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Research Article

Penalty Algorithm Based on Conjugate Gradient Method for Solving Portfolio Management Problem

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A new approach was proposed to reformulate the biobjectives optimization model of portfolio management into an unconstrained minimization problem, where the objective function is a piecewise quadratic polynomial. We presented some properties of such an objective function. Then, a class of penalty algorithms based on the well-known conjugate gradient methods was developed to find the solution of portfolio management problem. By implementing the proposed algorithm to solve the real problems from the stock market in China, it was shown that this algorithm is promising.

1. Introduction

Portfolio management problem concerns itself with allocating one's assets among alternative securities to maximize the return of assets and to minimize the investment risk. The pioneer work on this problem was Markowitz's mean variance model [1], and the solution of his mean-variance methodology has been the center of the consequent research activities and forms the basis for the development of modern portfolio management theory. Commonly, the portfolio management problem has the following mathematical description.

Assume that there are n kinds of securities. The return rate of the jth security is denoted as R_j, $j = 1, \ldots, n$. Let x_j be the proportion of total assets devoted to the jth security, then it is obvious that

$$\sum_{j=1}^{n} x_j = 1. \tag{1.1}$$

In the real setting, due to uncertainty, the return rates R_j, $j = 1,2,\ldots n$, are random parameters. Hence, the total return of the assets

$$R(x) = \sum_{j=1}^{n} R_j x_j \tag{1.2}$$

is also random. In this situation, the risk of investment has to be taken into consideration. In the classical model, this risk is measured by the variance of $R(x)$. If $V \in R^{n\times n}$ is the covariance matrix of the vector $R = (R_1, R_2, \ldots, R_n)^T$, then the variance of $R(x)$ is

$$V(R(x)) = x^T V x. \tag{1.3}$$

Therefore, a portfolio management problem can be formulated into the following biobjectives programming problem:

$$\begin{aligned} \text{maximize} \quad & R(x) = R^T x, \\ \text{minimize} \quad & V(R(x)) = x^T V x, \\ \text{subject to} \quad & e^T x = 1, \\ & 1 \geq x \geq 0, \end{aligned} \tag{1.4}$$

where e is a vector of all ones. Up to our knowledge, almost all of the existing models of portfolio management problems evolved from the basic model (1.4).

Summarily, the past attempts on the portfolio management problems concentrated on two major issues. The first one is to propose new models. In this connection, some recent notable contributions mainly include the following:

(i) mean-absolute deviation model (Simaan [2]),

(ii) maximizing probability model (Williams [3]),

(iii) different types of mean-variance models (Best and Jaroslava [4], Konna and Suzuki [5] and Yoshimoto [6]),

(iv) min-max models (Cai et al. [7], Deng et al. [8]),

(v) interval programming models (Giove et al. [9], Ida [10], Lai et al. [11]),

(vi) fuzzy goal programming model (Parra et al. [12]),

(vii) admissible efficient portfolio selection model (Zhang and Nie [13]),

(viii) possibility approach model with highest utility score (Carlsson et al. [14]),

(ix) upper and lower exponential possibility distribution based model (Tanaka and Guo [15]),

(x) model with fuzzy probabilities (Huang [16, 17], Tanaka and Guo [15]).

The second issue is about the numerical solution algorithms for the distinct models. One of the fundamental ways is to reformulate (1.4) into a deterministic single-objective optimization problem. For example, in the researches of Best [4, 18], Pang [19], Kawadai and Konno [20], Perold [21], Sharpe [22], Szegö [23] and Yoshimoto [6], they assumed that the return of each security, the variance, and the covariances among them can be estimated by the investor prior to decision. Under this assumption, the problem (1.4) is a deterministic problem. Furthermore, if an aversion coefficient λ is introduced, the problem (1.4) can be transformed into the following standard quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & f(x) = -(1-\lambda)\mu^T x + \lambda x^T V x \\ \text{subject to} \quad & e^T x = 1, \\ & b \geq x \geq a, \end{aligned} \tag{1.5}$$

where $\mu \in R^n$ is the expected value vector of R, and $a, b \in R^n$ are two given vectors denoting the lower and the upper bounds of decision vector, respectively.

