots \leq M^3_{s,t}(\mathbf{x},\mathbf{p};k) \leq \cdots \leq M^3_t. \tag{1.28}$$

Proof. Let $s,t \in \mathbb{R}$ such that $s \leq t$, if $s,t \neq 0$, then we set $f(x) = x^{t/s}$, $x_{i_j} = x^s_{i_j}$ in (1.24) and raising the power $1/t$, we get (1.27). Similarly we set $f(x) = x^{s/t}$, $x_{i_j} = x^t_{i_j}$ in (1.25) and raising the power $1/s$, we get (1.28).

When $s = 0$ or $t = 0$, we get the required results by taking limit. □

We define the quasiarithmetic means with respect to (1.25) as follows:

$$M^3_{h,g}(\mathbf{x},\mathbf{p};\mathbf{k}) = h^{-1}\left(\sum_{i_1,\ldots,i_k \in I} p_{i_1}\cdots p_{i_k} h \circ g^{-1}\left(\frac{1}{k}\sum_{j=1}^k g\left(x_{i_j}\right)\right)\right), \tag{1.29}$$

where $h \circ g^{-1}$ is the convex function.

We obtain these generalized means be setting $f = h \circ g^{-1}$, $x_{i_j} = g(x_{i_j})$ and applying h^{-1} to (1.25).

Corollary 1.13. *By similar setting in* (1.24)*, we get the monotonicity of generalized means as follows:*

$$M^3_h(\mathbf{x},\mathbf{p}) = M^3_{h,g}(\mathbf{x},\mathbf{p};1) \geq \cdots \geq M^3_{h,g}(\mathbf{x},\mathbf{p};k) \geq \cdots \geq M^3_g(\mathbf{x},\mathbf{p}), \tag{1.30}$$

where $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M^3_g(\mathbf{x},\mathbf{p}) = M^3_{g,h}(\mathbf{x},\mathbf{p};1) \leq \cdots \leq M^3_{g,h}(\mathbf{x},\mathbf{p};k) \leq \cdots \leq M^3_h(\mathbf{x},\mathbf{p}), \tag{1.31}$$

where $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

The following result is given at [4, page 97].

Theorem 1.14. *Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be a convex function, σ be an increasing function on $[0,1]$ such that $\int_0^1 d\sigma(x) = 1$, and $u : [0,1] \to I$ be σ-integrable on $[0,1]$. Then*

$$\begin{aligned} f\left(\int_0^1 u(x)d\sigma(x)\right) &\leq \int_0^1 \cdots \int_0^1 f\left(\frac{1}{k+1}\sum_{i=1}^{k+1} u(x_i)\right)\prod_{i=1}^{k+1} d\sigma(x_i) \\ &\leq \int_0^1 \cdots \int_0^1 f\left(\frac{1}{k}\sum_{i=1}^{k} u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i) \\ &\leq \cdots \\ &\leq \int_0^1 \cdots \int_0^1 f\left(\frac{1}{2}\sum_{i=1}^{2} u(x_i)\right)\prod_{i=1}^{2} d\sigma(x_i) \\ &\leq \int_0^1 f(u(x))d\sigma(x), \end{aligned} \tag{1.32}$$

for all positive integers k.

We write (1.32) in the way that $\Omega^5 \geq 0$, where

$$\Omega^5 := \int_0^1 \cdots \int_0^1 f\left(\frac{1}{m}\sum_{i=1}^{m} u(x_i)\right)\prod_{i=1}^{m} d\sigma(x_i) - \int_0^1 \cdots \int_0^1 f\left(\frac{1}{k}\sum_{i=1}^{k} u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i), \tag{1.33}$$

for any positive integer $k > m \geq 1$.

