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Research Article

Approximately n-Jordan Homomorphisms on Banach Algebras

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Let $n \in \mathbb{N}$, and let A, B be two rings. An additive map $h : A \to B$ is called n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. In this paper, we establish the Hyers-Ulam-Rassias stability of n-Jordan homomorphisms on Banach algebras. Also we show that (a) to each approximate 3-Jordan homomorphism h from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique 3-ring homomorphism near to f, (b) to each approximate n-Jordan homomorphism h between two commutative Banach algebras there corresponds a unique n-ring homomorphism near to f for all $n \in \{3, 4, 5\}$.

1. Introduction and Preliminaries

Let A, B be two rings (algebras). An additive map $h : A \to B$ is called n-Jordan homomorphism (n-ring homomorphism) if $h(a^n) = (h(a))^n$ for all $a \in A$, ($h(\prod_{i=1}^n a_i) = \prod_{i=1}^n h(a_i)$, for all $a_1, a_2, \ldots, a_n \in A$). If $h : A \to B$ is a linear n-ring homomorphism, we say that h is n-homomorphism. The concept of n-homomorphisms was studied for complex algebras by Hejazian et al. [1] (see also [2, 3]). A 2-Jordan homomorphism is a Jordan homomorphism, in the usual sense, between rings. Every Jordan homomorphism is an n-Jordan homomorphism, for all $n \geq 2$, (e.g., [4, Lemma 6.3.2]), but the converse is false, in general. For instance, let A be an algebra over $\mathbb{C}$ and let $h : A \to A$ be a nonzero Jordan homomorphism on A. Then, $-h$ is a 3-Jordan homomorphism. It is easy to check that $-h$ is not 2-Jordan homomorphism or 4-Jordan homomorphism. The concept of n-Jordan homomorphisms was studied by the first author [5]. A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$?" Such a problem was formulated by Ulam [6] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [7]. It gave rise to the *stability theory* for functional equations. Subsequently, various approaches to the problem have been introduced by several authors. For the history and various aspects of this theory we refer the reader to monographs [8–12]. Applying a theorem of Hyers [7], Rassias [13], and Gajda [14], Bourgin [15] proved the stability problem of ring homomorphisms between unital Banach algebras. Badora [16] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalizes the result of Bourgin. Recently, Miura et al. [17] proved the Hyers-Ulam-Rassias stability of Jordan homomorphisms. The stability problem of n-homomorphisms between Banach algebras, has been proved by the first author [18]. In this paper, we consider the stability, in the sense of Hyers-Ulam-Rassias, of n-Jordan homomorphisms on Banach algebras.

2. Main Result

By a following similar way as in [17], we obtain the next theorem.

Theorem 2.1. *Let A be a normed algebra, let B be a Banach algebra, let δ and ε be nonnegative real numbers, and let p,q be a real numbers such that $p,q<1$ or $p,q>1$, and that $q>0$. Assume that $f:A\to B$ satisfies the system of functional inequalities*

$$\left\|f(a+b)-f(a)-f(b)\right\|\leq\varepsilon\left(\|a\|^{p}+\|b\|^{p}\right), \tag{2.1}$$

$$\left\|f(a^{n})-f(a)^{n}\right\|\leq\delta\|a\|^{nq} \tag{2.2}$$

for all $a,b\in A$. Then, there exists a unique n-Jordan homomorphism $h:A\to B$ such that

$$\left\|f(a)-h(a)\right\|\leq\frac{2\varepsilon}{|2-2^{p}|}\|a\|^{p} \tag{2.3}$$

for all $a\in A$.

Proof. Put $s:=-\operatorname{sgn}(p-1)$, and $h(a):=\lim_{m}(1/2^{sm})f(2^{sm}a)$ for all $a\in A$. It follows from [13, 14] that h is additive map satisfies (2.3). We will show that h is n-Jordan homomorphism. Since $\lim_{m}2^{smn(q-1)}=0$, it follows from (2.2) that

$$\begin{aligned}
&\lim_{m}\frac{1}{2^{smn}}\left\{\left\|f\left((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a)\right)-\left(f(2^{sm}a)\right)^{n}\right\|\right\}\\
&\quad\leq\lim_{m}\frac{1}{2^{smn}}\delta\left\|2^{sm}a\right\|^{nq}\\
&\quad=\lim_{m}\left(2^{smn(q-1)}\right)\delta\|a\|^{nq}=0.
\end{aligned} \tag{2.4}$$