Obviously, if $\lambda = 0$ in (1.5), then it implies that the return is maximized regardless of the investment risk. On the other hand, if $\lambda = 1$, then the risk is minimized without consideration on the investment income. Increasing value of λ in the interval $[0, 1]$ indicates an increasingly weight of the invest risk, and vice versa.

For a fixed $\lambda \in (0, 1)$, it is noted that (1.5) is a quadratic programming problem. Since it has been shown that the matrix V is positive semidefinite, the problem (1.5) is a convex quadratic programming (CQP). For a CQP, there exist a lot of efficient methods to find its minimizers. Among them, active-set methods, interior-point methods, and gradient-projection methods have been widely used since the 1970s. For their detailed numerical performances, one can see [24–30] and the references therein. However, the efficiency of those methods seriously depends on the factorization techniques of matrix at each iteration, often exploiting the sparsity in V for a large-scale quadratic programming. So, from the viewpoint of smaller storage requirements and computation cost, the methods mentioned above must not be most suitable for solving the problem (1.5) if V is a dense matrix.

Fortunately, recent research shows that the conjugate gradient methods can remedy the drawback in factorization of Hessian matrix for an unconstrained minimization problem. At each conjugate-gradient iteration, it is only involved with computing the gradient of objective function. For details in this direction, see, for example, [31–34].

Motivated by the advantage of the conjugate gradient methods, the first aim of this paper is to reformulate problem (1.5) as an equivalent unconstrained optimization problem. Then, we are going to develop an efficient algorithm based on conjugate gradient methods to find its solution. The effectiveness of such algorithm will be tested by implementing the designed algorithm to solve some real problems from the stock market in China.

The lay out of the paper is as follows. Section 2 is devoted to the reformulation of the original constrained problem. Some features of the subproblem will be presented. Then, in Section 3, we are going to develop a penalty algorithm based on conjugate gradient methods. Section 4 will provide applications of the proposed algorithm. The last section concludes with some final remarks.

2. Reformulation

Firstly, for brevity, denote

$$c = (c_j)_{n\times 1} = -(1-\lambda)\mu, \qquad Q = (q_{ij})_{n\times n} = 2\lambda V. \tag{2.1}$$

Then, the problem (1.5) reads

$$\begin{aligned} \text{minimize} \quad & f(x) = c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} \quad & e^T x = 1, \\ & a \le x \le b. \end{aligned} \tag{2.2}$$

Since the covariance matrix V is symmetric positive semidefinite, Q also has such property. Thus, $f(x)$ is a convex function.

For the equality constraint $e^T x = 1$ and the inequality constraints $a \le x \le b$ in (2.2), we define a function $P : R^{n+1} \to R$, which is used to describe the constraints violation:

$$P(x;\theta) = \frac{\theta}{2}\left[\left(e^T x - 1\right)^2 + \|\min(x-a,0)\|^2 + \|\min(b-x,0)\|^2\right], \tag{2.3}$$

where $\theta > 0$ is called penalty parameter, and $\|\cdot\|$ denotes the 2-norm of vector. If x is a feasible point of problem (2.2), then

$$P(x;\theta) = 0. \tag{2.4}$$

Actually, the larger the absolute value of $P(x;\theta)$ is, the further x from the feasible region is.

The function $F : R^{n+1} \to R$,

$$\begin{aligned} F(x;\theta) = {} & c^T x + \frac{1}{2} x^T Q x \\ & + \frac{\theta}{2}\left[\left(e^T x - 1\right)^2 + \|\min(x-a,0)\|^2 + \|\min(b-x,0)\|^2\right] \end{aligned} \tag{2.5}$$

is said to be a penalty function of the problem (2.2). It is noted that F has the following features:

(1) F is a piecewise quadratic polynomial;

(2) F is piecewise continuously differentiable;

(3) If Q is positive semidefinite, then F is a piecewise convex quadratic function.