The mixed symmetric means with positive weights related to

$$\int_0^1 \cdots \int_0^1 f\left(\frac{1}{k}\sum_{i=1}^{k} u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i) \tag{1.34}$$

are defined as:

$$M_{s,t}^5(\mathbf{x};k) = \begin{cases} \left(\int_0^1 \cdots \int_0^1 M_t^s(u(x_1),\ldots,u(x_k))\prod_{i=1}^{k} d\sigma(x_i)\right)^{1/s}, & s \neq 0, \\ \exp\left(\left(\int_0^1 \cdots \int_0^1 \log M_t(u(x_1),\ldots,u(x_k))\prod_{i=1}^{k} d\sigma(x_i)\right)\right), & s = 0. \end{cases} \tag{1.35}$$

Corollary 1.15. *Let $s, t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Then, we have*

$$M_t^5 = M_{t,s}^5(\mathbf{x},\mathbf{p};1) \geq \cdots \geq M_{t,s}^5(\mathbf{x},\mathbf{p};k) \geq \cdots \geq M_s^5, \tag{1.36}$$

$$M_s^5 = M_{s,t}^5(\mathbf{x},\mathbf{p};1) \leq \cdots \leq M_{s,t}^5(\mathbf{x},\mathbf{p};k) \leq \cdots \leq M_t^5. \tag{1.37}$$

Proof. Let $s, t \in \mathbb{R}$ such that $s \le t$, if $s, t \ne 0$, then we set $f(x) = x^{t/s}$, $u = u^s$ in (1.32) and raising the power $1/t$, we get (1.36). Similarly we set $f(x) = x^{s/t}$, $u = u^t$ in (1.32) and raising the power $1/s$, we get (1.37).

When $s = 0$ or $t = 0$, we get the required results by taking limit. □

We define the quasiarithmetic means with respect to (1.32) as follows:

$$M^5_{h,g}(\mathbf{x}; k) = h^{-1}\left(\int_0^1 \cdots \int_0^1 h \circ g^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} g \circ u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i)\right), \tag{1.38}$$

where $h \circ g^{-1}$ is the convex function.

We obtain these generalized means by setting $f = h \circ g^{-1}$, $u(x) = g \circ u(x)$ and applying h^{-1} to (1.34).

Corollary 1.16. *By similar setting in* (1.32)*, we get the monotonicity of generalized means, given in* (1.38)*:*

$$M^5_h(\mathbf{x}, \mathbf{p}) = M^5_{h,g}(\mathbf{x}, \mathbf{p}; 1) \ge \cdots \ge M^5_{h,g}(\mathbf{x}, \mathbf{p}; k) \ge \cdots \ge M^5_g(\mathbf{x}, \mathbf{p}), \tag{1.39}$$

where $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M^5_g(\mathbf{x}, \mathbf{p}) = M^5_{g,h}(\mathbf{x}, \mathbf{p}; 1) \le \cdots \le M^5_{g,h}(\mathbf{x}, \mathbf{p}; k) \le \cdots \le M^5_h(\mathbf{x}, \mathbf{p}), \tag{1.40}$$

where $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

Remark 1.17. In fact unweighted version of these results were proved in [6], but in Remark 2.14 from [6], it is written that the same is valid for weighted case.

For convex function f, we define

$$\Omega^i(\mathbf{x}, \mathbf{p}, f) = f^i_{m,n}(\mathbf{x}, \mathbf{p}) - f^i_{k,n}(\mathbf{x}, \mathbf{p}), \quad \text{for } i = 1, 3;\ 1 \le m < k \le n,\ \text{for } i = 2, ;\ 1 \le m < k \tag{1.41}$$

from (1.4), (1.16), and (1.24), in the way that

$$\Omega^i(\mathbf{x}, \mathbf{p}, f) \ge 0, \quad i = 1, 2, 3, \tag{1.42}$$

combining (1.42) with (1.12) and (1.33), we have

$$\Omega^i(\mathbf{x}, \mathbf{p}, f) \ge 0, \quad i = 1, \ldots, 5, \tag{1.43}$$

for any convex function f.

The exponentially convex functions are defined in [7] as follows.

Definition 1.18. A function $f : (a,b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f(x_i + x_j) \geq 0 \tag{1.44}$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

We also quote here a useful propositions from [7].

Proposition 1.19. *Let $f : (a,b) \to \mathbb{R}$ be a function, then following statements are equivalent;*

(i) *f is exponentially convex.*

(ii) *f is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.45}$$

for every $\xi_i \in \mathbb{R}$ and every $x_i, x_j \in (a,b)$, $1 \leq i,j \leq n$.

