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Research Article

Some Starlikeness Criterions for Analytic Functions

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We determine the condition on α, μ, β, and λ for which $|(1-\alpha)(z/f(z))^{\mu}+\alpha(zf'(z)/f(z))(z/f(z))^{\mu}-1| < \lambda$ implies $f(z) \in S^*(\beta)$, where $S^*(\beta)$ is the class of starlike functions of order β. Some results of Obradović and Owa are extended. We also obtain some new results on starlikeness criterions.

1. Introduction

Let n be a positive integer, and let H_n denote the class of function

$$f(z) = z + \sum_{k=n}^{+\infty} a_{k+1} z^{k+1} \tag{1.1}$$

that are analytic in the unit disk $U = \{z : |z| < 1\}$. For $0 \le \beta < 1$, let

$$S^*(\beta) = \left\{ f \in H_1 : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta,\ z \in U \right\} \tag{1.2}$$

denote the class of starlike function of order β and $S^*(0) = S^*$.

Let $f(z)$ and $F(z)$ be analytic in U; then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)| \le |z|$, and $f(z) \equiv F(w(z))$, denoted that $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$ [1].

Let

$$S_\lambda = \{f \in H_1 : |f'(z) - 1| < \lambda, z \in U\}. \tag{1.3}$$

Singh [2] proved that $S_\lambda \subset S^*$ if $0 < \lambda \le 2/\sqrt{5}$. More recently, Fournier [3, 4] proved that

$$S_\lambda \subset S^* \Longleftrightarrow 0 \le \lambda \le \frac{2}{\sqrt{5}},$$
$$\rho_\lambda = \begin{cases} \dfrac{(1-\lambda)(1-\lambda/2)}{1-\lambda^2/4}, & \text{if } 0 \le \lambda \le \dfrac{2}{3}, \\ \dfrac{(1/2)(1-(5/4)\lambda^2)}{1-\lambda^2/4}, & \text{if } \dfrac{2}{3} \le \lambda \le 1, \end{cases} \tag{1.4}$$

is the order of starlikeness of S_λ. Now, we define

$$U(\lambda,\mu,n) = \left\{ f \in H_n : \left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < \lambda,\ z \in U \right\}. \tag{1.5}$$

Clearly, $U(\lambda,-1,1) = S_\lambda$. In 1998, Obradović [5] proved that

$$U(\lambda,\mu,1) \subset S^* \tag{1.6}$$

if $0 < \mu < 1$ and $0 < \lambda \le (1-\mu)/\sqrt{(1-\mu)^2 + \mu^2}$. Recently, Obradović and Owa [6] proved that

$$U(\lambda,\mu,n) \subset S^* \tag{1.7}$$

if $0 < \mu < 1$ and $0 < \lambda \le (n-\mu)/\sqrt{(n-\mu)^2 + \mu^2}$.

In this paper we find a condition on α, μ, β, and λ for which

$$\left| (1-\alpha)\left(\frac{z}{f(z)} \right)^{\mu} + \alpha \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < \lambda \tag{1.8}$$

implies $f(z) \in S^*(\beta)$ and extend some results of Obradović and Owa [5, 6]. Also, we obtain some new results on starlikeness criterions.

2. Main Results

For our results we need the following lemma.

Lemma 2.1 (see [6]). *Let* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ *be analytic in* U, $n \geq 1$, *and satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda z, \quad 0 < \mu < 1,\ 0 < \lambda \leq 1. \tag{2.1}$$

Then

$$p(z) \prec 1 + \lambda_1 z, \tag{2.2}$$

where

$$\lambda_1 = \frac{\lambda \mu}{n - \mu}. \tag{2.3}$$

Theorem 2.2. *Let* $0 \leq \mu < 1$, $n\alpha > \mu$, $0 \leq \beta < 1$, *and*

$$M_n(\alpha,\beta,\mu) = \begin{cases} \dfrac{\alpha(n\alpha-\mu)(1-\beta)}{\alpha(n+\mu-\mu\beta)-2\mu}, & \text{if } \alpha \geq \alpha_2, \\[2ex] \dfrac{(n\alpha-\mu)\sqrt{2\alpha(1-\beta)-1}}{\sqrt{n^2\alpha^2+2[\mu^2(1-\beta)-n\mu]\alpha}}, & \text{if } \alpha_1 < \alpha < \alpha_2, \\[2ex] \dfrac{(n\alpha-\mu)(1-\beta)}{n-\mu(1-\beta)}, & \text{if } 0 < \alpha \leq \alpha_1, \end{cases} \tag{2.4}$$

