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Research Article

Stability of a Bi-Jensen Functional Equation II

Kil-Woung Jun,[1] Il-Sook Jung,[1] and Yang-Hi Lee[2]

[1] *Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea*
[2] *Department of Mathematics Education, Gongju National University of Education, Gongju 314-711, South Korea*

Correspondence should be addressed to Kil-Woung Jun, kwjun@cnu.ac.kr and Yang-Hi Lee, yanghi2@hanmail.net

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We investigate the stability of the bi-Jensen functional equation II $f((x+y)/2, z) - f(x, z) - f(y, z) = 0$, $2f(x, (y+z)/2) - f(x, y) - f(x, z) = 0$ in the spirit of Găvruta.

1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms. Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(xy), h(x)h(y)\big) < \delta \tag{1.1}$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \varepsilon \tag{1.2}$$

for all $x \in G_1$. The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] gave a generalization of Hyers' theorem by allowing the Cauchy difference to be controlled by a sum of powers like

$$\big\|h(x+y) - h(x) - h(y)\big\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big). \tag{1.3}$$

Găvruta [4] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function.

Throughout this paper, let X and Y be a normed space and a Banach space, respectively. A mapping $g : X \to Y$ is called a Cauchy mapping (resp., a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (resp., $2g((x+y)/2) = g(x) + g(y)$).

For a given mapping $f : X \times X \to Y$, we define

$$\begin{aligned} J_1 f(x,y,z) &:= 2f\left(\frac{x+y}{2}, z\right) - f(x,z) - f(y,z), \\ J_2 f(x,y,z) &:= 2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z) \end{aligned} \tag{1.4}$$

for all $x, y, z \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if f satisfies the functional equations $J_1 f = 0$ and $J_2 f = 0$.

Bae and Park [5] obtained the generalized Hyers-Ulam stability of a bi-Jensen mapping. Jun et al. [6] improved the results of Bae and Park in the spirit of Rassias.

In this paper, we investigate the stability of a bi-Jensen functional equation $J_1 f = 0$, $J_2 f = 0$ in the spirit of Găvruta.

2. Stability of a Bi-Jensen Functional Equation

Jun et al. [7] established the basic properties of a bi-Jensen mapping in the following lemma.

Lemma 2.1. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then, the following equalities hold:*

$$\begin{aligned} f(x,y) &= \frac{1}{4^n} f(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)\left(f(2^n x, 0) + f(0, 2^n y)\right) + \left(1 - \frac{1}{2^n}\right)^2 f(0,0) \\ &= \frac{f(2^n x, 2^n y)}{2^n} + \frac{2^n - 1}{2^{2n+1}}\left(f(2^n x, -2^n y) + f(-2^n x, 2^n y)\right) + \left(1 - \frac{1}{2^n}\right)^2 f(0,0), \\ f(x,y) &= \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1)\left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) - \left(2^{n+1} - 3 + \frac{1}{4^n}\right) f(0,0), \\ f(x,y) &= 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + (2^n - 4^n)\left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) + (2^n - 1)^2 f(0,0), \\ f(x,y) &= \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) f(0, 2^n y) + \left(1 - \frac{1}{2^n}\right)^2 f(0,0) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we have the stability of a bi-Jensen mapping.

Theorem 2.2. *Let* $\varphi, \psi : X \times X \times X \to [0,\infty)$ *be two functions satisfying*

$$\sum_{j=1}^{\infty} \frac{\varphi(2^j x, 2^j y, 2^j z)}{2^j} < \infty, \tag{2.2}$$

$$\sum_{j=1}^{\infty} \frac{\psi(2^j x, 2^j y, 2^j z)}{2^j} < \infty \tag{2.3}$$

for all $x, y, z \in X$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\begin{gathered} \left\| J_1 f(x,y,z) \right\| \le \varphi(x,y,z), \\ \left\| J_2 f(x,y,z) \right\| \le \psi(x,y,z) \end{gathered} \tag{2.4}$$

for all $x, y, z \in X$. *Then, there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\left\| f(x,y) - F(x,y) \right\| \le \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=1}^{\infty} \frac{\psi(0,0,2^j y) + \varphi(2^j x,0,0)}{2^j} \tag{2.5}$$

for all $x, y \in X$ *with* $F(0,0) = f(0,0)$, *where*

$$\Phi(x,y) = \frac{\varphi(x,0,y) + \psi(x,0,y)}{2} + \psi\left(\frac{x}{2},0,y\right) + \varphi\left(x,0,\frac{y}{2}\right) + \frac{3\psi(0,0,y) + 3\varphi(x,0,0)}{2}. \tag{2.6}$$

The mapping $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty} \left[\frac{f(2^j x, 2^j y)}{4^j} + \frac{f(2^j x, 0) + f(0, 2^j y)}{2^j} \right] + f(0,0) \tag{2.7}$$

for all $x, y \in X$.

