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Research Article

Symmetrization of Functions and the Best Constant of 1-DIM L^p Sobolev Inequality

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The best constants $C(m,p)$ of Sobolev embedding of $W_0^{m,p}(-s,s)$ into $L^\infty(-s,s)$ for $m=1,2,3$ and $1<p$ are obtained. A lemma concerning the symmetrization of functions plays an important role in the proof.

1. Introduction

Let $W_0^{m,p}(-s,s)$ be a Sobolev space which consists of the functions whose derivatives up to $m-1$ vanish at $x=\pm s$, that is,

$$W_0^{m,p}(-s,s) := \left\{ u \mid u^{(i)} \in L^p(-s,s) \ \ (i=0,\dots,m),\ u^{(j)}(\pm s)=0 \ \left(j=0,\dots m-1\right) \right\}, \tag{1.1}$$

where $u^{(i)}$ denotes ith derivative of u in a distributional sense. The purpose of this paper is to investigate the best constant $C(m,p)$ of L^p Sobolev inequality

$$\left(\sup_{-s\le x\le s} |u(x)| \right) \ \le\ C \left(\int_{-s}^{s} \left| u^{(m)}(x) \right|^p dx \right)^{1/p}, \tag{1.2}$$

where $u \in W_0^{m,p}(-s,s)$ and $1<p$. The result is as follows.

Theorem 1.1. *The best constant of inequality* (1.2) *is*

$$C(1,p) = 2^{-(q-1)/q} s^{1/q}, \tag{1.3}$$

$$C(2,p) = \frac{s^{(q+1)/q}}{2^{(2q-1)/q}(q+1)^{1/q}}, \tag{1.4}$$

$$C(3,p) = 2^{(1-2q)/q}\left(\int_0^\alpha x^q(\alpha-x)^q dx + \int_\alpha^s x^q (x-\alpha)^q dx\right)^{1/q}, \tag{1.5}$$

where q satisfies $1/p+1/q=1$ and α in (1.5) *is the unique solution of the equation*

$$-\int_\alpha^s x^q(x-\alpha)^{q-1}dx + \int_0^\alpha x^q(\alpha-x)^{q-1}dx = 0, \tag{1.6}$$

satisfying $0<\alpha<s$.

For special values of p, the solution of (1.6) can be written explicitly, and $C(3,p)$ is expressed as follows.

Corollary 1.2. *One has*

$$C(3,2) = \frac{s^{5/2}}{2^{7/2}\cdot 5^{1/2}},$$
$$C\left(3,\frac{4}{3}\right) = \frac{s^{9/4}}{2^{11/4}\cdot 3^{1/2}\cdot 7^{1/2}}\left(-1-2^{1/3}5^{2/3}\left(13+3\sqrt{21}\right)^{-1/3}+2^{-1/3}5^{1/3}\left(13+3\sqrt{21}\right)^{1/3}\right)^{1/4}. \tag{1.7}$$

The best constants $C(1,p)$ and $C(2,p)$ were recently obtained by Oshime [1]. This paper gives an alternative proof which simplifies the derivation process of $C(1,p)$ and $C(2,p)$ and computes further constant $C(3,p)$. To compute these constants, the following lemma with respect to the symmetrization of functions plays an important role.

Lemma 1.3. *Let m be an integer satisfying $1\le m\le 3$ and let us define the functional S as follows:*

$$S(u) := \frac{\sup_{-s\le x\le s}|u(x)|}{\left(\int_{-s}^s |u^{(m)}(x)|^p dx\right)^{1/p}} \quad \left(u\in W_0^{m,p}(-s,s), u\neq 0\right). \tag{1.8}$$

Then, for an arbitrary $u\in W_0^{m,p}(-s,s)$, there exists an element u_* which belongs to the following sub-space $W_*^{m,p}$ of $W_0^{m,p}(-s,s)$:

$$W_*^{m,p} := \left\{u\in W_0^{m,p}(-s,s) \mid \max_{-a\le x\le a}|u(x)| = u(0),\ u(x)=u(|x|) \quad (-s\le x\le s)\right\} \tag{1.9}$$

such that

$$S(u) \leq S(u_*). \tag{1.10}$$

Remark 1.4. The proof of this lemma (see Section 3) does not apply to the case $m \geq 4$, since the relation

$$\tilde{u}^{(i)}(y-0) = \tilde{u}^{(i)}(y+0) \tag{1.11}$$

may fail to hold for $i \geq 4$ (see (3.4)–(3.6)). Hence the problem to obtain $C(m,p)$ for $m \geq 4$ (essentially) still remains.

