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Research Article

A Note on Strong Laws of Large Numbers for Dependent Random Sets and Fuzzy Random Sets

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This paper deals with a sequence of identically distributed random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space. The strong laws of large numbers for these two sequences are derived under Kuratowski-Mosco sense.

1. Introduction

Recently, great progress has been made towards the theories and applications of random sets and fuzzy random sets in the areas of information science, probability, and statistics. It is well known that Robbins [1, 2] first proposed the concept of random sets and investigated the relationships between random sets and geometric probabilities in his early work. After that, Kendall [3] and Matheron [4] provided a comprehensive mathematical theory of random sets which was greatly influenced by the geometric probability prospective. Their proposed framework exerted a strong influence on the limit theorems developed in the recent decades. Notice that strong laws of large numbers (SLLNs) play an important role in probability limit theorems, and several variants of SLLNs were built by Artstein and Vitale [5], Puri and Ralescu [6], Hiai [7], Inoue [8], Taylor et al. [9–11], Uemura [12], and so forth. Among them, Artstein and Vitale [5] proved limit theorems concerning random sets in $\mathfrak{R}$ and $\mathfrak{R}^d$. Puri and Ralescu [6] were the first to obtain the SLLNs for independent identically distributed (i.i.d.) Banach space-valued compact convex random sets. Among others, SLLNs were obtained under more relaxed conditions, and a detailed survey of these results is available in Taylor and Inoue [10].

The theory of fuzzy sets was introduced by Zadeh [13] (for an outline recently, one can refer to [14, 15]), and the concept of fuzzy random variables was promoted by Kwakernaak

[16], where useful basic properties were developed. Puri and Ralescu [17] used the concept of fuzzy random variables in generalizing results for random sets to fuzzy random sets. With respect to laws of large numbers, Kruse [18] proved an SLLN for i.i.d. fuzzy random variables. Klement et al. [19] considered fuzzy versions of random sets in Euclidean spaces and obtained an i.i.d. SLLN. Inoue [20] derived SLLNs for independent, tight fuzzy random sets, and i.i.d. fuzzy random sets in a separable Banach space. Recently, SLLNs have been established under various conditions, and one can refer to the following papers [8–11, 21–26]. Also for more detailed results about limit theorems of random sets and fuzzy random sets, we refer the readers to Li et al. [27] and references therein.

However, to the best of our knowledge, many limit theorems, especially the laws of large numbers, were obtained for independent random sets or fuzzy random sets in the past decades, and little is known of dependent random sets or fuzzy random sets except the exchangeable dependence involved in Inoue [8, 28], Taylor et al. [11], and Terán [26]. In this paper, we aim to propose a new kind of dependence for random sets and fuzzy random sets, and then establish several strong laws of large numbers in Kuratowski-Mosco convergence without the restriction of compactness, where random sets take values of closed subsets in separable Banach spaces.

The layout of this paper is as follows. In Section 2, we give some basic definitions and properties, and the new dependence is proposed in Section 3. In the last section we show several SLLNs for a sequence of dependent random sets and fuzzy random sets, and their proofs.

2. Definitions and Preliminaries

Throughout this paper, let $\mathbf{S}$ be a real separable Banach space with the norm $\|\cdot\|$ and the dual space $\mathbf{S}^*$. For each $A \subset \mathbf{S}$, $\operatorname{cl} A$ and $\overline{\operatorname{co}}A$ denote the norm-closure and the closed convex hull of A, respectively. Let $K(\mathbf{S})$ (resp., $K_c(\mathbf{S})$) denote the collections of all nonempty closed (resp., nonempty closed convex) subsets of $\mathbf{S}$. Define the Minkowski's addition and scalar multiplication, respectively, in $K(\mathbf{S})$ (or $K_c(\mathbf{S})$) by

$$\begin{gathered} A+B=\{a+b \mid a\in A,\ b\in B\}, \\ \lambda\cdot A=\{\lambda a \mid a\in A\}, \end{gathered} \tag{2.1}$$