Proof. Let f', f'', f''' be the maps defined by

$$\begin{gathered} f'(x,y) = f(x,y) - f(0,y), \\ f''(x,y) = f(x,y) - f(x,0), \\ f'''(x,y) = f(x,y) - f(x,0) - f(0,y) + f(0,0) \end{gathered} \tag{2.8}$$

for all $x, y \in X$. By (2.4), we get

$$\left\| \frac{f'(2^j x, 0)}{2^j} - \frac{f'(2^{j+1} x, 0)}{2^{j+1}} \right\| = \left\| \frac{J_1 f(2^{j+1} x, 0, 0)}{2^{j+1}} \right\| \leq \frac{\varphi(2^{j+1} x, 0, 0)}{2^{j+1}},$$
$$\left\| \frac{f''(0, 2^j y)}{2^j} - \frac{f''(0, 2^{j+1} y)}{2^{j+1}} \right\| = \left\| \frac{J_2 f(0, 0, 2^{j+1} y)}{2^{j+1}} \right\| \leq \frac{\varphi(0, 0, 2^{j+1} y)}{2^{j+1}},$$
$$\begin{aligned} &\left\| \frac{f'(2^j x, 2^j y)}{4^j} - \frac{f'(2^{j+1} x, 2^{j+1} y)}{4^{j+1}} \right\| \\ &\quad = \left\| \frac{J_1 f(2^{j+1} x, 0, 2^{j+1} y) + J_2 f(2^{j+1} x, 0, 2^{j+1} y) + 2 J_2 f(2^j x, 0, 2^{j+1} y)}{2 \cdot 4^{j+1}} \right. \\ &\qquad \left. + \frac{2 J_1 f(2^{j+1} x, 0, 2^j y) - 3 J_2 f(0, 0, 2^{j+1} y) - 3 J_1 f(2^{j+1} x, 0, 0)}{2 \cdot 4^{j+1}} \right\| \\ &\quad \leq \frac{\Phi(2^{j+1} x, 2^{j+1} y)}{4^{j+1}} \end{aligned} \tag{2.9}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \leq l < m)$,

$$\left\| \frac{f'(2^l x, 0)}{2^l} - \frac{f'(2^m x, 0)}{2^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\varphi(2^{j+1} x, 0, 0)}{2^{j+1}},$$
$$\left\| \frac{f''(0, 2^l y)}{2^l} - \frac{f''(0, 2^m y)}{2^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\varphi(0, 0, 2^{j+1} y)}{2^{j+1}}, \tag{2.10}$$
$$\left\| \frac{f'''(2^l x, 2^l y)}{4^l} - \frac{f'''(2^m x, 2^m y)}{4^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\Phi(2^{j+1} x, 2^{j+1} y)}{4^{j+1}}$$

for all $x, y \in X$. By (2.2) and (2.3), the sequences $\{(1/2^j)(f(2^j x, 0) - f(0,0))\}$, $\{(1/2^j)(f(0, 2^j y) - f(0,0))\}$, and $\{(1/4^j) f'(2^j x, 2^j y)\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{(1/2^j)(f(2^j x, 0) - f(0,0))\}$, $\{(1/2^j)(f(0, 2^j y) - f(0,0))\}$, and $\{(1/4^j) f'(2^j x, 2^j y)\}$ converge for all $x, y \in X$. Define $F_1, F_2, F_3 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j \to \infty} \frac{f(2^j x, 0)}{2^j},$$
$$F_2(x, y) := \lim_{j \to \infty} \frac{f(0, 2^j y)}{2^j}, \tag{2.11}$$
$$F_3(x, y) := \lim_{j \to \infty} \frac{f(2^j x, 2^j y)}{4^j}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.10), one can obtain the inequalities