For example, let $n = 2$, and denote

$$\overline{c}(x;\theta) = \begin{cases} \begin{pmatrix} c_1 - \theta \\ c_2 - \theta \end{pmatrix}, & x \in D_1 = \left\{ (x_1, x_2)^T : a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta a_1 \\ c_2 - \theta \end{pmatrix}, & x \in D_2 = \left\{ (x_1, x_2)^T : x_1 < a_1, a_2 \le x_2 \le b_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta b_1 \\ c_2 - \theta \end{pmatrix}, & x \in D_3 = \left\{ (x_1, x_2)^T : x_1 > b_1, a_2 \le x_2 \le b_2 \right\}, \\ \begin{pmatrix} c_1 - \theta \\ c_2 - \theta - \theta a_2 \end{pmatrix}, & x \in D_4 = \left\{ (x_1, x_2)^T : a_1 \le x_1 \le b_1, x_2 < a_2 \right\}, \\ \begin{pmatrix} c_1 - \theta \\ c_2 - \theta - \theta b_2 \end{pmatrix}, & x \in D_5 = \left\{ (x_1, x_2)^T : a_1 \le x_1 \le b_1, x_2 > b_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta a_1 \\ c_2 - \theta - \theta a_2 \end{pmatrix}, & x \in D_6 = \left\{ (x_1, x_2)^T : x_1 < a_1, x_2 < a_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta a_1 \\ c_2 - \theta - \theta b_2 \end{pmatrix}, & x \in D_7 = \left\{ (x_1, x_2)^T : x_1 < a_1, x_2 > b_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta b_1 \\ c_2 - \theta - \theta a_2 \end{pmatrix}, & x \in D_8 = \left\{ (x_1, x_2)^T : x_1 > b_1, x_2 < a_2 \right\}, \\ \begin{pmatrix} c_1 - \theta - \theta b_1 \\ c_2 - \theta - \theta b_2 \end{pmatrix}, & x \in D_9 = \left\{ (x_1, x_2)^T : x_1 > b_1, x_2 > b_2 \right\}. \end{cases}$$

$$
\overline{Q}(x;\theta)=\begin{cases}\begin{pmatrix} q_{11}+\theta & q_{12}+\theta \\ q_{21}+\theta & q_{22}+\theta \end{pmatrix}, & x\in D_1, \\ \begin{pmatrix} q_{11}+2\theta & q_{12}+\theta \\ q_{21}+\theta & q_{22}+\theta \end{pmatrix}, & x\in D_2\cup D_3, \\ \begin{pmatrix} q_{11}+\theta & q_{12}+\theta \\ q_{21}+\theta & q_{22}+2\theta \end{pmatrix}, & x\in D_4\cup D_5, \\ \begin{pmatrix} q_{11}+2\theta & q_{12}+\theta \\ q_{21}+\theta & q_{22}+2\theta \end{pmatrix}, & x\in D_6\cup D_7\cup D_8\cup D_9. \end{cases} \tag{2.6}
$$

Then, $F(x;\theta)$ has the following more compact form:

$$
F(x;\theta)=\overline{c}_0+\overline{c}(x;\theta)^T x+\frac{1}{2}x^T\overline{Q}(x;\theta)x, \tag{2.7}
$$

where $\overline{c}_0$ is a constant scalar.

Next, we are going to present some properties of F.

Proposition 2.1. *Given n, the piecewise function F consists of*

$$
2^0C_n^0+2C_n^1+2^2C_n^2+\cdots+2^nC_n^n \tag{2.8}
$$

pieces.

Proposition 2.2. *Assume that $k\le n$. For any $\{\ i_1,i_2,\ldots,i_k\ \}\subseteq\{\ 1,2\ldots,n\}$, define a matrix*

$$
A_{i_1i_2\cdots i_k}=\begin{pmatrix} \ddots & & & & & & & \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & \ddots & & & & & \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ & & & & & & & \ddots \end{pmatrix}\begin{matrix} \vdots \\ \cdots \\ \cdots\cdots i_1 \\ \vdots \\ \cdots\cdots\cdots i_2 \\ \vdots \\ \vdots \\ \cdots\cdots\cdots i_k \\ \vdots \end{matrix}, \tag{2.9}
$$

where each of the other rows is $(1\quad 1\quad \cdots\quad 1)$. *Then, the following results hold.*

(1) $R(A_{i_1 i_2 \cdots i_k}) = k + 1$, *where* $R(\cdot)$ *denotes the rank of a matrix.*

(2) *For a fixed* k, *all matrices* $A_{i_1 i_2 \cdots i_k}$, *where* $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, *have the same eigenvalues and eigenvectors.*

(3) *When* $k < n-1$, *all matrices* $A_{i_1 i_2 \cdots i_k}$, *where* $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, *have nonnegative eigenvalue, and hence they are positive semidefinite. When* $k \geq n - 1$, *they are positive definite matrices.*

Proof. From the construction of F and the linear algebra theory, it is not difficult to prove the above two propositions. We omit it. □

In the following, we turn to state the relation between the global minimizer of F and that of the original problem (1.5).

