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Research Article

A Hilbert's Inequality with a Best Constant Factor

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We give a new Hilbert's inequality with a best constant factor and some parameters.

1. Introduction

If $p > 1$, $1/p + 1/q = 1$, $a_n, b_n > 0$ such that $\infty > \sum_{n=1}^{\infty} a_n^p > 0$ and $\infty > \sum_{n=1}^{\infty} b_n^q > 0$, then the well-known Hardy-Hilbert's inequality and its equivalent form are given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\left\{\sum_{n=1}^{\infty}a_n^p\right\}, \tag{1.2}$$

where the constant factors are all the best possible [1]. It attracted some attention in the recent years. Actually, inequalities (1.1) and (1.2) have many generalizations and variants. Equation (1.1) has been strengthened by Yang and others (including integral inequalities) [2–11].

In 2006, Yang gave an extension of [2] as follows.

If $p > 1, 1/p + 1/q = 1, r > 1, 1/r + 1/s = 1, t \in [0,1], (2 - \min\{r,s\})t + \min\{r,s\} \geq \lambda > (2 - \min\{r,s\})t$, such that $\infty > \sum_{n=1}^{\infty} n^{p(1-t+(2t-\lambda)/r)-1} a_n^p > 0$, $\infty > \sum_{n=1}^{\infty} n^{q(1-t+(2t-\lambda)/s)-1} b_n^q > 0$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}\right) \left\{\sum_{n=1}^{\infty} n^{p(1-t+(2t-\lambda)/r)-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} n^{q(1-t+(2t-\lambda)/s)-1} b_n^q\right\}^{1/q}. \tag{1.3}$$

$B(u,v)$ is the Beta function.

In 2007 Xie gave a new Hilbert-type Inequality [3] as follows.

If $p > 1, 1/p + 1/q = 1, a, b, c > 0, 2/3 \geq \mu > 0$, and the right of the following inequalities converges to some positive numbers, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(n^{\mu} + a^2 m^{\mu})(n^{\mu} + b^2 m^{\mu})(n^{\mu} + a^2 m^{\mu})} < \frac{\pi}{\mu(a+b)(b+c)(c+a)} \left\{\sum_{n=1}^{\infty} n^{(1-3\mu/2)p-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} n^{(1-3\mu/2)q-1} b_n^q\right\}^{1/q}. \tag{1.4}$$

The main objective of this paper is to build a new Hilbert's inequality with a best constant factor and some parameters.

In the following, we always suppose that

(1) $1/p + 1/q = 1, p > 1$, $a \geq 0$, $-1 < \alpha < 1$,

(2) both functions $u(x)$ and $v(x)$ are differentiable and strict increasing in $(n_0 - 1, \infty)$ and $(m_0 - 1, \infty)$, respectively,

(3) $u'(x)/u^{\alpha}(x), v'(x)/v^{\alpha}(x)$ are strictly increasing in $(n_0 - 1, \infty)$ and $(m_0 - 1, \infty)$, respectively. $\{u'_n v'_m / [(u_n^2 + 2a u_n v_m + v_m^2) u_n^{\alpha} v_m^{\alpha}]\}$ is strict decreasing on n and m,

(4) $u(n) = u_n, u(n_0) = u_0, u((n_0 - 1)^+) = v((m_0 - 1)^+) = 0, u(\infty) = \infty, v(\infty) = \infty, u'(n) = u'_n, v(m) = v_m, v(m_0) = v_0, v'(m) = v'_m$.

2. Some Lemmas

Lemma 2.1. *Define the weight coefficients as follows:*

$$W(p,m) := \sum_{n=n_0}^{\infty} \frac{1}{u_n^2 + 2au_n v_m + v_m^2} \cdot \frac{v_m^{\alpha(p-1)}}{u_n^{\alpha}} \cdot \frac{u_n'}{(v_m')^{p-1}}, \tag{2.1}$$

$$\omega(p,m) := \int_{n_o-1}^{\infty} \frac{1}{u^2(x) + 2au(x)v_m + v_m^2} \cdot \frac{v_m^{\alpha(p-1)}}{u^{\alpha}(x)} \cdot \frac{u'(x)}{(v_m')^{p-1}} dx, \tag{2.2}$$