$$\begin{gathered}\|f(x,0)-f(0,0)-F_1(x,y)\| \le \sum_{j=1}^{\infty}\frac{\varphi(2^j x,0,0)}{2^j},\\ \|f(0,y)-f(0,0)-F_2(x,y)\| \le \sum_{j=1}^{\infty}\frac{\varphi(0,0,2^j y)}{2^j},\\ \|f(x,y)-f(x,0)-f(0,y)+f(0,0)-F_3(x,y)\| \le \sum_{j=1}^{\infty}\frac{\Phi(2^j x,2^j y)}{4^j}\end{gathered} \tag{2.12}$$

for all $x, y \in X$. By (2.4) and the definitions of F_1 and F_2, we get

$$\begin{gathered}J_1F_1(x,y,z) = \lim_{j\to\infty}\frac{1}{2^j}J_1f(2^j x,2^j y,0)=0,\\ J_2F_1(x,y,z)=0,\\ J_1F_2(x,y,z)=0,\\ J_2F_2(x,y,z)=\lim_{j\to\infty}\frac{1}{2^j}J_2f(0,2^j y,2^j z)=0,\\ J_1F_3(x,y,z)=\lim_{j\to\infty}\frac{J_1f(2^j x,2^j y,2^j z)}{4^j}=0,\\ J_2F_3(x,y,z)=\lim_{j\to\infty}\frac{J_2f(2^j x,2^j y,2^j z)}{4^j}=0\end{gathered} \tag{2.13}$$

for all $x, y, z \in X$. So F is a bi-Jensen mapping satisfying (2.5), where F is given by

$$F(x,y)=F_1(x,y)+F_2(x,y)+F_3(x,y)+f(0,0). \tag{2.14}$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.5) with $F'(0,0) = f(0,0)$. By Lemma 2.1, we have

$$\begin{aligned}&\|F(x,y)-F'(x,y)\|\\ &\quad=\left\|\frac{(F-F')(2^n x,2^n y)}{4^n}+\left(\frac{1}{2^n}-\frac{1}{4^n}\right)\left((F-F')(2^n x,0)+(F-F')(0,2^n y)\right)\right\|\\ &\quad\le\left\|\frac{(F-f)(2^n x,2^n y)}{4^n}\right\|+\left\|\frac{(F-f)(0,2^n y)}{2^n}\right\|+\left\|\frac{(F-f)(2^n x,0)}{2^n}\right\|\\ &\qquad+\left\|\frac{(f-F')(2^n x,2^n y)}{4^n}\right\|+\left\|\frac{(f-F')(0,2^n y)}{2^n}\right\|+\left\|\frac{(f-F')(2^n x,0)}{2^n}\right\|\end{aligned}$$

$$\leq \sum_{j=n+1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{2^{j-1}} + \sum_{j=n+1}^{\infty} \frac{\psi(0,0,2^j y) + \varphi(2^j x, 0, 0)}{2^{j-2}}$$
$$+ \sum_{j=n+1}^{\infty} \frac{\Phi(2^j x, 0) + \Phi(0, 2^j y)}{2^{j-1}} + \frac{\varphi(0,0,0)}{2^{n-1}} + \frac{\psi(0,0,0)}{2^{n-1}} \tag{2.15}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Remark 2.3. Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be the functions defined by

$$\varphi(x,y,z) = \psi(x,y,z) := \frac{\varepsilon}{3} \tag{2.16}$$

for all $x, y, z \in X$. Let $f, F, F' : X \times X \to Y$ be the bi-Jensen mappings defined by

$$f(x,y) := 0, \qquad F(x,y) := \varepsilon, \qquad F'(x,y) := -\varepsilon \tag{2.17}$$

for all $x, y \in X$. Then, φ, ψ, and f satisfy (2.2), (2.3), (2.4) for all $x, y, z \in X$. In addition, f, F satisfy (2.5) for all $x, y \in X$ and f, F' also satisfy (2.5) for all $x, y \in X$. But we get $F' \neq F$. Hence, the condition $F(0,0) = f(0,0)$ is necessary to show that the mapping F is unique.

We have another stability result applying for several cases.

Theorem 2.4. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} 4^j \left(\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) + \psi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) \right) < \infty \tag{2.18}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.4) *for all $x, y, z \in X$. Then, there exists a unique bi-Jensen mapping $F : X \times X \to Y$ satisfying*

$$\| f(x,y) - F(x,y) \| \leq \sum_{j=0}^{\infty} \left(4^j \Phi\left(\frac{x}{2^j}, \frac{y}{2^j} \right) + 2^j \psi\left(0, 0, \frac{y}{2^j} \right) + 2^j \varphi\left(\frac{x}{2^j}, 0, 0 \right) \right) \tag{2.19}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\left[4^j f\left(\frac{x}{2^j},\frac{y}{2^j}\right) - (4^j - 2^j)\left(f\left(\frac{x}{2^j},0\right) + f\left(0,\frac{y}{2^j}\right)\right) + \left(2^j - 1\right)^2 f(0,0)\right] \tag{2.20}$$

for all $x, y \in X$.