where $A, B \in K(\mathbf{S})$ (or $K_c(\mathbf{S})$) and λ is a real number. Note that neither $K(\mathbf{S})$ nor $K_c(\mathbf{S})$ are linear spaces even when $\mathbf{S} = \mathfrak{R}$, one-dimensional Euclidean space. For $A, B \in K(\mathbf{S})$, the distance $d(y, A)$ of A and $y \in \mathbf{S}$, the Hausdorff distance $d_H(A, B)$ of A and B, the norm $\|A\|$ of A and the support function $s(A, \cdot)$ of A are defined, respectively, by

$$d(y,A)=\inf_{a\in A}\|y-a\|,$$

$$d_H(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

$$\|A\| = d_H(A, \{0\}) = \sup_{a \in A} \|a\|,$$
$$s(A, a^*) = \sup_{a \in A} \langle a, a^* \rangle, \quad a^* \in \mathbf{S}^*. \tag{2.2}$$

Let $\{A_n\}$ be a sequence of closed sets in $K(\mathbf{S})$. We write $A_n \xrightarrow{H} A$ if $d_H(A_n, A) \to 0$ for some $A \in K(\mathbf{S})$. Rather than this Hausdorff convergence, we here use the Kuratowski-Mosco convergence. Let $s - \liminf A_n$ be the set of all $a \in \mathbf{S}$ such that $\|a_n - a\| \to 0$ for some $a_n \in A_n$, that is, $d_H(a, A_n) \to 0$, and let $w - \lim \sup A_n$ be the set of all $a \in \mathbf{S}$ such that $a_k \xrightarrow{w} a$ (i.e., a_k converges weakly to a) for some $a_k \in A_{n_k}$ and some subsequence $\{A_{n_k}\}$ of $\{A_n\}$. It is easily seen that $s - \liminf A_n \subset w - \limsup A_n$, and $s - \liminf A_n \in K(\mathbf{S})(\in K_c(\mathbf{S})$ if $\{A_n\} \subset K_c(\mathbf{S}))$. Thus we say A_n converges to A in the Kuratowski-Mosco sense if and only if

$$w - \limsup\ A_n \subset A \subset s - \liminf\ A_n. \tag{2.3}$$

Clearly, the Hausdorff convergence is generally stronger than Kuratowski-Mosco convergence, since the former implies the latter when $\mathbf{S}$ is infinite dimensional, and in finite dimensional spaces they coincide with bounded sets (cf. [29]).

Let $(\Omega, \mathcal{F}, \mathsf{P})$ denote a probability measure space. A random closed set is a Borel measurable function $F : \Omega \to K(\mathbf{S})$, that is, $F^{-1}(B) = \{\omega \in \Omega; X(\omega) \bigcap B \neq \emptyset\} \in \mathcal{F}$ for each $B \in K(\mathbf{S})$. Moreover, we assume that the random closed sets are $\mathcal{F} - B(K(\mathbf{S}))$ measurable in the sequel, where $B(K(\mathbf{S}))$ means the Borel subsets of $K(\mathbf{S})$. For a random set F in $K(\mathbf{S})$,there exists a corresponding set coF in $K_c(\mathbf{S})$, which can be used in defining an expected value. A measurable function $f : \Omega \to \mathbf{S}$ is called a measurable selection of F if $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$. Denote by

$$S_F = \left\{ f \in L^1(\Omega, \mathbf{S}); f(\omega) \in F(\omega), \text{a.e.} \right\}, \tag{2.4}$$

where $L^1(\Omega, \mathbf{S})$ denotes the space of measurable functions $f : \Omega \to \mathbf{S}$ such that $\int_\Omega \|f(\omega)\| d\mathsf{P} < \infty$. $S_F \neq \emptyset$ if and only if the random variable $\|F(\omega)\|$ is integrable. For each random set F, the expectation of F, denoted by $\mathsf{E}F$, is defined by

$$\mathsf{E}F = \int_\Omega F d\mathsf{P} = \left\{ \int_\Omega f d\mathsf{P}; f \in S_F \right\}, \tag{2.5}$$