$$\widetilde{W}(q,n) := \sum_{m=m_0}^{\infty} \frac{1}{u_n^2 + 2au_n v_m + v_m^2} \cdot \frac{u_n^{\alpha(q-1)}}{v_m^{\alpha}} \cdot \frac{v_m'}{(u_n')^{q-1}}, \tag{2.3}$$

$$\widetilde{\omega}(q,n) := \int_{m_0-1}^{\infty} \frac{1}{u_n^2 + 2au_n v(y) + v^2(y)} \cdot \frac{u_n^{\alpha(q-1)}}{v^{\alpha}(y)} \cdot \frac{v'(y)}{(u_n')^{q-1}} dy, \tag{2.4}$$

then

$$W(p,m) < \omega(p,m) = \frac{K v_m^{p\alpha-2\alpha-1}}{(v_m')^{p-1}}, \quad \widetilde{W}(q,n) < \widetilde{\omega}(q,n) = \frac{K u_n^{q\alpha-2\alpha-1}}{(u_n')^{q-1}}, \tag{2.5}$$

where

$$K = \int_0^{\infty} \frac{d\sigma}{(1+2a\sigma+\sigma^2)\sigma^{\alpha}}$$
$$= \begin{cases} \dfrac{\pi}{2\sqrt{a^2-1}\sin\alpha\pi}\left[\left(a+\sqrt{a^2-1}\right)^{\alpha} - \dfrac{1}{\left(a+\sqrt{a^2-1}\right)^{\alpha}}\right], & \text{if } \alpha \neq 0, a > 1, \\ |\alpha\pi| / \sin|\alpha\pi|, & \text{if } \alpha \neq 0, a = 1, \\ \pi csc\theta csc(\alpha\pi)\sin(\alpha\theta), & \text{if } \alpha \neq 0, a = \cos\theta,\ 0 < \theta < \pi, \\ \dfrac{1}{\sqrt{a^2-1}} \ln\left(a+\sqrt{a^2-1}\right), & \text{if } \alpha = 0, a > 1, \\ \theta csc\theta, & \text{if } \alpha = 0, a = \cos\theta,\ 0 < \theta < \dfrac{\pi}{2}, \\ 1, & \text{if } \alpha = 0, a = 1, \end{cases} \tag{2.6}$$

Proof. Let $f(z) = 1/[(1+2az+z^2)z^\alpha] = 1/[(z-z_1)(z-z_2)z^\alpha]$ then $K = (2\pi i/(1-e^{-2\alpha\pi i}))[\mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_2)]$ if $a>1$ then $z_1 = -a-\sqrt{a^2-1}, z_2 = -a+\sqrt{a^2-1}$

$$\begin{aligned} K &= \frac{2\pi i}{1-e^{-2\alpha\pi i}}\left[\frac{\left(-a-\sqrt{a^2-1}\right)^{-\alpha}}{-2\sqrt{a^2-1}} + \frac{\left(-a+\sqrt{a^2-1}\right)^{-\alpha}}{2\sqrt{a^2-1}}\right] \\ &= \frac{\pi}{2\sqrt{a^2-1}\sin\alpha\pi}\left[\left(a+\sqrt{a^2-1}\right)^{\alpha} - \frac{1}{\left(a+\sqrt{a^2-1}\right)^{\alpha}}\right], \end{aligned} \tag{2.7}$$

if $a=\cos\theta$ $(0<\theta<\pi/2)$, then $z_1 = -e^{i\theta}, z_2 = -e^{-i\theta}$

$$K = \frac{2\pi i}{1-e^{-2\alpha\pi i}}\left[\frac{1}{(-2i\sin\theta)(-e^{i\theta})^{\alpha}} + \frac{1}{(2i\sin\theta)(-e^{-i\theta})^{\alpha}}\right] = \pi\csc\theta\csc(\alpha\pi)\sin(\alpha\theta). \tag{2.8}$$

On the other hand, $W(p,m) < \omega(p,m)$. Setting $u(x) = v_m\sigma$, then $\omega(p,m) = Kv_m^{p\alpha-2\alpha-1}/(v'_m)^{p-1}$. Similarly, $\widetilde{W}(q,n) < \widetilde{\omega}(q,n) = Ku_n^{q\alpha-2\alpha-1}/(u'_n)^{q-1}$. □