Proof. By (2.4) and the similar method in Theorem 2.2, we define the maps $F_1, F_2, F_3 : X\times X \to Y$ by

$$\begin{aligned} F_1(x,y) &:= \lim_{j\to\infty} 2^j\left(f\left(\frac{x}{2^j},0\right) - f(0,0)\right),\\ F_2(x,y) &:= \lim_{j\to\infty} 2^j\left(f\left(0,\frac{y}{2^j}\right) - f(0,0)\right),\\ F_3(x,y) &:= \lim_{j\to\infty} 4^j\left[f\left(\frac{x}{2^j},\frac{y}{2^j}\right) - f\left(\frac{x}{2^j},0\right) - f\left(0,\frac{y}{2^j}\right) - f(0,0)\right] \end{aligned} \tag{2.21}$$

for all $x, y \in X$. By (2.4) and the definitions of F_1, F_2, and F_3, we get

$$\begin{aligned} J_1F_1(x,y,z) &= \lim_{j\to\infty} 2^j J_1 f\left(\frac{x}{2^j},\frac{y}{2^j},0\right) = 0,\\ J_2F_1(x,y,z) &= 0,\\ J_1F_2(x,y,z) &= 0,\\ J_2F_2(x,y,z) &= \lim_{j\to\infty} 2^j J_2 f\left(0,\frac{y}{2^j},\frac{z}{2^j}\right) = 0,\\ J_1F_3(x,y,z) &= \lim_{j\to\infty} 4^j\left[J_1 f\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) - J_1 f\left(\frac{x}{2^j},\frac{y}{2^j},0\right)\right] = 0,\\ J_2F_3(x,y,z) &= \lim_{j\to\infty} 4^j\left[J_2 f\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) - J_2 f\left(0,\frac{y}{2^j},\frac{z}{2^j}\right)\right] = 0 \end{aligned} \tag{2.22}$$

for all $x, y, z \in X$. By the similar method in Theorem 2.2, F is a bi-Jensen mapping satisfying (2.19), where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0). \tag{2.23}$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.19). Using Lemma 2.1, $\varphi(0,0,0) = \psi(0,0,0) = 0$, and $F'(0,0) = f(0,0) = F(0,0)$, we have

$$\begin{aligned}
&\left\|F(x,y) - F'(x,y)\right\| \\
&\quad = \left\| 4^n (F - F')\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + (2^n - 4^n)\left[(F - F')\left(\frac{x}{2^n}, 0\right) + (F - F')\left(0, \frac{y}{2^n}\right)\right]\right\| \\
&\quad \le 4^n \left[\left\|(F - f)\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + \left\|(f - F')\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + \left\|(F - f)\left(\frac{x}{2^n}, 0\right)\right\|\right. \\
&\qquad \left. + \left\|(f - F')\left(\frac{x}{2^n}, 0\right)\right\| + \left\|(F - f)\left(0, \frac{y}{2^n}\right)\right\| + \left\|(f - F')\left(0, \frac{y}{2^n}\right)\right\|\right] \\
&\quad \le 2\sum_{j=n}^{\infty} 4^j \left(\Phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) + 2\psi\left(0, 0, \frac{y}{2^j}\right) + 2\varphi\left(\frac{x}{2^j}, 0, 0\right) + \Phi\left(\frac{x}{2^j}, 0\right) + \Phi\left(0, \frac{y}{2^j}\right)\right)
\end{aligned} \tag{2.24}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Theorem 2.5. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \sum_{j=0}^{\infty} \frac{\varphi(2^j x, 2^j y, 2^j z)}{4^j} < \infty, \tag{2.25}$$

$$\sum_{j=0}^{\infty} 2^j \psi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \sum_{j=0}^{\infty} \frac{\psi(2^j x, 2^j y, 2^j z)}{4^j} < \infty \tag{2.26}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.4) *for all $x, y, z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \to Y$ satisfying*

$$\left\|f(x,y) - F(x,y)\right\| \le \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \left(\psi\left(0, 0, \frac{y}{2^j}\right) + \varphi\left(\frac{x}{2^j}, 0, 0\right)\right) \tag{2.27}$$

for all $x, y \in X$, where the mapping $F : X \times X \to Y$ is given by

$$\begin{aligned}
F(x,y) &:= \lim_{j\to\infty} \frac{1}{4^j}\left[f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)\right] \\
&\quad + \lim_{j\to\infty}\left[2^j\left(f\left(\frac{x}{2^j}, 0\right) + f\left(0, \frac{y}{2^j}\right)\right) - (2^{j+1} - 1) f(0,0)\right]
\end{aligned} \tag{2.28}$$

for all $x, y \in X$.

