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Research Article

An Inequality for the Beta Function with Application to Pluripotential Theory

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We prove in this paper an inequality for the beta function, and we give an application in pluripotential theory.

1. Introduction

A correspondence that started in 1729 between Leonhard Euler and Christian Goldbach was the dawn of the gamma function that is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{1.1}$$

(see, e.g., [1, 2]). One of the gamma function's relatives is the beta function, which is defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt\ . \tag{1.2}$$

The connection between these two Eulerian integrals is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ . \tag{1.3}$$

Since Euler's days the research of these special functions and their generalizations have had great impact on, for example, analysis, mathematical physics, and statistics. In this paper we prove the following inequality for the beta function.

Inequality A. For all $n \in \mathbb{N}$ and all $p \geq 0$ $(p \neq 0, p \neq 1)$ there exists a number $k > 0$ such that

$$k^{(n+p+np)/(n+p)} B(p+1, kn) > B(p+1, n). \tag{1.4}$$

If $p = 0$, then we have equality in (1.4), and if $p = 1$, then we have the opposite inequality for all $n \in \mathbb{N}$, $k > 0$.

In Section 3 we will give an application of Inequality A within the pluripotential theory.

2. Proof of Inequality A

A crucial tool in Lemma 2.2 is the following theorem.

Theorem 2.1. *Let $\psi(x) = \Gamma'(x)/\Gamma(x)$ be the digamma function. Then for $x > 0$ it holds that*

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2}, \qquad \psi''(x) > -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4}. \tag{2.1}$$

Proof. This follows from [3, Theorem 8] (see also [4, 5]). □

Lemma 2.2. *Let $\alpha : \mathbb{N} \times (0, +\infty) \to \mathbb{R}$ be a function defined by*

$$\alpha(n,p) = \frac{1}{n} + \frac{p}{n+p} + \psi(n) - \psi(n+p+1), \tag{2.2}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Then $\alpha(n,p) \neq 0$ for all $n \in \mathbb{N}$ and all $p > 0$ $(p \neq 1)$. Furthermore, $\alpha(n,1) = 0$ for all $n \in \mathbb{N}$.

Proof. Since $\psi(x+1) = \psi(x) + 1/x$, we have that $\alpha(n,1) = 0$, and

$$\alpha(n,p) = \frac{1}{n} + \frac{p-1}{n+p} + \psi(n) - \psi(n+p). \tag{2.3}$$

From the construction of α we also have that $\alpha(n,0) = 0$. By using (2.3) we get that

$$\frac{\partial \alpha}{\partial p} = \frac{n+1}{(n+p)^2} - \psi'(n+p) . \tag{2.4}$$

From Theorem 2.1 it follows that

$$\frac{\partial \alpha}{\partial p} < \frac{n+1}{(n+p)^2} - \frac{1}{n+p} - \frac{1}{2(n+p)^2} = \frac{1-2p}{2(n+p)^2}. \tag{2.5}$$

Thus,

$$\frac{\partial \alpha}{\partial p} < 0 \quad \text{for } p \in \left(\frac{1}{2}, +\infty\right). \tag{2.6}$$

Furthermore,

$$\frac{\partial^2 \alpha}{\partial p^2} = \frac{-2(n+1)}{(n+p)^3} - \psi''(n+p), \tag{2.7}$$

and since $\psi''(x) > -1/x^2 - 1/x^3 - 1/2x^4$ (Theorem 2.1), we get that

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial p^2} &< \frac{-2(n+1)}{(n+p)^3} + \frac{1}{(n+p)^2} + \frac{1}{(n+p)^3} + \frac{1}{2(n+p)^4} \\ &= \frac{-2n^2 - 2n - 2p + 2p^2 + 1}{2(n+p)^4}, \end{aligned} \tag{2.8}$$

which means that

$$\frac{\partial^2 \alpha}{\partial p^2} < 0 \quad \text{for } p \in (0,1). \tag{2.9}$$

From (2.6), (2.9), and the fact that $\alpha(n,1) = \alpha(n,0) = 0$, we conclude that $\alpha(n,p) \neq 0$ for all $n \in \mathbb{N}$ and all $p > 0$ $(p \neq 1)$. □

Proof of Inequality A.

Case 1 $(p = 0)$. The definition

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \tag{2.10}$$

yields that $B(a,1) = B(1,a) = 1/a$. Thus,

$$kB(1,kn) = \frac{1}{n} = B(1,n), \tag{2.11}$$

which is precisely the desired equality.