Proposition 1.20. *If $f : (a,b) \to \mathbb{R}^+$ is an exponentially convex function then f is a log-convex function.*

Consider $\varphi_s : (0,\infty) \to \mathbb{R}$, defined as

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{1.46}$$

and $\phi_s : \mathbb{R} \to [0,\infty)$, defined as

$$\phi_s = \begin{cases} \dfrac{1}{s^2} e^{sx}, & s \neq 0, \\ \dfrac{1}{2} x^2, & s = 0. \end{cases} \tag{1.47}$$

It is easy to see that both φ_s and ϕ_s are convex.

In this paper we prove the exponential convexity of (1.43) for convex functions defined in (1.46) and (1.47) and mean value theorems for the differences given in (1.43). We also define the corresponding means of Cauchy type and establish their monotonicity.

2. Main Result

The following theorems are the generalizations of results given in [6].

Theorem 2.1. (i) *Let the conditions of Theorem 1.1 be satisfied. Consider*

$$\Omega_t^i = \left(\varphi_t\right)_{m,n} - \left(\varphi_t\right)_{k,n}, \quad i = 1, \ldots, 5, \tag{2.1}$$

where Ω_s^i is obtained by replacing convex function f with φ_s for $s \in \mathbb{R}$, in $\Omega^i(\mathbf{x}, \mathbf{p}, f)$ $(i = 1, \ldots, 5)$. Then the following statements are valid.

(a) *For every $p \in \mathbb{N}$ and $s_1, \ldots, s_p \in \mathbb{R}$, the matrix $[\Omega^i_{(s_l+s_m)/2}]_{l,m=1}^p$ is a positive semidefinite matrix. Particularly*

$$\det\left[\Omega^i_{(s_l+s_m)/2}\right]_{l,m=1}^k \geq 0, \quad \text{for } k = 1, 2, \ldots, p. \tag{2.2}$$

(b) *The function $s \mapsto \Omega_s^i$ is exponentially convex on $\mathbb{R}$.*

Proof. (i) Consider a function

$$\mu(x) = \sum_{l,m=1}^{k} u_l u_m \varphi_{s_{lm}}(x), \tag{2.3}$$

for $k = 1, 2, \ldots, p$, $u_l \in \mathbb{R}$, u_l, and u_m are not simultaneously zero and $s_{lm} = (s_l + s_m)/2$. We have

$$\begin{aligned}
\mu''(x) &= \sum_{l,m=1}^{k} u_l u_m x^{s_{lm}-2}, \\
\Longrightarrow \mu''(x) &= \left(\sum_{l=1}^{k} u_l x^{s_l/2-1}\right)^2 \geq 0.
\end{aligned} \tag{2.4}$$

It follows that μ is a convex function. By taking $f = \mu$ in (1.43), we have

$$\begin{aligned}
0 &\leq \left(\sum_{l,m=1}^{k} u_l u_m \varphi^i_{s_{lm}}\right)_{m,n} - \left(\sum_{l,m=1}^{k} u_l u_m \varphi_{s_{lm}}\right)_{k,n} \\
&= \sum_{l,m=1}^{k} u_l u_m \Big(\left(\varphi_{s_{lm}}\right)_{m,n} - \left(\varphi_{s_{lm}}\right)_{k,n}\Big) \\
&= \sum_{l,m=1}^{k} u_l u_m \Omega^i_{s_{lm}}.
\end{aligned} \tag{2.5}$$

This means that the matrix $[\Omega^i_{(s_l+s_m)/2}]_{l,m=1}^p$ is a positive semidefinite, that is, (2.2) is valid.

(ii) It was proved in [6] that Ω_s^i is continuous for $s \in \mathbb{R}$. By using Proposition 1.19, we get exponential convexity of the function $s \mapsto \Omega_s^i$. □

Theorem 2.2. *Theorem 2.1 is still valid for convex functions $\phi_s = \varphi_s$.*

Theorem 2.3. *Let $n \geq 3$ and k be positive integers such that $2 \leq k \leq n-1$ and let $f \in C^2[a,b]$, $\Omega_s^i(\mathbf{x},\mathbf{p};x^2) \neq 0$, then there exists $\xi \in [a,b]$ such that*

$$\Omega^i(\mathbf{x},\mathbf{p},f) = \frac{1}{2}f''(\xi)\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right), \quad i = 1,\ldots,5. \tag{2.6}$$

Proof. Since $f \in C^2[a,b]$ therefore there exist real numbers $m = \min_{x\in[a,b]} f''(x)$ and $M = \max_{x\in[a,b]} f''(x)$. It is easy to show that the functions $\phi_1(x)$, $\phi_2(x)$ defined as

$$\begin{aligned} \phi_1(x) &= \frac{M}{2}x^2 - f(x), \\ \phi_2(x) &= f(x) - \frac{m}{2}x^2 \end{aligned} \tag{2.7}$$

are convex.