The existence of the maximizer of S can be seen in the proof of Theorem 1.1, where we construct it concretely, but here we would like to see this briefly though the proof of the following lemma.

Lemma 1.5. *Assume that the assertion of Lemma 1.3 holds, then the maximizer of S exists.*

Proof. Let R be sufficiently large, and let W' and W'' be as

$$\begin{aligned} W' &:= \left\{ u \in W_0^{m,p}(-s,s) \mid u(0) = 1 \right\}, \\ W'' &:= \left\{ u \in W_0^{m,p}(-s,s) \mid \left\| u^{(m)} \right\|_{L^p(-s,s)} \leq R \right\}. \end{aligned} \tag{1.12}$$

From Lemma 1.3, we see that if the maximizer exists, it is the element of $W := W' \cap W''$. So, we can restrict the definition domain of S to W. Since W' is convex and strongly closed (by Sobolev inequality) in $W_0^{m,p}(-s,s)$, it is weakly closed. In addition, W'' is weakly compact, so W is also weakly compact. Moreover, $\|\cdot\|$ is weakly lower-semicontinuous in $W_0^{m,p}(-s,s)$, and hence $1/S$ attains its minimum in W. This proves the lemma. □

Finally, we introduce some studies related to the present paper. When $p = 2$ (Hilbertian Sobolev space case), the best constants for the embeddings of $W^{m,2}(a,b)$ into $L^\infty(a,b)$ for various conditions were treated in Richardson [2], Kalyabin [3], and [4–8]; see also references of these literatures. On the other hand, for the case $p \neq 2$, few literature seems to be available. In [9], Kametaka, Oshime, Watanabe, Yamagishi, Nagai, and Takemura obtained the best constant of (1.2) when u belongs to a subspace $W_P^{m,p}$ of $W^{m,p}(0,1)$ which consists of periodic functions

$$W_P^{m,p} := \left\{ u \in W^{m,p}(0,1) \mid u^{(i)}(0) = u^{(i)}(1) \ (0 \leq i \leq m-1), \int_0^1 u(x)dx = 0 \right\}, \tag{1.13}$$

as

$$C(m,p) = \begin{cases} \|b_m(\cdot)\|_{L^{p/(p-1)}(0,1)} & (m = 2n-1, n = 1,2,3,\ldots), \\ \|b_m(\alpha;\cdot)\|_{L^{p/(p-1)}(0,1)} & (m = 2n, n = 1,2,3,\ldots), \end{cases} \tag{1.14}$$

where $b_m(\cdot)$ is a Bernoulli polynomial, $b_m(\alpha;\cdot) = b_m(\cdot) - b_m(\alpha)$, and α is an unique solution of the equation

$$\int_0^{\alpha}\left((-1)^{m-1}b_m(\alpha;x)dx\right)^{1/(p-1)} - \int_{\alpha}^{1/2}((-1)^m b_m(\alpha;x)dx)^{1/(p-1)} = 0 \tag{1.15}$$

in the interval $0 < \alpha < 1/2$. Moreover, in [1], Oshime obtained the best constant $C(1,p)$ and $C(2,p)$. Other topics on this subject, especially the best constant of Sobolev inequalities on Riemannian manifolds, are seen in Hebey [10].

2. Proof of Theorem 1.1

First, we prepare the following lemma.

Lemma 2.1. *Let $u \in W_0^{m,p}(-s,s)$ and H be a function satisfying*

$$H(x) := \begin{cases} \dfrac{1}{2}, & (-s \le x < 0), \\ -\dfrac{1}{2}, & (0 \le x \le s), \end{cases} \quad (m=1)$$

$$\begin{cases} -\dfrac{1}{2}\left(x + \dfrac{s}{2}\right), & (-s \le x < 0), \\ \dfrac{1}{2}\left(x - \dfrac{s}{2}\right), & (0 \le x \le s), \end{cases} \quad (m=2) \tag{2.1}$$

$$\begin{cases} \dfrac{1}{4}x(x+\alpha), & (-s \le x < 0), \\ -\dfrac{1}{4}x(x-\alpha), & (0 \le x \le s), \end{cases} \quad (m=3)$$

then, it holds that

$$u(0) = \int_{-s}^{s} u^{(m)}(x)H(x)dx, \tag{2.2}$$

where α is an arbitrary constant (later, in Lemma 2.3, one fixes the value of α to satisfy (1.6)*).*