Hence, we have

$$\begin{aligned}h(a^n) &= \lim_m \frac{1}{2^{smn}} f(2^{smn}(a^n)) \\ &= \lim_m \frac{1}{2^{smn}} f((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a)) \\ &= \lim_m \frac{1}{2^{smn}} \{f((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a)) - (f(2^{sm}a))^n + (f(2^{sm}a))^n\} \\ &= (h(a))^n\end{aligned} \tag{2.5}$$

for all $a \in A$. In other words, h is n-Jordan homomorphism. The uniqueness property of h follows from [13, 14]. □

Theorem 2.2. *Let A be a normed algebra, let B be a Banach algebra, let δ and ε be nonnegative real numbers, and let p, q be real numbers such that $p < 1$ and $q < 0$. If $f : A \longrightarrow B$ is a mapping, with $f(0) = 0$, such that the inequalities* (2.1) *and* (2.2) *are valid. Then, there exists a unique n-Jordan homomorphism $h : A \longrightarrow B$ such that*

$$\|f(a) - h(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p \tag{2.6}$$

for all $a \in A$.

Proof. Assume that $\|0\|^p = \infty$. It follows from [13] that there exists an additive map $h : A \longrightarrow B$ satisfies (2.6). It suffices to show that $h(a^n) = h(a)^n$ for all $a \in A$. Since h is additive, we get $h(0) = 0$, and so the case $a = 0$ is omitted. Let $a \in A - \{0\}$ be arbitrarily. If $a^n \neq 0$, then the proof of Theorem 2.1 works well, and $h(a^n) = h(a)^n$. Thus we need to consider only the case $a^n = 0$. Since $f(0) = 0$, it follows from (2.2), that

$$\left\|\frac{1}{2^{mn}}(f(2^m a))^n\right\| \leq \frac{1}{2^{mn}}\delta\|2^m a\|^{nq} = 2^{mn(q-1)}\delta\|a\|^{nq}. \tag{2.7}$$

Hence, we have

$$\lim_m \frac{1}{2^{mn}}(f(2^m a))^n = 0. \tag{2.8}$$

On the other hand, we have

$$h(a) = \lim_m \frac{1}{2^m}(f(2^m a)). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$h(a)^n = \lim_m \left\{ \frac{1}{2^{mn}} (f(2^m a))^n \right\} = 0, \tag{2.10}$$

which proves $h(a^n) = 0 = h(a)^n$, whenever $a^n = 0$. This completes the proof. □

By [17, Theorem 1.1] and [5, Theorem 2.5], we have the following theorem.

Theorem 2.3. *Let $n \in \{2,3\}$ be fixed. Suppose A is a Banach algebra, which needs not to be commutative, and suppose B is a semisimple commutative Banach algebra. Then, each n-Jordan homomorphism $h : A \longrightarrow B$ is a n-ring homomorphism.*

Let $n \in \{2,3\}$ be fixed. As a direct corollary, we show that to each approximate n-Jordan homomorphism f from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique n-ring homomorphism near to f.

Corollary 2.4. *Let $n \in \{2,3\}$ be fixed. Suppose A is a Banach algebra, which needs not to be commutative, and suppose B is a semisimple commutative Banach algebra. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$. Assume that $f : A \longrightarrow B$ satisfies the system of functional inequalities*

$$\begin{gathered} \|f(a+b) - f(a) - f(b)\| \leq \varepsilon\left(\|a\|^p + \|b\|^p\right), \\ \|f(a^n) - f(a)^n\| \leq \delta\|a\|^{nq} \end{gathered} \tag{2.11}$$

for all $a, b \in A$. Then, there exists a unique n-ring homomorphism $h : A \longrightarrow B$ such that

$$\|f(a) - h(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p \tag{2.12}$$

for all $a \in A$.

Proof. It follows from Theorems 2.1, 2.2, and 2.3. □

Theorem 2.5. *Let $n \in \{3,4,5\}$ be fixed, A, B be two commutative algebras, and let $h : A \longrightarrow B$ be a n-Jordan homomorphism. Then, h is n-ring homomorphism.*

Proof. For $n = 3, 4$, (see [5, Theorem 2.2]). Now suppose $n = 5$. Then, h is additive and $h(a^5) = (h(a))^5$ for all $a \in A$. Replacing a by $a + b$ to get

$$\begin{aligned} &h\left(5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4\right) \\ &\quad = 5h\left(a^4\right)h(b) + 10h\left(a^3\right)h\left(b^2\right) + 10h\left(a^2\right)h\left(b^3\right) + 5h(a)h\left(b^4\right). \end{aligned} \tag{2.13}$$