where $\int_\Omega f d\mathsf{P}$ is the usual Bochner integral in $L^1(\Omega, \mathbf{S})$. Define $\int_A F d\mathsf{P} = \{\int_A f d\mathsf{P}; f \in S_F\}$ for $A \in \mathcal{F}$. This definition was introduced by Aumann in 1965 as a natural generalization of the integral of real-valued random variables in [30]. If $\mathsf{E}\|\operatorname{co} X\| < \infty$, a Bochner integral can be defined as $\mathsf{E}(\operatorname{co} X) = \int_\Omega \operatorname{co} X d\mathsf{P}$ and $\mathsf{E}(\operatorname{co} X) \in K(\mathbf{S})$ (cf. [31]). The random set X is said to be integrably bounded if the real-valued random variable $\|X(\omega)\|$ is integrable (cf. [27, 32]). Hiai and Umegaki [32] showed that a random set is integrably bounded if and only if S_X is bounded in $L^1(\Omega, \mathbf{S})$. Thus an integrably bounded random set may take unbounded sets.

Now we introduce some notions of fuzzy random sets. A fuzzy set in $\mathbf{S}$ is a function $u : \mathbf{S} \to [0, 1]$. Let $F(\mathbf{S})$ denote the family of the fuzzy subset u satisfying the following conditions:

(a) u is upper semicontinuous, that is, the α-level set of u, that is, $u_\alpha = \{x \in \mathbf{S}; u(x) \geq \alpha\}$ is a closed subset of $\mathbf{S}$ for each $\alpha \in (0,1]$,

(b) $\{x \in \mathbf{S} : u(x) > 0\}$ has compact closure,

(c) $\{x \in \mathbf{S} : u(x) = 1\} \neq \emptyset$.

A linear structure in $F(\mathbf{S})$ is defined by the following operations:

$$(u+v)(x) = \sup_{y+z=x} \min[u(y), v(z)],$$
$$(\lambda u)(x) \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0, \\ I_0(x) & \text{if } \lambda = 0, \end{cases} \tag{2.6}$$

where $u, v \in F(\mathbf{S}), \lambda \in \mathfrak{R}$. This of course implies $(u+v)_\alpha = u_\alpha + v_\alpha$, and $(\lambda u)_\alpha = \lambda u_\alpha$. Then we adopt the metric $d_r(d_\infty)$ (see [17, 19, 33]) as a generalization of the Hausdorff metric from $K(\mathbf{S})$ to $F(\mathbf{S})$, where

$$d_r(u,v) = \left[\int_0^1 d_H^r(u_\alpha, v_\alpha)d\alpha\right]^{1/r} \quad \text{if } 1 \leq r < \infty,$$
$$d_\infty(u,v) = \sup_{\alpha\in(0,1]} d_H(u_\alpha, v_\alpha) \quad \text{if } r = \infty, \tag{2.7}$$

where $u, v \in F(\mathbf{S})$. The concept of a fuzzy random set as a generation for a random set was extensively studied by Puri and Ralescu [17]. A fuzzy random set is a function $X : \Omega \to F(\mathbf{S})$ such that for each $\alpha \in (0,1], X_\alpha = \{x \in \mathbf{S}; X(\omega)(x) \geq \alpha\}$ is a random closed set. The expectation of a fuzzy random set X, denoted by $\mathsf{E}[X]$, is an element in $F(\mathbf{S})$ such that for each $\alpha \in (0,1]$,

$$(\mathsf{E}[\mathrm{X}])_\alpha = \mathrm{cl}\int_\Omega \mathrm{X}_\alpha d\mathsf{P} = \mathrm{cl}\{\mathsf{E}(f); f \in S_{\mathrm{X}_\alpha}\}, \tag{2.8}$$

where the closure is taken in $\mathbf{S}$ and $S_{X_\alpha} = \{f \in L^1(\Omega, \mathbf{S}); f(\omega) \in X_\alpha(\omega) \text{ a.e.}\}$. By virtue of the existence theorem (cf. [27]), we have an equivalent definition as follows:

$$\mathsf{E}[X](x) = \sup\{\alpha \in (0,1]; x \in \mathsf{E}[\mathrm{X}_\alpha]\}. \tag{2.9}$$

Furthermore, $(\mathsf{E}[\mathrm{co}\, X])_\alpha = \mathsf{E}[(\mathrm{co}\, X)_\alpha]$ for any $\alpha \in (0,1]$.