Case 2 $(p = 1)$. We will now prove that for all $k > 0$ it holds that

$$k^{(2n+1)/(n+1)}B(2,kn) \leq B(2,n). \tag{2.12}$$

Inequality (2.12) is equivalent to

$$k^{(2n+1)/(n+1)} \frac{1}{kn+1} \frac{1}{kn} \leq \frac{1}{n(n+1)}. \tag{2.13}$$

Hence, to complete this case we need to prove that for all $k > 0$ we have that

$$k^{n/(n+1)} \frac{1}{kn+1} \leq \frac{1}{n+1}. \tag{2.14}$$

Let $h : [0, +\infty) \to \mathbb{R}$ be defined by

$$h(k) = kn + 1 - k^{n/(n+1)} n - k^{n/(n+1)}. \tag{2.15}$$

To obtain (2.14) it is sufficient to prove that $h \geq 0$. The definition of h yields that

$$h(0) = 1, \qquad \lim_{k \to \infty} h(k) = +\infty \ , \qquad h'(k) = n\left(1 - k^{-1/(n+1)}\right). \tag{2.16}$$

Thus,

(a) h has a minimum point in $k = 1$;

(b) h is decreasing on $(0, 1)$;

(c) h is increasing on $(1, +\infty)$;

(d) $h(1) = 0$.

Thus, $h(k) \geq 0$ for $k \geq 0$.

Case 3 ($p > 0, p \neq 1$). Fix $n \in \mathbb{N}$. Let $F : (0, +\infty) \to \mathbb{R}$ be the function defined by

$$F(k) = k^{(n+p+np)/(n+p)} B(p+1, kn) - B(p+1, n). \tag{2.17}$$

This construction implies that F is continuously differentiable, and $F(1) = 0$. To prove this case it is enough to show that $F'(1) \neq 0$. By rewriting $B(p+1, kn)$ with (1.3) the function F can be written as

$$F(k) = k^{(n+p+np)/(n+p)} \frac{\Gamma(p+1)\Gamma(kn)}{\Gamma(kn+p+1)} - B(p+1, n), \tag{2.18}$$

and therefore we get that

$$\begin{aligned}F'(k) &= \Gamma(p+1)\Bigg(\frac{n+p+np}{n+p}k^{np/(n+p)}\frac{\Gamma(kn)}{\Gamma(kn+p+1)}\\&\qquad + nk^{(n+p+np)/(n+p)}\frac{\Gamma'(kn)\Gamma(kn+p+1)-\Gamma(kn)\Gamma'(kn+p+1)}{\Gamma^2(kn+p+1)}\Bigg)\\&= nk^{np/(n+p)}B(kn,p+1)\left(\frac{1}{n}+\frac{p}{n+p}+k(\psi(kn)-\psi(kn+p+1))\right).\end{aligned} \tag{2.19}$$

Thus

$$F'(1) = nB(n,p+1)\left(\frac{1}{n}+\frac{p}{n+p}+\psi(n)-\psi(n+p+1)\right), \tag{2.20}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. This proof is then completed by using Lemma 2.2. □

3. The Application

We start this section by recalling some definitions and needed facts. A *domain* is an open and connected set, and a bounded domain $\Omega \subseteq \mathbb{C}^n$ is *hyperconvex* if there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$; that is, for every $c < 0$ the level set $\{z \in \Omega : \varphi(z) < c\}$ is relatively compact in Ω. The geometric condition that our underlying domain should be hyperconvex is to ensure that we have a satisfying quantity of plurisubharmonic functions. By $\mathcal{E}_0(\Omega)$ we denote the family of all bounded plurisubharmonic functions φ defined on Ω such that

$$\lim_{z\to\xi}\varphi(z) = 0 \quad \text{for every } \xi \in \partial\Omega, \qquad \int_\Omega (dd^c\varphi)^n < +\infty, \tag{3.1}$$

where $(dd^c\cdot)^n$ is the complex Monge-Ampère operator. Next let $\mathcal{E}_p(\Omega)$, $p > 0$, denote the family of plurisubharmonic functions u defined on Ω such that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0$, that converges pointwise to u on Ω, as j tends to $+\infty$, and

$$\sup_{j\geq 1}\int_\Omega (-u_j)^p (dd^c u_j)^n = \sup_{j\geq 1} e_p(u_j) < +\infty. \tag{3.2}$$

If $u \in \mathcal{E}_p(\Omega)$, then $e_p(u) < +\infty$ ([6, 7]). It should be noted that it follows from [6] that the complex Monge-Ampère operator is well defined on $\mathcal{E}_p$. For further information about pluripotential theory and the complex Monge-Ampère operator we refer to [8, 9].

The convex cone $\mathcal{E}_p$ has applications in dynamical systems and algebraic geometry (see, e.g., [10, 11]). A fundamental tool in working with $\mathcal{E}_p$ is the following energy estimate (the proof can be found in [12], see also [6, 13, 14]).