Now, replacing a by $x+y$ in (2.13), we obtain that

$$\begin{aligned} &h\{5(x^4+4x^3y+6x^2y^2+4xy^3+y^4)b+10(x^3+3x^2y+3xy^2+y^3)b^2 \\ &\quad +10(x^2+2xy+y^2)b^3+5(x+y)b^4\} \\ &= 5\left[h(x)^4+4h(x)^3h(y)+6h(x)^2h(y)^2+4h(x)h(y)^3+h(y)^4\right]h(b) \\ &\quad +10\left[h(x)^3+3h(x)^2h(y)+3h(x)h(y)^2+h(y)^3\right]h(b)^2 \\ &\quad +10\left[h(x)^2+2h(x)h(y)+h(y)^2\right]h(b)^3+5\left[h(x)+h(y)\right]h(b)^4. \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), we get

$$\begin{aligned} &h\{(20x^3yb+30x^2y^2b+20xy^3b+30x^2yb^2+30xy^2b^2+20xyb^3)\} \\ &= 20h(x)^3h(y)h(b)+30h(x)^2h(y)^2h(b)+20h(x)h(y)^3h(b) \\ &\quad +30h(x)^2h(y)h(b)^2+30h(x)h(y)^2h(b)^2+20h(x)h(y)h(b)^3. \end{aligned} \tag{2.15}$$

By (2.15) it follows that

$$\begin{aligned} &h\{xyb\left[20(x^2+y^2+b^2)+30(xy+xb+yb)\right]\} \\ &= h(x)h(y)h(b)\left[20h(x)^2+30h(x)h(y)+20h(y)^2\right. \\ &\quad \left.+30h(x)h(b)+30h(y)h(b)+20h(b)^2\right]. \end{aligned} \tag{2.16}$$

Replacing b by $z+w$ in (2.16), we obtain

$$\begin{aligned} &h\{xyz\left[20(w^2+2zw)+30(xy+xw+yw)\right]+xyw\left[20(z^2+2zw)+30(xy+xz+yz)\right]\} \\ &= h(x)h(y)h(z)\left[20\left(h(w)^2+2h(z)h(w)\right)+30\left(h(x)h(y)+h(x)h(w)+h(y)h(w)\right)\right] \\ &\quad +h(x)h(y)h(w)\left[20\left(h(z)^2+2h(z)h(w)\right)+30\left(h(x)h(y)+h(x)h(z)+h(y)h(z)\right)\right]. \end{aligned} \tag{2.17}$$

Replacing z by $t+s$ in (2.17), we get

$$\begin{aligned} &h\{xy(t+s)\left[20(w^2+2(t+s)w)+30(xy+xw+yw)\right] \\ &\quad +xyw\left[20((t+s)^2+2(t+s)w)+30(xy+x(t+s)+y(t+s))\right]\} \\ &= h(x)h(y)h(t+s)\left[20\left(h(w)^2+2h(t+s)h(w)\right)\right. \\ &\qquad \left.+30\left(h(x)h(y)+h(x)h(w)+h(y)h(w)\right)\right] \\ &\quad +h(x)h(y)h(w)\left[20\left(h(t+s)^2+2h(t+s)h(w)\right)\right. \\ &\qquad \left.+30\left(h(x)h(y)+h(x)h(t+s)+h(y)h(t+s)\right)\right]. \end{aligned} \tag{2.18}$$

Hence, we get

$$\begin{aligned}
&h\left[40xywts - 30x^2y^2w - 30xys(xy + xw + yw)\right]\\
&\quad = 40h(x)h(y)h(w)h(t)h(s) - 30h(x)^2h(y)^2h(w)\\
&\qquad - 30h(x)h(y)h(s)\left[h(x)h(y) + h(x)h(w) + h(y)h(w)\right],\\
&h\left[40xywts - 30x^2y^2w - 30xyt(xy + xw + yw)\right]\\
&\quad = 40h(x)h(y)h(w)h(t)h(s) - 30h(x)^2h(y)^2h(w)\\
&\qquad - 30h(x)h(y)h(t)\left[h(x)h(y) + h(x)h(w) + h(y)h(w)\right].
\end{aligned} \tag{2.19}$$

By (2.19) it follows that

$$\begin{aligned}
&h\left[xy(s - t)(xy + xw + yw)\right]\\
&\quad = h(x)h(y)\left(h(s) - h(t)\right)\left[h(x)h(y) + h(x)h(w) + h(y)h(w)\right].
\end{aligned} \tag{2.20}$$

Replacing t by $-s$ in (2.20), we obtain

$$h\left[2xys(xy + xw + yw)\right] = 2h(x)h(y)h(s)\left[h(x)h(y) + h(x)h(w) + h(y)h(w)\right]. \tag{2.21}$$