3. Mixing Dependence

Many statistical results are concerned with independent and identically distributed (i.i.d.) random sets or fuzzy random sets. While it is not always possible to assume that random sets or fuzzy random sets are independent, the sequence can be often dependent. However, for dependent case, it seems that only the exchangeability is involved. In what follows, we propose a new kind of dependence for random sets which is popular with random variables and random elements. Similarly, it can be defined for fuzzy random sets.

Given two σ-fields $\mathcal{U}, \mathcal{V}$ in $\mathcal{F}$, write

$$\varphi(\mathcal{U}, \mathcal{V}) := \sup\{|\mathsf{P}(B \mid A) - \mathsf{P}(B)|; A \in \mathcal{U}, B \in \mathcal{V}, \mathsf{P}(A) \neq 0\}. \tag{3.1}$$

Let $\{X_n; n \geq 1\}$ be a sequence of random closed sets on $(\Omega, \mathcal{F}, \mathsf{P})$. Denote $S_L = \sum_{n \in L} X_n$, where $L \subset \aleph$ (the set of all nature numbers). For two nonempty disjoint sets $S, T \subset \aleph$, define $\operatorname{dist}(S, T)$ as $\min\{|j - k|; j \in S, k \in T\}$. Let $\sigma(S)$ and $\sigma(T)$ be the σ-fields generated by $\{X_n; n \in S\}$ and $\{X_n; n \in T\}$, respectively. Now we define two mixing coefficients for the sequence of $\{X_n; n \geq 1\}$. For any real number $s \geq 1$, set

$$\begin{aligned} \varphi(s) &= \sup\{\varphi(\sigma(S), \sigma(T)); S, T \subset \aleph, \operatorname{dist}(S, T) \geq s\}, \\ \varphi^*(s) &= \sup\{\max(\varphi(\sigma(S), \sigma(T)), \varphi(\sigma(T), \sigma(S))); S, T \subset \aleph, \operatorname{dist}(S, T) \geq s\}. \end{aligned} \tag{3.2}$$

If $\varphi(s)$ (resp. $\varphi^*(s)$) tends to zero as $s \to \infty$, then we say that the sequence is φ-mixing (resp., φ^*-mixing). Obviously, a φ^*-mixing sequence is a φ-mixing sequence. Also it is well known that many limit results were derived for real-valued mixing random sequences and random fields in the past thirty years (cf. [34, 35] and references therein). Zhang [36, 37] extended them to the Banach space-valued mixing random fields and established some moment inequalities. As far as we know, there is little concerning the dependent random sets or fuzzy random sets except the exchangeability dependence. The main purpose of this paper is to establish limit theorems for mixing dependent random sets or fuzzy random sets which extend the results of independent case.

4. Limit Theorems

Lemma 4.1. *Let $\mathcal{A}_X, \mathcal{A}_Y$ be the smallest sub-σ-filed of $\mathcal{F}$ to which X and Y are measurable, respectively. Let X be a random closed set and $\mathcal{F} - B(K(\mathbf{S}))$ measurable. Then one has the following.*

(1) *For each X with $S_X \neq \emptyset$,*

$$\overline{\text{co}}\mathsf{E}(X) = \overline{\text{co}}\mathsf{E}_{\mathcal{A}_X}(X), \tag{4.1}$$

where $\mathsf{E}_{\mathcal{A}_X}(X) = \{\mathsf{E}(x) : x \in S_X(\mathcal{A}_X)\}$.

(2) *Let $\{X_n; n \geq 1\}$ be a sequence of φ-mixing and identically distributed random closed sets. For each $x_1 \in S_{X_1}(\mathcal{A}_{X_1})$, where $S_{X_1}(\mathcal{A}_{X_1})$ denotes the set of all $\mathcal{A}_{X_1}$ measurable functions in S_{X_1}, there exists $\{x_i \in S_{X_i}(\mathcal{A}_{X_i});\ i \geq 2\}$ such that $\{x_n; n \geq 1\}$ is φ-mixing.*

(3) *For each φ-mixing and identically distributed random closed sets X_1, X_2 with $S_{X_1} \neq \emptyset$, one has*

$$\mathsf{E}_{\mathcal{A}_{X_1}}(X_1) = \mathsf{E}_{\mathcal{A}_{X_2}}(X_2). \tag{4.2}$$