where

$$\alpha_1 = \frac{n-\mu(1-\beta)}{n(1-\beta)},$$
$$\alpha_2 = \frac{n+3\mu(1-\beta)+\sqrt{[n+3\mu(1-\beta)]^2-8n\mu(1-\beta)}}{2n(1-\beta)}. \tag{2.5}$$

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ *and* $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ *are analytic in* U, *satisfy*

$$q(z) \prec 1 + \frac{\mu\lambda}{n\alpha - \mu} z, \tag{2.6}$$

$$q(z)[1 - \alpha + \alpha p(z)] \prec 1 + \lambda z, \tag{2.7}$$

where $0 < \lambda \le M_n(\alpha, \beta, \mu)$, *then*

$$\operatorname{Re} p(z) > \beta, \quad \text{for } z \in U. \tag{2.8}$$

Proof. If $\mu = 0$, it is easy to see the result is true. Now, assume $\mu > 0$. Let

$$N = \frac{\mu\lambda}{n\alpha - \mu}. \tag{2.9}$$

If there exists $z_0 \in U$, such that $\operatorname{Re} p(z_0) = \beta$, then we will show that

$$\left| q(z_0)\left[1 - \alpha + \alpha p(z_0)\right] - 1 \right| \ge \lambda \tag{2.10}$$

for $0 < \lambda \le M_n(\alpha, \beta, \mu)$. Note that $|q(z_0) - 1| \le N$ for $z \in U$; it is sufficient to show that

$$\alpha\left|p(z_0) - 1\right| - N\left|1 - \alpha + \alpha p(z_0)\right| \ge \lambda \tag{2.11}$$

for $0 < \lambda \le M_n(\alpha, \beta, \mu)$. Let $p(z_0) = \beta + iy$, $y \in \mathbb{R}$; then, the left-hand side of (2.11) is

$$\begin{aligned} &\alpha\sqrt{(\beta - 1)^2 + y^2} - N\sqrt{(\alpha\beta + 1 - \alpha)^2 + \alpha^2 y^2} \\ &\quad = \alpha\sqrt{\beta^2 + y^2 + 1 - 2\beta} - N\sqrt{\alpha^2\beta^2 + \alpha^2 y^2 + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}. \end{aligned} \tag{2.12}$$

Suppose that $x = \beta^2 + y^2$ and note that $(n\alpha - \mu)N = \mu\lambda$; then inequality (2.11) is equivalent to

$$N \le \frac{\alpha\mu\sqrt{x + 1 - 2\beta}}{n\alpha - \mu + \mu\sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}} \tag{2.13}$$

for all $x \ge \beta^2$ and $0 < \lambda \le M_n(\alpha, \beta, \mu)$. Now, if we define

$$\varphi(x) = \frac{\sqrt{x + 1 - 2\beta}}{n\alpha - \mu + \mu\sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}}, \quad x \ge \beta^2, \tag{2.14}$$

then we have

$$\varphi'(x) = \frac{(n\alpha - \mu)\psi(x) + \mu\left[1 - 2\alpha(1 - \beta)\right]}{2\psi(x)\sqrt{x + 1 - 2\beta}\left[(n\alpha - \mu) + \mu\psi(x)\right]^2}, \quad x > \beta^2, \tag{2.15}$$

where

$$\psi(x) = \sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}. \tag{2.16}$$