Replacing y, w by x in (2.21), we get

$$h\left(x^4s\right) = h(x)^4h(s). \tag{2.22}$$

Replacing x by $x + y$ in above equality to get

$$h\left(\left(4x^3y + 6x^2y^2 + 4xy^3\right)s\right) = \left(4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3\right)h(s). \tag{2.23}$$

Replacing x by $x + z$ in (2.23), we obtain

$$\begin{aligned}
&h\left\{\left[\left(4x^3y + 6x^2y^2 + 4xy^3\right) + \left(4z^3y + 6z^2y^2 + 4zy^3\right) + 12\left(x^2zy + xz^2y + xzy^2\right)\right]s\right\}\\
&\quad = \left\{\left(4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3\right)\right.\\
&\qquad + \left(4h(z)^3h(y) + 6h(z)^2h(y)^2 + 4h(z)h(y)^3\right)\\
&\qquad \left. + 12\left(h(x)^2h(z)h(y) + h(x)h(z)^2h(y) + h(x)h(z)h(y)^2\right)\right\}h(s).
\end{aligned} \tag{2.24}$$

Combining (2.23) by (2.24), we get

$$h\left\{(xyz)(x + y + z)s\right\} = \left[\left(h(x)h(y)h(z)\right)\left(h(x) + h(y) + h(z)\right)\right]h(s). \tag{2.25}$$

Replacing z by $-x$ in (2.25) to obtain

$$h(x^2y^2s) = h(x)^2h(y)^2h(s), \tag{2.26}$$

replacing y by $y + w$ in (2.26), we get

$$h(x^2yws) = h(x)^2h(y)h(w)h(s). \tag{2.27}$$

Now, replace x by $x + t$ in (2.27), we obtain

$$h(xtyws) = h(x)h(t)h(y)h(w)h(s). \tag{2.28}$$

Hence, h is 5-ring homomorphism. □

Corollary 2.6. *Let $n \in \{3, 4, 5\}$ be fixed. Suppose A, B are commutative Banach algebras. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$, and $f(0) = 0$. Assume that $f : A \longrightarrow B$ satisfies the system of functional inequalities*

$$\begin{gathered} \| f(a+b) - f(a) - f(b) \| \leq \varepsilon(\|a\|^p + \|b\|^p), \\ \| f(a^n) - f(a)^n \| \leq \delta \|a\|^{nq} \end{gathered} \tag{2.29}$$

for all $a, b \in A$. Then, there exists a unique n-ring homomorphism $h : A \longrightarrow B$ such that

$$\| f(a) - h(a) \| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \tag{2.30}$$

for all $a \in A$.

Proof. It follows from Theorems 2.1, 2.2, and 2.5. □

References

[1] S. Hejazian, M. Mirzavaziri, and M. S. Moslehian, "n-homomorphisms," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 13–23, 2005.

[2] J. Bračič and M. S. Moslehian, "On automatic continuity of 3-homomorphisms on Banach algebras," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 30, no. 2, pp. 195–200, 2007.

[3] E. Park and J. Trout, "On the nonexistence of nontrivial involutive n-homomorphisms of C^*-algebras," to appear in *Transactions of the American Mathematical Society*.

[4] Th. W. Palmer, *Banach Algebras and the General Theory of *-Algebras. Vol. I: Algebras and Banach Algebra*, vol. 49 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, Mass, USA, 1994.

[5] M. Eshaghi Gordji, "n-Jordan homomorphisms," preprint.

[6] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1940.

[7] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[8] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[9] V. A. Faĭziev, Th. M. Rassias, and P. K. Sahoo, "The space of (ψ, γ)-additive mappings on semigroups," *Transactions of the American Mathematical Society*, vol. 354, no. 11, pp. 4455–4472, 2002.
[10] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser, Boston, Mass, USA, 1998.
[11] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
[12] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[13] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[14] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
[15] D. G. Bourgin, "Approximately isometric and multiplicative transformations on continuous function rings," *Duke Mathematical Journal*, vol. 16, pp. 385–397, 1949.
[16] R. Badora, "On approximate derivations," *Mathematical Inequalities & Applications*, vol. 9, no. 1, pp. 167–173, 2006.
[17] T. Miura, S.-E. Takahasi, and G. Hirasawa, "Hyers-Ulam-Rassias stability of Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, vol. 2005, no. 4, pp. 435–441, 2005.
[18] M. Eshaghi Gordji, "On approximate n-ring homomorphisms and n-ring derivations," preprint.