Proof. Note that for $X \in L^1(\Omega, \mathbf{S})$, the conditional expectation of X with respect to $\mathcal{B} \in \mathcal{F}$ is given as a function $\mathsf{E}(X \mid \mathcal{B}) \in L^1(\Omega, \mathcal{B}, \mathbf{S})$ such that

$$\int_B \mathsf{E}(\mathrm{X} \mid \mathcal{B}) d\mathsf{P} = \int_B X d\mathsf{P} \quad \forall\, \mathrm{B} \in \mathcal{B}. \tag{4.3}$$

If $X \in K(\mathbf{S})$ with $S_X \neq \emptyset$, then from Hiai and Umegaki [32] it follows that there exists a $\mathcal{B}$-measurable $\mathsf{E}(X \mid \mathcal{B}) \in K(\mathbf{S})$ satisfying that

$$S_{\mathsf{E}(X|\mathcal{B})}(\mathcal{B}) = \mathrm{cl}\{\mathsf{E}(x \mid \mathcal{B}) : x \in S_X\} \text{in } L^1(\Omega, \mathbf{S}). \tag{4.4}$$

(1) Noting that $\overline{\mathrm{co}}X$ is $\mathcal{A}_X$ measurable, it follows that

$$S_{\overline{\mathrm{co}}X}(\mathcal{A}_X) = \{\mathsf{E}(x \mid \mathcal{A}_X) : x \in S_{\overline{\mathrm{co}}X}\}. \tag{4.5}$$

Recall that $S_{\overline{\mathrm{co}}X} = \overline{\mathrm{co}}S_X$ and $S_{\overline{\mathrm{co}}X}(\mathcal{A}_X) = \overline{\mathrm{co}}S_X(\mathcal{A}_X)$, and hence

$$\begin{aligned} \overline{\mathrm{co}}\mathsf{E}(X) &= \mathrm{cl}\,\mathsf{E}(\overline{\mathrm{co}}X) \\ &= \mathrm{cl}\{\mathsf{E}(\mathsf{E}(x \mid \mathcal{A}_X)) : x \in S_{\overline{\mathrm{co}}}X\} \\ &= \mathrm{cl}\{\mathsf{E}(x) : x \in S_{\overline{\mathrm{co}}X}(\mathcal{A}_X)\} \\ &= \overline{\mathrm{co}}\mathsf{E}_{\mathcal{A}_X}(\mathrm{X}). \end{aligned} \tag{4.6}$$

(2) Since $\mathbf{S}$ is separable and x_1 is $\mathcal{A}_{X_1}$-measurable, there exists a $(B(K(\mathbf{S}), B(\mathbf{S})))$-measurable function $\Psi : K(\mathbf{S}) \to \mathbf{S}$ satisfying that $x(\omega) = \Psi(X(\omega))$ for every $\omega \in \Omega$, where $B(\mathbf{S})$ means the Borel subsets of $\mathbf{S}$. Now define $x_n(\omega) = \Psi(X_n(\omega)), \omega \in \Omega$. Note that $\{X_n; n \geq 1\}$ is an identically distributed sequence of φ-mixing random sets, and this leads to $\{x_n; n \geq 1\}$ that is also of mixing dependence, since the definitions of mixing dependence rely on σ-fields. In fact, the mixing coefficients of $\{x_n; n \geq 1\}$ are less than those of $\{X_n; n \geq 1\}$. Thus we have

$$\int_\Omega \|x_1\| d\mathsf{P} = \int_{K(\mathbf{S})} \|\Psi(\mathrm{X})\| d\mathsf{P}_X = \int_{K(\mathbf{S})} \|\Psi(\mathrm{X}_n)\| d\mathsf{P}_{X_n} = \int_\Omega \|x_n(\omega)\| d\mathsf{P} < \infty. \tag{4.7}$$

Noting that the function $d(x, X)$ of $\mathbf{S} \times K(\mathbf{S})$ into R is $(B(\mathbf{S}), B(K(\mathbf{S})))$-measurable, thus $\{d(x_n(\cdot), X_n(\cdot))\}$ is φ-mixing and identically distributed. Hence, $d(x(\omega), X(\omega)) = 0$ a.s. implies $d(x_n(\omega), X_n(\omega)) = 0$ a.s., which leads to $x_n \in S_{X_n}(\mathcal{A}_{X_n})$.