Since

$$\varphi(x)=\sqrt{\alpha^2x^2+2\alpha(1-\alpha)\beta+(1-\alpha)^2}>\left|1-\alpha(1-\beta)\right|,\quad \text{for } x>\beta^2, \tag{2.17}$$

the denominator of $\varphi'(x)$ is positive. Further, let

$$T(x)=(n\alpha-\mu)\varphi(x)+\mu\left[1-2\alpha(1-\beta)\right],\quad x\geq\beta^2. \tag{2.18}$$

We have

$$T(x)\geq(n\alpha-\mu)\left|1-\alpha(1-\beta)\right|+\mu\left[1-2\alpha(1-\beta)\right]. \tag{2.19}$$

If

$$\frac{1}{1-\beta}\leq\alpha, \tag{2.20}$$

we get

$$\begin{aligned}T(x)&\geq n\alpha^2(1-\beta)-\left[n+3\mu(1-\beta)\right]\alpha+2\mu\\&=n(1-\beta)(\alpha-r_1)(\alpha-r_2),\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}r_1&=\frac{n+3\mu(1-\beta)-\sqrt{\left[n+3\mu(1-\beta)\right]^2-8n\mu(1-\beta)}}{2n(1-\beta)},\\r_2&=\frac{n+3\mu(1-\beta)+\sqrt{\left[n+3\mu(1-\beta)\right]^2-8n\mu(1-\beta)}}{2n(1-\beta)}.\end{aligned} \tag{2.22}$$

Note that

$$r_1<\frac{1}{1-\beta}<r_2. \tag{2.23}$$

We obtain

$$T(x)\geq 0\quad \text{for } \alpha\geq\alpha_2=r_2. \tag{2.24}$$

If

$$\frac{1}{2(1-\beta)}\leq\alpha<\frac{1}{1-\beta}, \tag{2.25}$$

we have

$$T(x) \geq \alpha\left[n - \mu(1-\beta) - n(1-\beta)\alpha\right]. \tag{2.26}$$

Hence we obtain

$$T(x) \geq 0 \quad \text{for } \frac{1}{2(1-\beta)} \leq \alpha \leq \alpha_1, \tag{2.27}$$

where

$$\alpha_1 = \frac{n - \mu(1-\beta)}{n(1-\beta)} < \frac{1}{1-\beta}. \tag{2.28}$$

If

$$0 < \alpha < \frac{1}{2(1-\beta)}, \tag{2.29}$$

we have $1 - 2\alpha(1-\beta) > 0$. It follows that $T(x) > 0$.

Therefore we obtain $\varphi'(x) \geq 0$ for $x > \beta^2$ if $0 < \alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$. It follows that

$$\min_{x \geq \beta^2} \varphi(x) = \varphi\left(\beta^2\right) = \begin{cases} \dfrac{(1-\beta)}{\alpha(n+\mu-\mu\beta) - 2\mu}, & \text{if } \alpha \geq \alpha_2, \\ \dfrac{(1-\beta)}{\alpha\left[n - \mu(1-\beta)\right]}, & \text{if } 0 < \alpha \leq \alpha_1. \end{cases} \tag{2.30}$$

If $\alpha_1 < \alpha < \alpha_2$, we have

$$\lim_{x \to (\beta^2)^+} T(x) = T\left(\beta^2\right) = (n\alpha - \mu)\left|1 - \alpha(1-\beta)\right| + \mu\left[1 - 2\alpha(1-\beta)\right] < 0 \tag{2.31}$$

by (2.13) and (2.21) for $1/(1-\beta) \leq \alpha < \alpha_2$ and by (2.23) for $\alpha_1 < \alpha < 1/(1-\beta)$. Note that $T(x)$ is an continuous increasing function for $x \geq \beta^2$, and

$$\lim_{x \to \infty} T(x) > 0. \tag{2.32}$$

Then there exists a unique $x_0 \in (\beta^2, +\infty)$, such that

$$T(x_0) = 0, \quad \text{or} \quad \varphi'(x_0) = 0. \tag{2.33}$$

Thus, x_0 is the global minimum point of $\varphi(x)$ on $[\beta^2, +\infty)$. It follows from (2.33) that

$$(n\alpha - \mu)\sqrt{\alpha^2 x_0 + 2\alpha(1-\alpha)\beta + (1-\alpha)^2} = \mu\left[2\alpha(1-\beta) - 1\right], \tag{2.34}$$

or

$$x_0 = \frac{1}{\alpha^2}\left\{\frac{\mu^2[2\alpha(1-\beta)-1]^2}{(n\alpha-\mu)^2} - 2\alpha(1-\alpha)\beta - (1-\alpha)^2\right\}. \tag{2.35}$$