Theorem 2.3. *For a given sequence* $\{\theta^k\}$, *suppose that* $\theta^k \to +\infty$ *as* $k \to +\infty$. *Let* $x^{(k)}$ *be an exact global minimizer of* $F(x; \theta^{(k)})$. *Then, every accumulation point* x^* *of* $\{x^{(k)}\}$ *is a solution of problem* (1.5).

Proof. Let $\overline{x}$ be a global solution of problem (1.5). Then, for any feasible point x, we have

$$f(\overline{x}) \leq f(x). \tag{2.10}$$

Since $x^{(k)}$ is an exact global minimizer of $F(x; \theta^{(k)})$ for the fixed θ^k, it follows that

$$F\left(x^{(k)}; \theta^{(k)}\right) \leq F\left(\overline{x}; \theta^{(k)}\right). \tag{2.11}$$

By definition, (2.11) is equivalent to

$$\begin{aligned} &f\left(x^{(k)}\right) + \frac{\theta^{(k)}}{2}\left[\left(e^T x^{(k)} - 1\right)^2 + \left\|\min\left(x^{(k)} - a, 0\right)\right\|^2 + \left\|\min\left(b - x^{(k)}, 0\right)\right\|^2\right] \\ &\quad \leq f(\overline{x}) + \frac{\theta^{(k)}}{2}\left[\left(e^T \overline{x} - 1\right)^2 + \|\min(\overline{x} - a, 0)\|^2 + \|\min(b - \overline{x}, 0)\|^2\right] \\ &\quad = f(\overline{x}), \end{aligned} \tag{2.12}$$

where the last equality is from the feasibility of $\overline{x}$. So, it is obtained that

$$\left(e^T x^{(k)} - 1\right)^2 + \left\|\min(x^{(k)} - a, 0)\right\|^2 + \left\|\min(b - x^{(k)}, 0)\right\|^2 \leq \frac{2}{\theta^{(k)}}\left[f(\overline{x}) - f\left(x^{(k)}\right)\right]. \tag{2.13}$$

Let x^* be an accumulation point of $\{x^{(k)}\}$. Without loss of generality, assume that

$$\lim_{k \to +\infty} x^{(k)} = x^*. \tag{2.14}$$

Then, by taking the limit of $k \to \infty$ on both sides of (2.13), we have

$$0 \le \left(e^T x^* - 1\right)^2 + \|\min(x^* - a, 0)\|^2 + \|\min(b - x^*, 0)\|^2 \\ \le \lim_{k \to +\infty} \frac{2}{\theta^{(k)}} \left[f(\overline{x}) - f\left(x^{(k)}\right) \right] = 0, \tag{2.15}$$

where the last equality follows from $\theta^{(k)} \to +\infty$. It follows that

$$e^T x^* = 1, \quad a \le x^* \le b. \tag{2.16}$$

Therefore, we have proved that x^* is a feasible point.

In the following, we prove that x^* is a global minimizer of problem (1.5).

Because

$$f(x^*) \le f(x^*) + \lim_{k \to +\infty} \frac{\theta^{(k)}}{2} \left[\left(e^T x^{(k)} - 1\right)^2 + \left\|\min\left(x^{(k)} - a, 0\right)\right\|^2 + \left\|\min\left(b - x^{(k)}, 0\right)\right\|^2 \right] \\ \le f(\overline{x}), \tag{2.17}$$

x^* is a global minimizer of f.

The desired result has been proved. □

Without difficulty, the following result can be proved.

Theorem 2.4. *Suppose that $\overline{x}$ is a solution of problem* (1.5)*. Then, $\overline{x}$ is a global minimizer of $F(\cdot;\theta)$ for any θ.*

Based on Theorems 2.3 and 2.4, we will develop an algorithm to search for a solution of problem (1.5) by solving a sequence of piecewise quadratical programming problems.

3. Penalty Algorithm Based on Conjugate Gradient Method

Among all methods for the unconstrained optimization problems, the conjugate gradient method is regarded as one of the most powerful approaches due to its smaller storage requirements and computation cost. Its priorities over other methods have been addressed in many literatures. For example, in [27, 32, 34–38], the global convergence theory and the detailed numerical performances on the conjugate gradient methods have been extensively investigated.