(3) It follows from (2) easily. □

Remark 4.2. The lemma also holds for φ^*-mixing random closed sets in a similar way.

Hiai [7] proved a strong law of large numbers of i.i.d. random variables in $K(\mathbf{S})$ in Kuratowski-Mosco convergence. Recently, Inoue and Taylor [28] replaced i.i.d. by exchangeability and obtain a strong law of large numbers. Here we replace the i.i.d. by $\varphi(\varphi^*)$-mixing dependence which is a more extensive dependence and derive strong laws of large numbers for random sets and fuzzy random sets, respectively.

Theorem 4.3. *Let $\{X_n; n \geq 1\}$ be a sequence of mixing and identically distributed random closed sets in $K(\mathbf{S})$ with $\mathsf{E}\|X\| < \infty$. Suppose that one of the following conditions is satisfied:*

$$\text{(a)} \lim_{\tau \to \infty} \varphi^*(\tau) < \frac{1}{2}, \quad \text{(b)} \sum_{i=1}^{\infty} \varphi^{1/2}\left(2^i\right) < \infty. \tag{4.8}$$

Then one has

$$\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathsf{E}\,\mathrm{co}\,X_1 \quad \textit{in } K\textit{-}M \textit{ sense}. \tag{4.9}$$

Proof. Here we only consider the φ-mixing case, since the φ^*-mixing case can be proved similarly. By Lemma 4.1(2), for a sequence of φ-mixing random set $\{X_n; n \geq 1\}$ in $K(\mathbf{S})$, there exists a $(B(K(\mathbf{S})), B(\mathbf{S}))$-measurable function $f : K(\mathbf{S}) \to \mathbf{S}$ and the corresponding random elements $\{x_n; n \geq 1\}$ such that $x_n(\omega) = f(X_n(\omega))$ for all $\omega \in \Omega$.

Let $D = \overline{\mathrm{co}}\mathsf{E}(X_1)$ and $G_n(\omega) = n^{-1}\,\mathrm{cl}\sum_{i=1}^{n} X_i(\omega)$. Since $\{X_n; n \geq 1\}$ is φ-mixing and identically distributed random sets, it follows that $\{x_n; n \geq 1\}$ is φ-mixing and identically distributed. For any $\varepsilon > 0$ and $d \in D$, we can choose $x_i \in S_{X_i}(\mathcal{A}_{X_i}), 1 \leq i \leq m$, such that

$$\left\| \frac{1}{m}\sum_{i=1}^{m} \mathsf{E}(x_i) - d \right\| < \varepsilon, \tag{4.10}$$

where $\mathcal{A}_{X_i}$ is the smallest sub σ-field of $\mathcal{F}$ with respect to which X_i is measurable and $S_{X_i}(\mathcal{A}_{X_i}) = \mathrm{cl}\{\mathsf{E}(x \mid \mathcal{A}_{X_i}); x \in S_{X_i}\}$. By Lemma 4.1, there exists a sequence $\{x_n\}$ of $x_n \in S_{X_n}(\mathcal{A}_{X_n}), n \geq 1$ such that $\{x_{(k-1)m+j}; k \geq 1\}$ is φ-mixing and identically distributed for each $1 \leq j \leq m$. Let $d_i = \mathsf{E}(x_i)$. If $n = (k-1)m + l$, where $1 \leq l \leq m$, then we have

$$\left\| \frac{1}{n}\sum_{i=1}^{n} x_i(\omega) - \frac{1}{m}\sum_{i=1}^{m} d_i \right\| = \left\| \frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{k} x_{(i-1)m+j}(\omega) - \frac{1}{n}\sum_{j=l+1}^{m} x_{(k-1)m+j}(\omega) - \frac{1}{m}\sum_{i=1}^{m} d_i \right\|$$