Lemma 2.2. *For* $0<\varepsilon<\min\{p, p(1-\alpha)\}$ *one has*

$$\int_0^\infty \frac{d\sigma}{(1+2a\sigma+\sigma^2)\sigma^{\alpha+\varepsilon/p}} = K + o(1) \quad (\varepsilon \longrightarrow 0^+). \tag{2.9}$$

Proof.

$$\begin{aligned} &\left|\int_0^\infty \frac{1}{(1+2a\sigma+\sigma^2)\sigma^{\alpha+\varepsilon/p}}d\sigma - K\right| \\ &\le \left|\int_0^1 \frac{\sigma^{-\alpha}(1-\sigma^{-\varepsilon/p})}{1+2a\sigma+\sigma^2}d\sigma\right| + \left|\int_1^\infty \frac{\sigma^{-\alpha}(1-\sigma^{-\varepsilon/p})}{1+2a\sigma+\sigma^2}d\sigma\right| \\ &\le \left|\int_0^1 \sigma^{-\alpha}\left(1-\sigma^{-\varepsilon/p}\right)d\sigma\right| + \left|\int_1^\infty \sigma^{-2-\alpha}\left(1-\sigma^{-\varepsilon/p}\right)d\sigma\right| \\ &= \left|\frac{1}{1-\alpha} - \frac{1}{1-\alpha-\varepsilon/p}\right| + \left|\frac{1}{1+\alpha} - \frac{1}{1+\alpha+\varepsilon/p}\right| \longrightarrow 0 \quad \text{for } \varepsilon \longrightarrow 0^+. \end{aligned} \tag{2.10}$$

The lemma is proved. □

Lemma 2.3. *Setting $w_n = u_n$ (or v_m) and $w_0 = n_0$ (or m_0, resp.), then $k > 0$. $\{\tau'_w/\tau^k_w\}$ is strictly decreasing, then*

$$\sum_{w=w_0}^{N} \frac{\tau'_w}{\tau^k_w} = \int_{w_0}^{N} \frac{\tau'(x)}{\tau^k(x)} dx + A. \tag{2.11}$$

There $A \in (0, \tau'_{w_0}/\tau^k_{w_0})$, (for any N).

Proof. We have

$$\int_{w_0}^{N} \frac{\tau'(x)}{\tau^k(x)} dx < \sum_{w=w_0}^{N} \frac{\tau'_w}{\tau^k_w} = \frac{\tau'_{w_0}}{\tau^k_{w_0}} + \sum_{w=w_0+1}^{N} \frac{\tau'_w}{\tau^k_w} < \frac{\tau'_{w_0}}{\tau^k_{w_0}} + \int_{w_0}^{N} \frac{\tau'(x)}{\tau^k(x)} dx. \tag{2.12}$$

Easily, A had up bounded when $N \to \infty$. □

3. Main Results

Theorem 3.1. *If $a_n > 0$, $b_n > 0$, $0 < \sum_{n=1}^{\infty} v_m^{p\alpha-2\alpha-1}/(v'_m)^{p-1} a_n^p < \infty$, $0 < \sum_{n=n_0}^{\infty} u_n^{q\alpha-2\alpha-1}/(u'_n)^{q-1} b_n^q < \infty$, then*

$$\sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_m b_n}{u_n^2 + 2a u_n v_m + v_m^2} < K \left\{ \sum_{m=m_0}^{\infty} \frac{v_m^{p\alpha-2\alpha-1}}{(v'_m)^{p-1}} a_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{q\alpha-2\alpha-1}}{(u'_n)^{q-1}} b_n^q \right\}^{1/q}, \tag{3.1}$$

$$\sum_{n=n_0}^{\infty} u_n^{p\alpha+p-2\alpha-1} u'_n \left\{ \sum_{m=m_0}^{\infty} \frac{a_m}{u_n^2 + 2a u_n v_m + v_m^2} \right\}^p < K^p \sum_{m=m_0}^{\infty} \frac{v_m^{p\alpha-2\alpha-1}}{(v'_m)^{p-1}} a_m^p. \tag{3.2}$$

K is defined by Lemma 2.1.