Since the number of the possible selected securities in the investment management is large and the matrix $\overline{Q}(x;\theta)$ may be dense, it is natura that the conjugate gradient method is selected to find the minimizer of F for some given θ. However, it is noted that (2.7) is not a classical quadratic function. The standard procedures of minimizing a quadratic function can not be directly employed. To develop a new algorithm, we first propose an rule of updating the coefficients in F.

Regarding the coefficients of the quadratic terms in

$$\frac{\theta}{2}\left[\|\min(x-a,0)\|^2+\|\min(b-x,0)\|^2\right], \tag{3.1}$$

we modify $Q=(q_{ij})$ according to the following update rule:

$$q_{ij}=\begin{cases} q_{ij}, & \text{if } i\neq j, \\ q_{ii}, & \text{if } i=j, \quad a_i\le x_i\le b_i, \\ (q_{ii}+\theta), & \text{if } i=j, \quad a_i>x_i \text{ or } x_i>b_i. \end{cases} \tag{3.2}$$

Regarding the coefficients of the linear terms in

$$\frac{\theta}{2}\left[\|\min(x-a,0)\|^2+\|\min(b-x,0)\|^2\right], \tag{3.3}$$

we modify $c=(c_i)$ according to the following update rule:

$$c_i=\begin{cases} c_i, & \text{if } a_i\le x_i\le b_i, \\ (c_i-\theta a_i), & \text{if } x_i<a_i, \\ (c_i-\theta b_i), & \text{if } x_i>b_i. \end{cases} \tag{3.4}$$

Define

$$\overline{Q}\triangleq Q+\theta ee^T, \qquad \overline{c}\triangleq c-\theta e. \tag{3.5}$$

The conjugate gradient method will be employed into an ordinary minimization of quadratic function:

$$\min F(x;\theta)=c_0+\overline{c}^T x+\frac{1}{2}x^T\overline{Q}x, \tag{3.6}$$

where θ is a given parameter. It is easy to see that

$$\nabla F_x(x;\theta)=\overline{Q}x+\overline{c}. \tag{3.7}$$

Although there exist several variants on the conjugate gradient method, the fundamental computing procedures for the solution of (3.6) include the following two steps.

(1) At the current iterate point $x^{(l)}$, determinate a search direction:

$$d^{(l)} = \begin{cases} -\left(\overline{Q}x^{(0)} + \overline{c}\right) & \text{for } l = 0, \\ -\left(\overline{Q}x^{(l)} + \overline{c}\right) + \beta_l d^{(l-1)} & \text{for } l \geq 1, \end{cases} \tag{3.8}$$

where β_l is chosen such that $d^{(l)}$ is a conjugate direction of $d^{(l-1)}$ with respect to the matrix $\overline{Q}$.

(2) Along the direction $d^{(l)}$, choose a step size α_l such that, at the new iterate point

$$x^{(l+1)} = x^{(l)} + \alpha_l d^{(l)}, \tag{3.9}$$

the absolute value of the function $F(\cdot;\theta)$ decreases sufficiently.

The following lemma presents a method to determine the search direction.

Lemma 3.1. *If*

$$\beta_l = \frac{\left(\overline{Q}x^{(l)} + \overline{c}\right)^T \overline{Q}d^{(l-1)}}{\left(d^{(l-1)}\right)^T \overline{Q}d^{(l-1)}}, \tag{3.10}$$

$$d^{(l)} = -\left(\overline{Q}x^{(l)} + \overline{c}\right) + \beta_l d^{(l-1)}, \tag{3.11}$$

then $d^{(l)}$ *in* (3.8) *is a conjugate direction of* $d^{(l-1)}$ *with respect to* $\overline{Q}$.

Proof. Owing to

$$\begin{aligned} \left(d^{(l)}\right)^T \overline{Q}d^{(l-1)} &= \left(-\left(\overline{Q}x^{(l)} + \overline{c}\right) + \beta_l d^{(l-1)}\right)^T \overline{Q}d^{(l-1)} \\ &= \left(-(\overline{Q}x^{(l)} + \overline{c}) + \frac{(\overline{Q}x^{(l)} + \overline{c})^T \overline{Q}d^{(l-1)}}{(d^{(l-1)})^T \overline{Q}d^{(l-1)}} d^{(l-1)}\right)^T \overline{Q}d^{(l-1)} \\ &= 0, \end{aligned} \tag{3.12}$$

the desired result is obtained. □

Actually, the formula (3.10) is called HS method.