$$\leq \frac{k}{n}\sum_{j=1}^{m}\left\|\frac{1}{k}\sum_{i=1}^{k}x_{(i-1)m+j}(\omega)-d_j\right\|+\frac{k}{n}\sum_{j=1}^{m}\left\|x_{(k-1)m+j}(\omega)\right\|$$
$$+\left(\frac{k}{n}-\frac{1}{m}\right)\left\|\sum_{i=1}^{m}d_i\right\|\longrightarrow 0 \text{ a.s.} \tag{4.11}$$

by a similar way of Theorem 3.2 of Hiai [7] and Theorem 3.1 of Zhang [37]. Note that $G_n(\omega)$ are closed sets in $K(\mathbf{S})$, which implies $n^{-1}\sum_{i=1}^{n} x_i(\omega) \in G_n(\omega)$ a.s. Thus, it follows that $m^{-1}\sum_{i=1}^{m} d_i \in s - \liminf G_n(\omega)$ a.s., and hence $D \subset s - \liminf G_n(\omega)$ a.s.

In the following we show $w - \limsup G_n(\omega) \subset D$ a.s. Let $\{d_i, i \geq 1\}$ be a sequence dense in $\mathbf{S} \setminus d$. Since $\mathbf{S}$ is separable, by the separation theorem there exists a sequence $\{d_i^*\}$ in $\mathbf{S}^*$ with $\|d_i^*\| = 1$ such that

$$\langle d_i, d_i^*\rangle \leq s\left(D, d_i^*\right), \quad \forall i. \tag{4.12}$$

Thus it follows $d \in D$ if and only if $\langle d, d_i^*\rangle \leq s(D, d_i^*)$, for all $i \geq 1$. Notice for each $i \geq 1$, $\{s(X_n(\cdot), d_i^*)\}$ is a sequence of φ-mixing random variables in L^1 since $\{X_n; n \geq 1\}$ is φ-mixing, and hence there exists a P-null set $N \in \mathscr{A}$ such that for every $\omega \in \Omega \setminus N$ and $i \geq 1$,

$$s\left(G_n(\omega), d_i^*\right) = \frac{1}{n}\sum_{i=1}^{n} s\left(X_i(\omega), d_i^*\right) \longrightarrow s\left(D, d_i^*\right) \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

If $d \in w - \limsup G_n(\omega)$ for $\omega \in \Omega \setminus N$, then $d_k \overset{w}{\to} d$ for some $d_k \in G_{n_k}(\omega)$. So, for each $i \geq 1$, we have

$$\langle d, d_i^*\rangle = \lim_{k\to\infty}\langle d_k, d_i^*\rangle \leq \lim_{k\to\infty} s\left(G_{n_k}(\omega), d_i^*\right) = s\left(D, d_i^*\right), \tag{4.14}$$

which implies $d \in D$. Thus $w - \limsup G_n(\omega) \subset D$ a.s. follows. □

Remark 4.4. If $\mathbf{S}$ is a finite dimensional Banach space and $\{X_n; n \geq 1\}$ are compact sets, then Theorem 4.3 still holds in the Hausdorff convergence.

The next theorem describes a strong law of large numbers for mixing fuzzy random sets in $F(\mathbf{S})$.

Theorem 4.5. *Let $\{X_n; n \geq 1\}$ be a sequence of mixing fuzzy random sets taking values in $F(\mathbf{S})$. If $\mathsf{E}\|X\| < \infty$, and one of the following conditions is satisfied:*

$$\text{(a)} \lim_{\tau\to\infty}\varphi^*(\tau) < \frac{1}{2}, \qquad \text{(b)} \sum_{i=1}^{\infty}\varphi^{1/2}\left(2^i\right) < \infty, \tag{4.15}$$

then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i = \mathsf{E}X_1 \quad \text{in } K\text{-}M \text{ sense}. \tag{4.16}$$

Proof. Since $\{X_n; n \geq 1\}$ is a sequence of mixing fuzzy random sets in $F(\mathbf{S})$, by the definitions of mixing we have that $\{X_n\alpha; n \geq 1\}$ is mixing random closed sets with $S_{X_n\alpha} \neq \emptyset$ for any $\alpha \in (0,1]$. Thus the desired result follows from Theorem 4.3 immediately. □

Remark 4.6. By the definitions of mixing dependence and σ-fields, it follows that the mixing coefficients in Theorem 4.5 is less than those in Theorem 4.3.

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