We use ϕ_1 in (1.43),

$$\begin{gathered} \Omega^i\left(\mathbf{x},\mathbf{p}, \ \frac{M}{2}x^2 - f(x)\right) \geq 0, \\ \Omega^i(\mathbf{x},\mathbf{p},f(x)) \leq \frac{M}{2}\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right). \end{gathered} \tag{2.8}$$

Similarly, by using ϕ_2 in (1.43), we get

$$\begin{gathered} \Omega^i\left(\mathbf{x},\mathbf{p},f(x) - \frac{m}{2}x^2\right) \geqslant 0, \\ \frac{m}{2}\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right) \leq \Omega^i(\mathbf{x},\mathbf{p},f(x)). \end{gathered} \tag{2.9}$$

From (2.8) and (2.9), we get

$$\frac{m}{2}\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right) \leq \Omega^i(\mathbf{x},\mathbf{p},f(x)) \leq \frac{M}{2}\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right). \tag{2.10}$$

Since $\Omega^i(\mathbf{x},\mathbf{p},x^2) \neq 0$, therefore

$$\Longrightarrow m \leq \frac{2\Omega^i(\mathbf{x},\mathbf{p},f(x))}{\Omega^i(\mathbf{x},\mathbf{p},x^2)} \leq M. \tag{2.11}$$

Hence, we have

$$\Omega^i(\mathbf{x},\mathbf{p},f) = \frac{1}{2}f''(\xi)\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right). \tag{2.12}$$

□

Theorem 2.4. *Let $n \geq 3$ and k be positive integer such that $2 \leq k \leq n-1$ and $f, g \in C^2[a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\frac{\Omega^i(\mathbf{x},\mathbf{p},f)}{\Omega^i(\mathbf{x},\mathbf{p},g)} = \frac{f''(\xi)}{g''(\xi)}, \tag{2.13}$$

provided that the denominators are non zero.

Proof. Define $h \in C^2[a,b]$ in the way that

$$h = c_1 f - c_2 g, \tag{2.14}$$

where c_1 and c_2 are as follow;

$$\begin{aligned} c_1 &= \Omega^i(\mathbf{x},\mathbf{p},g) \\ c_2 &= \Omega^i(\mathbf{x},\mathbf{p},f). \end{aligned} \tag{2.15}$$

Now using Theorem 2.3 with $f = h$, we have

$$\left(c_1\frac{f''(\xi)}{2} - c_2\frac{g''(\xi)}{2}\right)\Omega^i\left(\mathbf{x},\mathbf{p},x^2\right) = 0. \tag{2.16}$$

Since $\Omega^i_{k,n}(\mathbf{x},\mathbf{p},x^2) \neq 0$, therefore (2.16) gives

$$\frac{\Omega^i(\mathbf{x},\mathbf{p},f)}{\Omega^i(\mathbf{x},\mathbf{p},g)} = \frac{f''(\xi)}{g''(\xi)}\ . \tag{2.17}$$

□

Corollary 2.5. *Let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples, then for distinct real numbers l and r, different from zero and 1, there exists $\xi \in [a,b]$, such that*

$$\xi^{l-r} = \frac{r(r-1)}{l(l-1)}\frac{\Omega^i(\mathbf{x},\mathbf{p};x^l)}{\Omega^i(\mathbf{x},\mathbf{p};x^r)}. \tag{2.18}$$

Proof. Taking $f(x) = x^l$ and $g(x) = x^r$, in (2.13), for distinct real numbers l and r, different from zero and 1, we obtain (2.18). □

Remark 2.6. Since the function $\xi \to \xi^{l-r}$, $l \neq r$ is invertible, then from (2.18), we get

$$m \leq \left(\frac{r(r-1)}{l(l-1)}\frac{\Omega^i(\mathbf{x},\mathbf{p};x^l)}{\Omega^i(\mathbf{x},\mathbf{p};x^r)}\right)^{1/(l-r)} \leq M, \quad r \neq l,\ r,l \neq 0,1. \tag{2.19}$$