Proof. By integration by parts, we obtain the result. □

From Lemma 1.3, to obtain the best constant of (1.2), we can restrict the definition domain of the functional S to the nonzero element of $W_*^{m,p}$. Now, let $u \in W_*^{m,p}$, then from Lemma 2.1 and Hölder's inequality, we have for $m = 1, 2, 3$,

$$\sup_{-s \le x \le s} |u(x)| = u(0) \le \|H\|_{L^q(-s,s)} \left\| u^{(m)} \right\|_{L^p(-s,s)}. \tag{2.3}$$

So, we have

$$C(m,p) \le \|H\|_{L^q(-s,s)}, \tag{2.4}$$

and the equality holds in (2.4) if and only if there exists $u \in W_*^{m,p}$ satisfying

$$u^{(m)}(x) = (\operatorname{sgn} H(x))|H(x)|^{q/p} = (\operatorname{sgn} H(x))|H(x)|^{q-1}. \tag{2.5}$$

To confirm the existence of such u, we use the following lemmas.

Lemma 2.2. *Let $f \in C[-s,s]$ satisfy*

$$\int_{-s}^{s} x^i f(x)dx = 0 \quad (i = 0, 1, \ldots, m-1), \tag{2.6}$$

then the solution u of

$$u^{(m)}(x) = f(x) \tag{2.7}$$

exists in $W_0^{m,p}(-s,s)$.

Proof. Let us define u as

$$u(x) := \int_{-s}^{x} \frac{(x-t)^{m-1}}{(m-1)!} f(t)dt. \tag{2.8}$$

Clearly u is $C^m[-s,s]$, and

$$u^{(i)}(x) = \begin{cases} \displaystyle\int_{-s}^{x} \frac{(x-t)^{m-1-i}}{(m-1-i)!} f(t)dt & (0 \le i \le m-1), \\ f(x) & (i = m). \end{cases} \tag{2.9}$$

Moreover, from the assumption, it holds that $u^{(i)}(\pm s) = 0$ $(0 \le i \le m-1)$. □

Lemma 2.3. *The solution α of* (1.6) *uniquely exists.*

Proof. Let

$$f(\alpha) := -\int_{\alpha}^{s} x^q (x-\alpha)^{q-1} dx + \int_{0}^{\alpha} x^q (\alpha - x)^{q-1} dx. \tag{2.10}$$

Since

$$\begin{aligned} f'(\alpha) &= (q-1)\int_{\alpha}^{s} x^q (x-\alpha)^{q-2} dx + (q-1)\int_{0}^{\alpha} x^q (\alpha - x)^{q-2} dx > 0, \\ f(0) &= -\int_{0}^{s} x^{2q-1} dx < 0, \\ f(s) &= \int_{0}^{s} x^q (s-x)^{q-1} dx > 0 \end{aligned} \tag{2.11}$$

the assertion is proved. □

Using Lemma 2.2 and 5, we obtain the following lemma.

Lemma 2.4. *Let α be a solution of* (1.6) *(when $m = 3$), then the solution of* (2.5) *belongs to $W_*^{m,p}$ for $m = 1, 2, 3$.*

Proof. First, we prove the case $m = 2$ and 3. For simplicity, let us put $\widetilde{H}(x) = (\operatorname{sgn} H(x))|H(x)|^{q-1}$. Note that in these cases $\widetilde{H}$ is a continuous function on $[-s, s]$.

(1) In the case $m = 2$, $\widetilde{H}$ is an even function, so integration of $x\widetilde{H}(x)$ over the interval $[-s, s]$ vanishes. In addition,

$$\int_{s/2}^{s} \frac{1}{2^{q-1}} \left(x + \frac{s}{2}\right)^{q-1} dx - \int_{0}^{s/2} \frac{1}{2^{q-1}} \left(-x - \frac{s}{2}\right)^{q-1} dx = 0 \tag{2.12}$$

holds, so the integration of $\widetilde{H}(x)$ over the interval $[-s, s]$ also vanishes. Hence, by Lemma 2.2, the solution u of (2.5) belongs to $W_0^{m,p}(-s, s)$. Properties $\max_{-s \le x \le s}|u(x)| = u(0)$ and $u(x) = u(-x)$ follow from the fact that $\widetilde{H}$ is an even function and (2.12). So, we have proven $u \in W_*^{m,p}$.