In the case that the step size α_l is chosen by exact linear search along the direction $d^{(l)}$, that is,

$$\alpha_l = -\frac{\left(\overline{Q}x^{(l)} + \overline{c}\right)^T d^{(l)}}{\left(d^{(l)}\right)^T \overline{Q}d^{(l)}}, \tag{3.13}$$

we have the following global convergence theorem.

Theorem 3.2. *Let $x^{(0)}$ be an arbitrary initial vector. Let $\{x^{(l)}, l = 1, 2, \ldots\}$ be a sequence generated by the conjugate gradient algorithm defined by* (3.8)–(3.13). *Then, either*

$$\lim_{l \to +\infty} F\left(x^{(l)}; \theta\right) = -\infty, \tag{3.14}$$

or

$$\liminf \nabla_x F\left(x^{(l)}; \theta\right) = 0. \tag{3.15}$$

In particular, if x^* is an accumulation point of the sequence $\{x^{(l)} : l = 1, 2, \ldots\}$, then x^* is a global minimizer of $F(\cdot; \theta)$.

Remark 3.3. If β_l is computed by

$$\beta_l = \frac{\left\|\overline{Q}x^{(l)} + \overline{c}\right\|^2}{\left\|\overline{Q}x^{(l-1)} + \overline{c}\right\|^2}, \tag{3.16}$$

then the results in Theorem 3.2 still hold. Equation (3.16) is called FR method.

Based on the discussion above, we now come to develop a penalty algorithm based on conjugate gradient method in the last of this section.

Algorithm 1 (Penalty Algorithm Based on Conjugate Gradient Method).

Step 0 (Initialization). Given constant scalars $\theta > 1$, $\lambda \in [0, 1]$, $\delta > 0$, $\varepsilon > 0$ and ρ. Input the expected return vector μ, and compute Q and c. Choose an initial solution $x^{(0)}$. Set $k := 0$, $l := 0$, and $x^{(l)} := x^{(k)}$.

Step 1 (Reformulation). If

$$\left\|\overline{Q}x^{(l)} + \overline{c}\right\| \leq \varepsilon, \tag{3.17}$$

then set

$$x^{(k)} := x^{(l)}, \tag{3.18}$$

and go to Step 4; otherwise, go to Step 2.

Step 2 (Search Direction). Compute the search direction $d^{(l)}$ by (3.8) and (3.10).

Step 3 (Exact Line Search). Compute α_l by (3.13), and update

$$x^{(l)} := \left(x^{(l)} + \alpha_l d^{(l)}\right). \tag{3.19}$$

Return to Step 1.

Table 1: Numerical performance of Algorithm 1.

Problem	n	CPU of HS	CPU of FR	k	θ	$P(x^*;\theta^*)$
1	10	1″	1″	3	10^4	$2.5652e-005$
2	20	1″	8″	3	10^4	$1.3314e-005$
3	30	1″	5″	3	10^4	$5.9817e-005$
4	40	4″	5′8″	4	10^5	$1.1302e-005$
5	50	2″	11″	4	10^5	$2.4787e-005$
6	60	2″	46″	3	10^4	$8.5817e-005$
7	70	4″	12″	4	10^5	$1.7759e-005$
8	80	4″	1′43″	4	10^5	$2.8436e-005$
9	90	2″	1′26″	4	10^5	$9.6836e-005$
10	100	3″	6′48″	4	10^5	$4.8977e-005$

Step 4 (Feasibility Test). Check feasibility of $x^{(k)}$ in problem (2.2). If

$$P\left(x^{(k)};\theta\right) \le \delta, \tag{3.20}$$

the algorithm terminates; otherwise, go to Step 5.

Step 5 (Update). Set $l := 0$, $x^{(l)} := x^{(k)}$, $\theta := \rho\theta$. At the new iterate point $x^{(k)}$, modify the matrix Q and the vector c by (3.2) and (3.4), respectively. Set $k := k+1$, and return to Step 1.