3. Cauchy Mean

In fact, similar result can also be find for (2.13). Suppose that f''/g'' has inverse function. Then (2.13) gives

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Omega^i(\mathbf{x},\mathbf{p},f)}{\Omega^i(\mathbf{x},\mathbf{p},g)}\right). \tag{3.1}$$

We have that the expression on the right hand side of above, is also a mean. We define Cauchy means

$$\begin{aligned} M^i_{l,r} &= \left(\frac{r(r-1)}{l(l-1)}\frac{\Omega^i(\mathbf{x},\mathbf{p};x^l)}{\Omega^i(\mathbf{x},\mathbf{p};x^r)}\right)^{1/(l-r)}, \quad r\neq l,\ r,l\neq 0,1,\\ &= \left(\frac{\Omega^i(\mathbf{x},\mathbf{p};\varphi_l)}{\Omega^i(\mathbf{x},\mathbf{p};\varphi_r)}\right)^{1/(l-r)}, \quad r\neq l. \end{aligned} \tag{3.2}$$

Also, we have continuous extensions of these means in other cases. Therefore by limit, we have the following:

$$\begin{aligned} M^i_{r,r} &= \exp\left(\frac{1-2r}{r(r-1)} - \frac{\Omega^i(\mathbf{x},\mathbf{p};\varphi_r\varphi_0)}{\Omega^i(\mathbf{x},\mathbf{p};\varphi_r)}\right), \quad r\neq 0,1,\\ M^i_{1,1} &= \exp\left(-1-\frac{\Omega^i(\mathbf{x},\mathbf{p};\varphi_o\varphi_1)}{2\Omega^i(\mathbf{x},\mathbf{p};\varphi_1)}\right),\\ M^i_{0,0} &= \exp\left(1-\frac{\Omega^i(\mathbf{x},\mathbf{p};\varphi_0^2)}{2\Omega^i(\mathbf{x},\mathbf{p};\varphi_0)}\right). \end{aligned} \tag{3.3}$$

The following lemma gives an equivalent definition of the convex function [4, page 2].

Lemma 3.1. *Let f be a convex function defined on an interval $I\subset\mathbb{R}$ and $l\le v,\ r\le u,\ l\neq r,\ u\neq v$. Then*

$$\frac{f(l)-f(r)}{l-r} \le \frac{f(v)-f(u)}{v-u}. \tag{3.4}$$

Now, we deduce the monotonicity of means given in (3.2) in the form of Dresher's inequality, as follows.

Theorem 3.2. *Let $M^i_{r,l}$ be given as in (3.2) and $r,l,u,v\in\mathbb{R}$ such that $r\le v$, $l\le u$, then*

$$M^i_{r,l} \le M^i_{v,u}. \tag{3.5}$$

Proof. By Proposition 1.20 Ω_l^i is log-convex. We set $f(l) = \log \Omega_l^i$ in Lemma 3.1 and get

$$\frac{\log \Omega_l^i - \log \Omega_r^i}{l-r} \leq \frac{\log \Omega_v^i - \log \Omega_u^i}{v-u}. \tag{3.6}$$

This together with (2.1) follows (3.5). □

Corollary 3.3. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be positive* n*-tuples, then for distinct real numbers* l, r, *and* s, *all are different from zero and 1, there exists* $\xi \in I$, *such that*

$$\xi^{l-r} = \frac{r(r-s)}{l(l-s)} \frac{\left(M_{l,s}^i(\mathbf{x},\mathbf{p};k)\right)^l - \left(M_{l,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^l}{\left(M_{r,s}^i(\mathbf{x},\mathbf{p};k)\right)^r - \left(M_{r,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^r}. \tag{3.7}$$

Proof. Set $f(x) = x^{l/s}$ and $g(x) = x^{r/s}$, then taking $x_i \to x_i^s$ in (2.13), we get (3.7) by the virtue of (1.2), (1.18), (1.26) and (1.35) for non zero, distinct real numbers l, r and s. □

Remark 3.4. Since the function $\xi \to \xi^{l-r}$ is invertible, then from (3.7) we get

$$m \leq \left(\frac{r(r-s)}{l(l-s)} \frac{\left(M_{l,s}^i(\mathbf{x},\mathbf{p};k)\right)^l - \left(M_{l,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^l}{\left(M_{r,s}^i(\mathbf{x},\mathbf{p};k)\right)^r - \left(M_{r,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^r} \right)^{1/(l-r)} \leq M, \tag{3.8}$$

where l, r, and s are non zero, distinct real numbers.