Theorem 3.1. *Let $p > 0$, and $n \geq 1$. Then there exists a constant $D(n,p) \geq 1$, depending only on n and p, such that for any $u_0, u_1, \ldots, u_n \in \mathcal{E}_p$ it holds that*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq D(n,p) e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)} \ldots e_p(u_n)^{1/(p+n)}. \tag{3.3}$$

Moreover,

$$D(n,p) \leq \begin{cases} \left(\dfrac{1}{p}\right)^{n/(n-p)}, & \text{if } 0 < p < 1, \\ p^{pa(n,p)/(p-1)}, & \text{if } p > 1, \end{cases} \tag{3.4}$$

$D(n,1) = 1$ *and* $a(n,p) = (p+2)((p+1)/p)^{n-1} - (p+1)$. *If* $n = 1$, *then one follows [12] and interprets* (3.3) *as*

$$\int_\Omega (-u)^p \Delta v \leq D(1,p) \left(\int_\Omega (-u)^p \Delta u \right)^{p/(p+1)} \left(\int_\Omega (-v)^p \Delta v \right)^{1/(p+1)}. \tag{3.5}$$

If $D(n,p) = 1$ for all functions in $\mathcal{E}_p$, then the methods in [15] would immediately imply that the vector space $\mathcal{E}_p - \mathcal{E}_p$, with certain norm, is a Banach space. Furthermore, proofs in [15] (see also [6]) could be simplified, and some would even be superfluous. Therefore, it is important to know for which n, p the constant $D(n,p)$ is equal or strictly greater than one. With the help of Inequality A we settle this question. In Example 3.2, we show that there are functions such that, for all $n \in \mathbb{N}$ and all $p > 0$ $(p \neq 1)$, the constant $D(n,p)$, in (3.3), is strictly greater than 1.

Example 3.2. Let $B(0,1) \subset \mathbb{C}^n$ be the unit ball, and for $\alpha > 0$ set

$$u_\alpha(z) = |z|^{2\alpha} - 1. \tag{3.6}$$

Hence,

$$(dd^c u_\alpha)^n = n! 4^n \alpha^{n+1} |z|^{2n(\alpha-1)} d\lambda_n, \tag{3.7}$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. For $\beta > 0$ we then have that

$$\begin{aligned}\int_{B(0,1)} (-u_\alpha)^p \left(dd^c u_\beta\right)^n &= n!4^n \beta^{n+1} \int_{B(0,1)} \left(1 - |z|^{2\alpha}\right)^p |z|^{2n(\beta-1)} d\lambda_n \\ &= n!4^n \beta^{n+1} \int_{\partial B(0,1)} d\sigma_n \int_0^1 \left(1 - t^{2\alpha}\right)^p t^{2n(\beta-1)} t^{2n-1} dt \\ &= n!4^n \beta^{n+1} \sigma_n(\partial B(0,1)) \int_0^1 \left(1 - t^{2\alpha}\right)^p t^{2n\beta-1} dt \\ &= n!4^n \beta^{n+1} 2 \frac{\pi^n}{(n-1)!} \frac{1}{2\alpha} \int_0^1 (1-s)^p s^{n\beta/\alpha-1} ds \\ &= n(4\pi)^n \frac{\beta^{n+1}}{\alpha} \; B\left(p+1, \frac{\beta}{\alpha} n\right),\end{aligned} \tag{3.8}$$

where $d\sigma_n$ is the Lebesgue measure on $\partial B(0,1)$. If $\alpha = \beta$, then

$$\int_{B(0,1)} (-u_\alpha)^p \left(dd^c u_\alpha\right)^n = n(4\pi)^n \alpha^n \; B(p+1, n). \tag{3.9}$$

If we assume that $D(n,p) = 1$ in Theorem 3.1, then it holds that

$$n(4\pi)^n \frac{\beta^{n+1}}{\alpha} B\left(p+1, \frac{\beta}{\alpha} n\right) \le \left(n(4\pi)^n \alpha^n B(p+1,n)\right)^{p/(n+p)} \left(n(4\pi)^n \beta^n B(p+1,n)\right)^{n/(n+p)}. \tag{3.10}$$

Hence,

$$\left(\frac{\beta}{\alpha}\right)^{(n+p+np)/(n+p)} B\left(p+1, \frac{\beta}{\alpha} n\right) \le B(p+1,n) \quad \forall \alpha, \beta > 0. \tag{3.11}$$

In particular, if $\beta/\alpha = k$, then we get that

$$k^{(n+p+np)/(n+p)} B(p+1, kn) \le B(p+1,n). \tag{3.12}$$

This contradicts Inequality A. Thus, there are functions such that $D(n,p) > 1$ for all $n \in \mathbb{N}$ and all $p > 0$ $(p \neq 1)$.

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