(2) In the case $m = 3$, $\widetilde{H}$ is an odd function, so integrations of $\widetilde{H}(x)$ and $x^2\widetilde{H}(x)$ over the interval $[-s, s]$ vanish. Moreover, from (1.6), the integration of $x\widetilde{H}(x)$ over the interval $[-s, s]$ also vanishes. Hence, again by Lemma 2.2, we have the solution of (2.5) which belongs to $W_0^{m,p}(-s, s)$. The remaining part is the same as case (i).

(3) In the case $m = 1$, let us define u as follows:

$$u(x) = \begin{cases} \dfrac{1}{2^{q-1}}(x+s) & (-s \le x < 0), \\ -\dfrac{1}{2^{q-1}}(x-s) & (0 \le x \le s). \end{cases} \tag{2.13}$$

Clearly it holds that u satisfies (2.5) (a.e.), $u(\pm s) = 0$, $\max_{-s\le x\le s}|u(x)| = u(0)$ and $u(x) = u(-x)$. To see $u \in W^{1,p}(-s,s)$, let φ be an arbitrary element of $C_0^\infty(-s,s)$. Since

$$\begin{aligned}
&\int_{-s}^{s} u'(x)\varphi(x)dx \\
&= \int_{-s}^{0} -\frac{1}{2^{q-1}}(x+s)\varphi'(x)dx + \int_{0}^{s}\frac{1}{2^{q-1}}(x-s)\varphi'(x)dx \int_{-s}^{0}\frac{1}{2^{q-1}}\varphi(x)dx \\
&- \int_{0}^{s}\frac{1}{2^{q-1}}\varphi(x)dx,
\end{aligned} \tag{2.14}$$

we have, in a distributional sense,

$$u'(x) = \begin{cases} \dfrac{1}{2^{q-1}} & (-s \le x < 0), \\ -\dfrac{1}{2^{q-1}} & (0 \le x \le s). \end{cases} \tag{2.15}$$

Therefore, $u \in W^{1,p}(-s,s)$. This proves the case $m = 1$. □

Proof of Theorem 1.1. From Lemma 1.3, and the argument of this section, especially Lemma 2.4, $C(m,p) = \|H\|_{L^q(-s,s)}$. This proves Theorem 1.1. □

Proof of Corollary 1.2 . In this case, (1.6) becomes

$$35s^3 - 120\alpha s^2 + 140\alpha^2 s - 56\alpha^3 = 0. \tag{2.16}$$

So, we can explicitly solve this equation with respect to α. Substituting to (1.5), we obtain the result. □

3. Proof of Lemma 1.3

Now, all we have to do is to prove Lemma 1.3.

Proof of Lemma 1.3. To avoid the complexity of notation, in the follwings, we fix $s = 1$. Let u be an arbitrary element of $W_0^{m,p}(-1,1)$ $(1 \le m \le 3)$, and let

$$\max_{-1\le x\le 1}|u(x)| = u(y). \tag{3.1}$$

Here, we can assume $y \ge 0$, since if it does not, $u(x)$ can be replaced by $u(-x)$.

(i) There is the case

$$\int_{-1}^{y}\left|u^{(m)}(x)\right|^p dx \ge \int_{y}^{1}\left|u^{(m)}(x)\right|^p dx. \tag{3.2}$$

Let us define $\tilde{u}$ as

$$\tilde{u}(x) := \begin{cases} 0 & (-1 \le x < -1+2y), \\ u(2y-x) & (-1+2y \le x < y), \\ u(x) & (y \le x \le 1). \end{cases} \tag{3.3}$$

We have

$$\tilde{u}(y-0) = \tilde{u}(y+0) = u(y), \tag{3.4}$$

$$\tilde{u}'(y-0) = \tilde{u}'(y+0) = 0, \tag{3.5}$$

$$\tilde{u}''(y-0) = \tilde{u}''(y+0) = u''(y), \tag{3.6}$$

when $m = 3$, (3.4) and (3.5) when $m = 2$, and (3.4) when $m = 1$. Further, let us define

$$u_*(x) := \begin{cases} \tilde{u}(x+y) & (-1+y \le x \le 1-y), \\ 0 & (1-y < |x| \le 1). \end{cases} \tag{3.7}$$

Then $u_* \in W_*^{m,p}$, since $\max_{-1\le x\le 1}|u_*(x)| = u_*(0)$, $u_*^{(i)}(\pm 1) = 0 (0 \le i \le m-1)$. Moreover, from (3.2), we have $\|u_*^{(m)}\|_{L^p(-1,1)} \le \|u^{(m)}\|_{L^p(-1,1)}$. In addition, clearly $u_*(0) = \max_{-1\le x\le 1}|u_*(x)| = u(y) = \max_{-1\le x\le 1}|u(x)|$. So, in the case (i), we have proven the lemma.