The corresponding Cauchy means are given by

$$M_{l,r;s}^i = \left(\frac{r(r-s)}{l(l-s)} \frac{\left(M_{l,s}^i(\mathbf{x},\mathbf{p};k)\right)^l - \left(M_{l,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^l}{\left(M_{r,s}^i(\mathbf{x},\mathbf{p};k)\right)^r - \left(M_{r,s}^i(\mathbf{x},\mathbf{p};k+1)\right)^r} \right)^{1/(l-r)}, \tag{3.9}$$

where l, r, and s are non zero, distinct real numbers. We write (3.9) as

$$M_{l,r;s}^i = \left(\frac{\Omega^i(\mathbf{x}^s,\mathbf{p};\varphi_{l/s})}{\Omega^i(\mathbf{x}^s,\mathbf{p};\varphi_{r/s})} \right)^{1/(l-r)}, \quad l \neq r, \tag{3.10}$$

where $\mathbf{x}^s = (x_1^s, \ldots, x_n^s)$ and the limiting cases are as follows:

$$
\begin{aligned}
M^i_{r,r;s} &= \exp\left(\frac{(s-2r)}{r(r-s)} - \frac{\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_{r/s}\varphi_0)}{s\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_{r/s})}\right), \quad r(r-s) \neq 0,\ s \neq 0,\\
M^i_{0,0;s} &= \exp\left(\frac{1}{s} - \frac{\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_0^2)}{2s\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_0)}\right), \quad s \neq 0,\\
M^i_{s,s;s} &= \exp\left(\frac{-1}{s} - \frac{\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_0\varphi_1)}{2s\Omega^i(\mathbf{x}^s, \mathbf{p}; \varphi_1)}\right), \quad s \neq 0,\\
M^i_{r,r;0} &= \exp\left(\frac{-2}{r} + \frac{\Omega^i(\log \mathbf{x}, \mathbf{p}; x\phi_r)}{\Omega^i(\log \mathbf{x}, \mathbf{p}; \phi_r)}\right), \quad r \neq 0,\\
M^i_{0,0;0} &= \exp\left(\frac{\Omega^i(\log \mathbf{x}, \mathbf{p}; x\phi_0)}{3\Omega^i(\log \mathbf{x}, \mathbf{p}; \phi_0)}\right),
\end{aligned}
\tag{3.11}
$$

where $\log \mathbf{x} = (\log x_1, \ldots, \log x_n)$.

Now, we give the monotonicity of new means given in (3.10), as follows:

Theorem 3.5. *Let $l, r, u, v \in \mathbb{R}$ such that $l \leq v$, $r \leq u$, then*

$$
M^i_{l,r;s} \leq M^i_{v,u;s}, \qquad i = 1, \ldots, n, \tag{3.12}
$$

where $M^i_{l,r}$ is given in (3.10).

Proof. We take Ω^i_l as defined in Theorem 2.1. Ω^i_l are log-convex by Proposition 1.20, therefore by Lemma 3.1 for $l, r, u, v \in \mathbb{R}$, $l \leq v$, $r \leq u$, we get

$$
\left(\frac{\Omega^i_l}{\Omega^i_r}\right)^{1/(l-r)} \leq \left(\frac{\Omega^i_v}{\Omega^i_u}\right)^{1/(v-u)}. \tag{3.13}
$$

For $s > 0$, we set $x_i = x_i^s$, $l = l/s$, $r = r/s$, $u = u/s$, $v = v/s \in \mathbb{R}$ such that $l/s \leq v/s$, $r/s \leq u/s$, in (2.1) to obtain (3.12) with the help of (3.13).

Similarly for $s < 0$, we set $x_i = x_i^s$, $l = l/s$, $r = r/s$, $u = u/s$, $v = v/s \in \mathbb{R}$ such that $v/s \leq l/s$, $u/s \leq r/s$, in (2.1) and get (3.12) again, by the virtue of (3.13).

In the case $s = 0$, since $s \to \Omega^i_s$ for $s \in \mathbb{R}$ is continuous therefore We get required result by taking limit. □

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