Remark 3.4. (1) In Algorithm 1, the index k denotes the number of updating penalty parameter, and l denotes the number of iterations of conjugate gradient method for unconstrained subproblem (3.6).

(2) For some fixed θ, it is easy to see that the condition

$$P\left(x^{(k)};\theta\right) = \frac{\theta^{(k)}}{2}\left[\left(e^T x^{(k)} - 1\right)^2 + \left\|\min\left(x^{(k)} - a, 0\right)\right\|^2 + \left\|\min\left(b - x^{(k)}, 0\right)\right\|^2\right] \le \delta \tag{3.21}$$

implies that $x^{(k)}$ is feasible. From Theorem 2.3, it leads that $x^{(k)}$ is a global minimizer of the original problem (1.5) if $x^{(k)}$ is a global minimizer of problem (3.6).

4. Numerical Experiments

In this section, we are going to test the effectiveness of Algorithm 1. All the test problems come from the real stock market in China, in 2007. The computer procedures are implemented on MATLAB 6.5.

In our numerical experiments, the initial solution is chosen to satisfy

$$e^T x^{(0)} = 1, \tag{4.1}$$

Table 2: Optimal solutions of the ten problems.

	the optimal solution x^*
Problem 1	$x^*(3) = 0.6691; x^*(6) = 0.3311;$ other components of x^* are zeros
Problem 2	$x^*(18) = 1.0000;$ other components of x^* are zeros
Problem 3	$x^*(4) = 0.6985; x^*(28) = 0.3021;$ other components of x^* are zeros
Problem 4	$x^*(38) = 1.0000;$ other components of x^* are zeros
Problem 5	$x^*(38) = 0.8859; x^*(49) = 0.1142;$ other components of x^* are zeros
Problem 6	$x^*(14) = 1.0000;$ other components of x^* are zeros
Problem 7	$x^*(51) = 0.8486; x^*(61) = 0.1516;$ other components of x^* are zeros
Problem 8	$x^*(21) = 0.1175; x^*(78) = 0.8827;$ other components of x^* are zeros
Problem 9	$x^*(9) = 0.2215; x^*(25) = 0.6690;$ other components of x^* are zeros
Problem 10	$x^*(67) = 0.3086; x^*(79) = 0.5138; x^*(97) = 0.1797;$ other components of x^* are zeros

the bound vector a is a vector of all zeros, and b is a vector of all ones. We take the initial penalty parameter $\theta = 10$ and the aversion coefficient $\lambda = 0.5$. The tolerance of error is taken as

$$\varepsilon = 10^{-7}, \qquad \delta = 10^{-4}. \tag{4.2}$$

We implement Algorithm 1 to solve ten real problems. Each of them has a different dimension ranging from 10 to 100. In these problems, the expected return rates of each stock come from the monthly data in the stock market of China, in 2007. In Table 3 , we list the data used to form a real problem whose size of dimension is 30.

In Table 1, we report the numerical behavior of Algorithm 1 for all ten problems.

In Table 1, n is the dimensional size of each problem; the third and the forth columns report the CPU time when β_l is evaluated by HS method and FR method, respectively. k indicates the number of updating penalty parameter, θ is the penalty parameter, and $P(x^*;\theta^*)$ denotes the value of penalty term.

In Table 2, we list the obtained optimal solution for each problem.

5. Final Remarks

In this paper, the biobjectives optimization model of portfolio management was reformulated as an unconstrained minimization problem. We also presented the properties of the obtained quadratic function.

Table 3: The return rates collected from the stock market in China, 2007.