Proof. We can obtain F_1, F_2 as in Theorem 2.4 and F_3 as in Theorem 2.2. Hence, F is a bi-Jensen mapping satisfying (2.27), where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0). \tag{2.29}$$

□

Theorem 2.6. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions satisfying (2.2) and (2.26) for all $x,y,z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.4) for all $x,y,z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \to Y$ satisfying*

$$\|f(x,y) - F(x,y)\| \le \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \psi\left(0,0,\frac{y}{2^j}\right) + \sum_{j=1}^{\infty} \frac{\varphi(2^j x,0,0)}{2^j} \tag{2.30}$$

for all $x,y \in X$, where the mapping $F : X \times X \to Y$ is given by

$$\begin{aligned} F(x,y) := &\lim_{j\to\infty} \frac{1}{4^j}\left[f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)\right] \\ &+ \lim_{j\to\infty}\left[\frac{1}{2^j} f(2^j x, 0) + 2^j f\left(0,\frac{y}{2^j}\right) - (2^j - 1) f(0,0)\right] \end{aligned} \tag{2.31}$$

for all $x,y \in X$.

Theorem 2.7. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions satisfying (2.3) and (2.25) for all $x,y,z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2.4) for all $x,y,z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \to Y$ satisfying*

$$\|f(x,y) - F(x,y)\| \le \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j},0,0\right) \tag{2.32}$$

for all $x,y \in X$, where the mapping $F : X \times X \to Y$ is given by

$$\begin{aligned} F(x,y) := &\lim_{j\to\infty} \frac{1}{4^j}\left[f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)\right] \\ &+ \lim_{j\to\infty}\left[2^j f\left(\frac{x}{2^j},0\right) + \frac{1}{2^j} f(0, 2^j y) - (2^j - 1) f(0,0)\right] \end{aligned} \tag{2.33}$$

for all $x,y \in X$.

Applying Theorems 2.2–2.7, we easily get the following corollaries.

Corollary 2.8. *Let $0 < p(\neq 1,2)$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned} \|J_1 f(x,y,z)\| &\le \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right), \\ \|J_2 f(x,y,z)\| &\le \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \end{aligned} \tag{2.34}$$

for all $x, y, z \in X$*. Then, there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \le \varepsilon\left(\frac{7 \cdot 2^p + 2}{2|4 - 2^p|} + \frac{2^p}{|2 - 2^p|}\right)(\|x\|^p + \|y\|^p) \tag{2.35}$$

for all $x, y \in X$.

Proof. Applying Theorem 2.2 (Theorems 2.4 and 2.5, resp.) for the case $0 < p < 1$ ($2 < p$ and $1 < p < 2$, resp.), we obtain the desired result. □

Corollary 2.9. *Let* $0 < p,\ q < 2$ $(p, q \neq 1)$*, and* $\varepsilon > 0$*. Let* $f : X \times X \to Y$ *be a mapping such that*

$$\begin{aligned} \|J_1 f(x,y,z)\| &\le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p), \\ \|J_2 f(x,y,z)\| &\le \varepsilon(\|x\|^q + \|y\|^q + \|z\|^q) \end{aligned} \tag{2.36}$$

for all $x, y, z \in X$*. Then, there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\begin{aligned} &\|f(x,y) - F(x,y)\| \\ &\quad \le \varepsilon\left(\frac{3 \cdot 2^p}{4-2^p}\|x\|^p + \frac{2^q+2}{2(4-2^q)}\|x\|^q + \frac{2^p}{|2-2^p|}\|x\|^p + \frac{2^p+2}{2(4-2^p)}\|y\|^p + \frac{3 \cdot 2^q}{4-2^q}\|y\|^q + \frac{2^q}{|2-2^q|}\|y\|^q\right) \end{aligned} \tag{2.37}$$

for all $x, y \in X$.

Proof. Applying Theorems 2.6 and 2.7, we obtain the desired result. □

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