By a simple calculation, we may obtain

$$\min_{x\geq\beta^2}\varphi(x) = \varphi(x_0) = \frac{\sqrt{2\alpha(1-\beta)-1}}{\alpha\sqrt{n^2\alpha^2 + 2[\mu^2(1-\beta)-n\mu]\alpha}} \tag{2.36}$$

for $\alpha_1 < \alpha < \alpha_2$. It follows from (2.30) and (2.36) that that inequality (2.13) holds. This shows that inequality (2.10) holds, which contradicts with (2.7). Hence we must have

$$\operatorname{Re} p(z) > \beta, \quad z \in U. \tag{2.37}$$

□

Theorem 2.3. *Let α, μ, β, λ and $M_n(\alpha,\beta,\mu)$ be defined as in Theorem 2.2. If $f(z) \in H_n$ satisfies*

$$\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < \lambda, \tag{2.38}$$

where $0 < \lambda \leq M_n(\alpha,\beta,\mu)$, then $f(z) \in S^*(\beta)$.

Proof. If $\mu = 0$, $M_n(\alpha,\beta,0) = \alpha(1-\beta)$ and the result is trivial. Now, assume $\mu > 0$. If we put

$$q(z) = \left(\frac{z}{f(z)}\right)^{\mu}, \tag{2.39}$$

then by some transformations and (2.38) we get

$$q(z) - \frac{\alpha}{\mu}zq'(z) \prec 1 + \lambda z. \tag{2.40}$$

By Lemma 2.1, we obtain

$$q(z) \prec 1 + \frac{\mu\lambda}{n\alpha - \mu}z. \tag{2.41}$$

Let

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.42}$$

Then we have

$$q(z)\left[1-\alpha+\alpha p(z)\right] \prec 1+\lambda z. \tag{2.43}$$

By Theorem 2.2, we get

$$\operatorname{Re} p(z) > \beta, \quad z \in U. \tag{2.44}$$

It follows that $f(z) \in S^*(\beta)$. □

For $\beta = 0$, we get the following corollary.

Corollary 2.4. *Let $0 \le \mu < 1$, $n\alpha > \mu$, and let*

$$M_n(\alpha,\mu) = \begin{cases} \dfrac{\alpha(n\alpha-\mu)}{\alpha(n+\mu)-2\mu}, & \text{if } \alpha \ge \alpha_2, \\[2ex] \dfrac{(n\alpha-\mu)\sqrt{2\alpha-1}}{\sqrt{n^2\alpha^2+2\left[\mu^2-n\mu\right]\alpha}}, & \text{if } \alpha_1 \le \alpha < \alpha_2, \\[2ex] \dfrac{(n\alpha-\mu)}{n-\mu}, & \text{if } 0 < \alpha < \alpha_1, \end{cases} \tag{2.45}$$

where

$$\alpha_1 = \frac{n-\mu}{n}, \qquad \alpha_2 = \frac{n+3\mu+\sqrt{(n+3\mu)^2-8n\mu}}{2n}. \tag{2.46}$$

If $f(z) \in H_n$ satisfies

$$\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < \lambda, \tag{2.47}$$

where $0 < \lambda \le M_n(\alpha,\mu)$, then $f(z) \in S^*$.

Corollary 2.5. *Let $0 \le \mu < 1$, $0 \le \beta < 1$, and let*

$$M_n(\beta,\mu) = \begin{cases} \dfrac{(n-\mu)(1-\beta)}{n-\mu(1+\beta)}, & \text{if } 1 > \beta \ge \dfrac{\mu}{n+\mu}, \\[2ex] \dfrac{(n-\mu)\sqrt{1-2\beta}}{\sqrt{n^2+2\left[\mu^2(1-\beta)-n\mu\right]}}, & \text{if } 0 \le \beta < \dfrac{\mu}{n+\mu}. \end{cases} \tag{2.48}$$

If $f(z) \in H_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < \lambda, \tag{2.49}$$

where $0 < \lambda \le M_n(\beta, \mu)$, then $f(z) \in S^*(\beta)$.