	1	2	3	4	5	6	7	8	9	10	11	12
No.1	0.4600	0.1900	0.1800	0.1130	0.2400	0.4600	0.4200	0.1500	0.1700	0.1140	0.2100	0.4200
No.2	0.6420	0.6560	0.6630	0.6990	0.6080	0.5420	0.6210	0.5550	0.6590	0.5810	0.6850	0.6210
No.3	0.1190	0.0590	0.2100	0.1100	0.1200	0.1190	0.1280	0.0580	0.2100	0.1100	0.1300	0.1280
No.4	0.0800	−0.0350	−0.2540	0.0830	0.0960	0.0800	0.1000	−0.0340	−0.2440	0.1100	0.1200	0.1000
No.5	0.7170	0.0940	0.4400	0.1430	0.6880	0.7170	0.7080	0.0190	0.3100	0.1470	0.6810	0.7080
No.6	0.0151	0.0105	0.0749	0.0081	0.0133	0.0151	0.0179	0.0083	0.0309	0.0090	0.1390	0.0179
No.7	0.2530	0.2430	0.3100	0.0480	0.1500	0.2530	0.2470	0.2440	0.3000	0.0480	0.1500	0.2470
No.8	0.3400	0.3006	0.3500	0.2280	0.4800	0.3400	0.3400	0.3026	0.3500	0.2270	0.4800	0.3400
No.9	0.0804	0.0579	0.1190	0.0420	0.0600	0.0804	0.0833	0.0597	0.1070	0.0430	0.0600	0.0833
No.10	0.0360	0.0230	0.0300	0.0140	0.0360	0.0360	0.0740	0.0210	0.0420	0.0150	0.0540	0.0740
No.11	0.0050	0.0130	0.0234	0.0020	0.0020	0.0050	0.0046	0.0130	0.0187	0.0020	0.0014	0.0046
No.12	0.2897	0.3100	0.4303	0.1153	0.1930	0.2897	0.2893	0.3200	0.3893	0.1151	0.1927	0.2893
No.13	0.7690	0.8060	0.9050	0.5340	0.4980	0.7690	0.7670	0.8090	0.8600	0.4350	0.5700	0.7670
No.14	0.0160	0.0110	0.0258	0.0006	0.0050	0.0160	−0.0230	0.0170	0.0171	−0.0242	−0.0220	−0.0230
No.15	0.0820	0.0370	0.0640	0.0200	0.0550	0.0820	0.0770	0.0370	0.0690	0.0200	0.0490	0.0770
No.16	0.4714	0.3607	0.6000	0.1275	0.2700	0.4714	0.4295	0.3585	0.5700	0.1275	0.2600	0.4295
No.17	0.2280	0.0950	0.1240	0.0820	0.1650	0.2280	0.2150	0.0970	0.1250	0.0812	0.1520	0.2150
No.18	0.0107	0.0053	0.0120	0.0040	0.0070	0.0107	0.0108	0.0416	−0.0400	0.0040	0.0120	0.0108
No.19	0.1400	0.2000	0.2400	0.0518	0.1100	0.1400	0.1400	0.2000	0.2300	0.0512	0.1200	0.1400
No.20	0.1500	0.1600	0.2100	0.0400	0.1100	0.1500	0.1500	0.1500	0.2000	0.0400	0.1100	0.1500
No.21	0.9850	1.3137	1.3200	0.2567	0.6100	0.9850	0.8130	1.3179	1.2900	0.1336	0.4300	0.8130
No.22	0.4717	0.4800	0.5730	0.0150	0.4271	0.4717	0.4285	0.2500	0.2338	0.0130	0.3816	0.4285
No.23	0.2500	0.1250	0.3300	0.0780	0.2000	0.2500	0.2400	0.1260	0.3400	0.0770	0.2000	0.2400
No.24	0.0310	0.0600	0.0880	0.0060	0.0300	0.0310	0.0240	0.0600	0.0950	0.0060	0.0190	0.0240
No.25	0.1190	0.0590	0.2100	0.1100	0.1200	0.1190	0.1280	0.0580	0.2100	0.1100	0.1300	0.1280
No.26	0.0110	0.0139	0.0140	0.0040	0.0090	0.0110	0.0020	0.0137	0.0080	0.0010	0.0010	0.0020
No.27	0.0100	0.0640	0.0515	−0.0350	0.0040	0.0100	−0.0700	0.0630	−0.0061	−0.0690	−0.0733	−0.0700
No.28	0.2680	0.2770	0.4320	0.0230	0.1900	0.2680	0.2680	0.2740	0.4320	0.0220	0.1880	0.2680
No.29	0.0061	−0.0060	−0.6742	0.0033	0.0050	0.0061	0.0131	−0.0220	−0.4474	0.0039	0.0099	0.0131
No.30	0.0600	−0.2200	−2.2300	0.0250	0.0400	0.0600	0.0450	−0.0100	−1.5700	0.0080	0.0300	0.0450

Regarding the features of the optimization models in portfolio management, a class of penalty algorithms based on the conjugate gradient method was developed. The numerical performance of the proposed algorithm in solving the real problems verifies its effectiveness.

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