(ii) There is the case

$$\int_{-1}^{y} \left|u^{(m)}(x)\right|^p dx < \int_{y}^{1} \left|u^{(m)}(x)\right|^p dx. \tag{3.8}$$

Let t be an element satisfying $0 \le t \le y$, and let

$$x' = -1 + \frac{t+1}{y+1}(x+1) = -1 + a(x+1). \tag{3.9}$$

Further, let $U(x')$ be

$$U(x') := u\left(\frac{x'+1}{a} - 1\right) \quad (-1 \le x' \le t). \tag{3.10}$$

So,

$$\partial_{x'}^m U(x') = a^{-m}\partial_x^m u(x)\big|_{x=(x'+1)/a-1} = a^{-m}u^{(m)}\left(\frac{x'+1}{a} - 1\right), \tag{3.11}$$

and hence we obtain

$$\int_{-1}^{t} \left|\partial_{x'}^{m} U(x')\right|^{p} dx' = a^{-mp} \int_{-1}^{t} \left|u^{(m)}\left(\frac{x'+1}{a} - 1\right)\right|^{p} dx'. \tag{3.12}$$

By putting $x' = -1 + a(x+1)$ the right-hand side of (3.12) becomes

$$a^{-mp+1} \int_{-1}^{y} \left|u^{(m)}(x)\right|^{p} dx. \tag{3.13}$$

Similarly, let us put

$$x' = 1 + \frac{1-t}{1-y}(x-1) = 1 + b(x-1) \tag{3.14}$$

and define $U(x')$ as

$$U(x') := u\left(\frac{x'-1}{b} + 1\right) \quad (t \leq x' \leq 1). \tag{3.15}$$

So,

$$\partial_{x'}^{m} U(x') = b^{-m} \partial_{x}^{m} u(x)|_{x=(x'-1)/b+1} = b^{-m} u^{(m)}\left(\frac{x'-1}{b} + 1\right), \tag{3.16}$$

and hence we obtain

$$\int_{t}^{1} \left|\partial_{x'}^{m} U(x')\right|^{p} dx' = b^{-mp} \int_{t}^{a} \left|u^{(m)}\left(\frac{x'-1}{b} + 1\right)\right|^{p} dx'. \tag{3.17}$$

By putting $x' = 1 + b(x-1)$ the right-hand side of (3.17) becomes

$$b^{-mp+1} \int_{y}^{1} \left|u^{(m)}(x)\right|^{p} dx. \tag{3.18}$$

Let us put

$$A := \int_{-1}^{y} \left|u^{(m)}(x)\right|^{p} dx, \quad B := \int_{y}^{1} \left|u^{(m)}(x)\right|^{p} dx \tag{3.19}$$

and define

$$f(t) := \left(\frac{t+1}{y+1}\right)^{-mp+1} A + \left(\frac{1-t}{1-y}\right)^{-mp+1} B. \tag{3.20}$$

Note that

$$f(y) = A + B = \left\| u^{(m)} \right\|_{L^p(-1,1)}^p. \tag{3.21}$$

The derivative of f is

$$f'(t) = (mp-1)\left\{ -\frac{1}{y+1}\left(\frac{t+1}{y+1}\right)^{-mp} A + \frac{1}{1-y}\left(\frac{1-t}{1-y}\right)^{-mp} B \right\}. \tag{3.22}$$

(a) The case

$$1 \leq \left(\frac{1-y}{1+y}\right)^{mp-1} \frac{B}{A}. \tag{3.23}$$

In this case, we have

$$\begin{aligned} & f'(t) \geq 0 \\ & \quad \longleftrightarrow (y+1)^{mp-1}(t+1)^{-mp} A \leq (1-y)^{mp-1}(1-t)^{-mp} B \\ & \quad \longleftrightarrow \left(\frac{1-t}{1+t}\right)^{mp} \leq \left(\frac{1-y}{1+y}\right)^{mp-1} \frac{B}{A}. \end{aligned} \tag{3.24}$$

Since

$$\max_{0 \leq t \leq y} \left(\frac{1-t}{1+t}\right)^{mp} = 1, \tag{3.25}$$

from the assumption (3.23), f is monotone increasing. So, we have

$$\min_{0 \leq t \leq y} f(t) = f(0) = \int_{-1}^{0} \left| U^{(m)}(x) \right|^p dx + \int_{0}^{1} \left| U^{(m)}(x) \right|^p dx. \tag{3.26}$$