Proof. Note that

$$\begin{aligned} \alpha_1 &= \frac{n - \mu(1-\beta)}{n(1-\beta)} \ge 1, \quad \text{for } \beta \ge \frac{\mu}{n+\mu}, \\ \alpha_1 &= \frac{n - \mu(1-\beta)}{n(1-\beta)} < 1, \quad \text{for } \beta < \frac{\mu}{n+\mu}, \\ \alpha_2 &= \frac{n + 3\mu(1-\beta) + \sqrt{[n + 3\mu(1-\beta)]^2 - 8n\mu(1-\beta)}}{2n(1-\beta)} \ge 1. \end{aligned} \tag{2.50}$$

Putting $\alpha = 1$ in Theorem 2.3, we obtain the above corollary. □

Remark 2.6. Our results extend the results given by Obradović [5], and Obradović and Owa [6].

Theorem 2.7. *Let $0 < \mu < 1$, $0 \le \beta < 1$, $\mathrm{Re}\{c\} > -\mu$, and let*

$$\beta_n(\beta, \mu) = \begin{cases} \dfrac{(n-\mu)(1-\beta)|n + c - \mu|}{[n - \mu(1-\beta)]|c - \mu|}, & \text{if } \beta \ge \dfrac{\mu}{n+\mu}, \\ \dfrac{(n-\mu)\sqrt{1-2\beta}|n + c - \mu|}{\sqrt{n^2 + 2[\mu^2(1-\beta) - n\mu]}|c - \mu|}, & \text{if } \beta < \dfrac{\mu}{n+\mu}. \end{cases} \tag{2.51}$$

If $f(z) \in H_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < \lambda, \tag{2.52}$$

where $0 < \lambda \le \beta_n(\beta, \mu)$, and

$$F(z) = z \left[\frac{c - \mu}{z^{c-\mu}} \int_0^z \left(\frac{t}{f(t)} \right)^{\mu} t^{c-\mu-1} dt \right]^{-1/\mu}, \tag{2.53}$$

then $F(z) \in S^*(\beta)$.

Proof. Let

$$Q(z) = F'(z)\left(\frac{z}{F(z)}\right)^{1+\mu}. \tag{2.54}$$

Then from (2.52) and (2.53) we obtain

$$Q(z) + \frac{1}{c-\mu}Q'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + \lambda z. \tag{2.55}$$

Hence, by using Theorem 1 given by Hallenbeck and Ruscheweyh [7], we have that

$$Q(z) \prec 1 + \lambda_1 z, \qquad \lambda_1 = \frac{|c-\mu|\lambda}{|n+c-\mu|}z, \tag{2.56}$$

and the desired result easily follows from Corollary 2.5. □

For $c = \mu + 1$, we have the following corollary.

Corollary 2.8. *Let* $0 < \mu < 1$, $0 \leq \beta < 1$, *and let*

$$\beta_n(\beta,\mu) = \begin{cases} \dfrac{(n-\mu)(1-\beta)(n+1)}{[n-\mu(1-\beta)]}, & \text{if } 1 > \beta \geq \dfrac{\mu}{n+\mu}, \\ \dfrac{(n-\mu)\sqrt{1-2\beta}(n+1)}{\sqrt{n^2+2[\mu^2(1-\beta)-n\mu]}}, & \text{if } 0 \leq \beta < \dfrac{\mu}{n+\mu}. \end{cases} \tag{2.57}$$

If $f(z) \in H_n$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < \lambda, \tag{2.58}$$

where $0 < \lambda \leq \beta_n(\beta,\mu)$, *and*

$$F(z) = z\left[\frac{1}{z}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} dt\right]^{-1/\mu}, \tag{2.59}$$

then $F(z) \in S^*(\beta)$.

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