But, from (3.23), it holds that

$$\int_{-1}^{0} \left| U^{(m)}(x) \right|^p dx = \left(\frac{1}{y+1}\right)^{-mp+1} A \leq \left(\frac{1}{1-y}\right)^{-mp+1} B = \int_{0}^{1} \left| U^{(m)}(x) \right|^p dx. \tag{3.27}$$

So, if we put

$$u_*(x) := \begin{cases} U(x) & (-1 \le x \le 0), \\ U(-x) & (0 \le x \le 1), \end{cases} \tag{3.28}$$

as case (i), we have $u_* \in W_*^{m,p}$ and $\|u_*^{(m)}\|_{L^p(-1,1)}^p \le f(0) \le f(y) = \|u^{(m)}\|_{L^p(-1,1)}^p$. In addition, $u_*(0) = \max_{-1\le x\le 1}|u_*(x)| = u(y) = \max_{-1\le x\le 1}|u(x)|$. So, we have proven the case (ii)-(a).

(b) The case

$$\left(\frac{1-y}{1+y}\right)^{mp-1}\frac{B}{A} < 1. \tag{3.29}$$

In this case, we have

$$f''(t) = mp(mp-1)\left\{\frac{1}{y+1}\left(\frac{t+1}{y+1}\right)^{-mp-1}A + \frac{1}{1-y}\left(\frac{1-t}{1-y}\right)^{-mp-1}B\right\} > 0. \tag{3.30}$$

Moreover

$$\begin{aligned} f'(0) &= (mp-1)\left\{-(1+y)^{mp-1}A + (1-y)^{mp-1}B\right\} < 0, \\ f'(y) &= (mp-1)\left(-\frac{1}{y+1}A + \frac{1}{1-y}B\right) > 0, \end{aligned} \tag{3.31}$$

since we have (3.29) and the assumption (3.8) ($A < B$), respectively. Therefore, there exists t_0 $(0 < t_0 < y)$ such that $f'(t_0) = 0$. Let us define the constant M as

$$M := \left(\frac{1-y}{1+y}\right)^{(mp-1)/mp}\left(\frac{B}{A}\right)^{1/mp}, \tag{3.32}$$

then $t_0 = (1-M)/(1+M)$. Now we have

$$\int_{t_0}^{1}\left|U^{(m)}(x)\right|^p dx < \int_{-1}^{t_0}\left|U^{(m)}(x)\right|^p dx, \tag{3.33}$$

since

$$\begin{aligned} &\int_{t_0}^{1}\left|U^{(m)}(x)\right|^p dx < \int_{-1}^{t_0}\left|U^{(m)}(x)\right|^p dx \\ &\longleftrightarrow \left(\frac{1-t_0}{1-y}\right)^{-mp+1}B < \left(\frac{t_0+1}{y+1}\right)^{-mp+1}A \longleftrightarrow \left(\frac{1-y}{1+y}\right)^{mp-1}\frac{B}{A} < \left(\frac{1-t_0}{1+t_0}\right)^{mp-1} \\ &\longleftrightarrow M^{mp} < M^{mp-1} \longleftrightarrow M < 1 \longleftrightarrow (3.29). \end{aligned} \tag{3.34}$$

Let us define $\widetilde{u}$ as

$$\widetilde{u}(x) := \begin{cases} 0 & (-1 \le x \le 2t_0 - 1), \\ U(2t_0 - x) & (2t_0 - 1 \le x \le t_0), \\ U(x) & (t_0 \le x \le 1), \end{cases}$$
$$u_* := \begin{cases} \widetilde{u}(x + t_0) & (-1 + t_0 \le x \le 1 - t_0), \\ 0 & (1 - t_0 < |x| \le 1). \end{cases} \tag{3.35}$$

Then, again as case (i), we have $u_* \in W_*^{m,p}$ and by (3.33), $\|u_*^{(m)}\|^p_{L^p(-1,1)} = \|\widetilde{u}^{(m)}\|^p_{L^p(-1,1)} \le \|U^{(m)}\|^p_{L^p(-1,1)} = f(t_0) < f(y) = \|u^{(m)}\|^p_{L^p(-1,1)}$. In addition, $u_*(0) = \max_{-1\le x\le 1}|u_*(x)| = u(y) = \max_{-1\le x\le 1}|u(x)|$. So, we have proven the case (ii)-(b). This completes the proof. □

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