Proof. By Hölder's inequality [12] and (2.5),

$$\begin{aligned} J :&= \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_m b_n}{u_n^2 + 2a u_n v_m + v_m^2} \\ &= \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{1}{u_n^2 + 2a u_n v_m + v_m^2} \cdot \frac{v_m^{\alpha/q}}{u_n^{\alpha/p}} \cdot \frac{(u'_n)^{1/p}}{(v'_m)^{1/q}} a_m \cdot \frac{u_n^{\alpha/p}}{v_m^{\alpha/q}} \cdot \frac{(v'_m)^{1/q}}{(u'_n)^{1/p}} b_n \\ &\le \left\{ \sum_{m=m_0}^{\infty} W(p,m) a_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \widetilde{W}(q,n) b_n^q \right\}^{1/q} \\ &< K \left\{ \sum_{m=m_0}^{\infty} \frac{v_m^{p\alpha-2\alpha-1}}{(v'_m)^{p-1}} a_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{q\alpha-2\alpha-1}}{(u'_n)^{q-1}} b_n^q \right\}^{1/q}, \end{aligned} \tag{3.3}$$

setting $b_n = u_n^{p\alpha-2\alpha+p-1} u_n' (\sum_{m=m_0}^{\infty} a_m/(u_n^2 + 2au_n v_m + v_m^2))^{p-1} > 0$. By(3.1) we have

$$\sum_{n=n_0}^{\infty} \frac{u_n^{q\alpha-2\alpha-1}}{(u_n')^{q-1}} b_n^q = \sum_{n=n_0}^{\infty} u_n^{p\alpha-2\alpha+p-1} u_n' \left(\sum_{m=m_0}^{\infty} \frac{a_m}{u_n^2 + 2au_n v_m + v_m^2} \right)^p$$
$$= J \leq K \left\{ \sum_{m=m_0}^{\infty} \frac{v_m^{p\alpha-2\alpha-1}}{(v_m')^{p-1}} a_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{q\alpha-2\alpha-1}}{(u_n')^{q-1}} b_n^q \right\}^{1/q}. \tag{3.4}$$

By $0 < \sum_{n=n_0}^{\infty} (u_n^{q\alpha-2\alpha-1}/(u_n')^{q-1}) b_n^q < \infty$ and (3.4) taking the form of strict inequality, we have (3.1). By Hölder's inequality[12], we have

$$J = \sum_{n=n_0}^{\infty} \left\{ u_n^{-\alpha+2\alpha/q+1/q} (u_n')^{-1+1/q} \sum_{m=m_0}^{\infty} \frac{a_m}{u_n^2 + 2au_n v_m + v_m^2} \right\} \left(u_n^{\alpha-2\alpha/q-1/q} b_n \right) (u_n')^{1-1/q}$$
$$\leq \left\{ \sum_{n=n_0}^{\infty} u_n^{p\alpha-2\alpha+p-1} u_n' \left[\sum_{m=m_0}^{\infty} \frac{a_m}{u_n^2 + 2au_n v_m + v_m^2} \right]^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{q\alpha-2\alpha-1}}{(u_n')^{q-1}} b_n^q \right\}^{1/q}. \tag{3.5}$$

as $0 < \{\sum_{n=n_0}^{\infty} (u_n^{q\alpha-2\alpha-1}/(u_n')^{q-1}) b_n^q\}^{1/q} < \infty$. By (3.2), (3.5) taking the form of strict inequality, we have (3.1). □

Theorem 3.2. *If $\alpha = 0$, then both constant factors, K and K^p of* (3.1) *and* (3.2), *are the best possible.*

Proof. We only prove that K is the best possible. If the constant factor K in (3.1) is not the best possible, then there exists a positive H (with $H < K$), such that

$$J < H \left\{ \sum_{m=m_0}^{\infty} \frac{v_m^{-1}}{(v_m')^{p-1}} a_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{-1}}{(u_n')^{q-1}} b_n^q \right\}^{1/q}. \tag{3.6}$$

For $0 < \varepsilon < min\{p, q\}$, setting $\widetilde{a}_m = v_m^{-\varepsilon/p} v_m'$, $\widetilde{b}_n = u_n^{-\varepsilon/q} u_n'$, then

$$\left\{ \sum_{m=m_0}^{\infty} \frac{v_m^{-1}}{(v_m')^{p-1}} \widetilde{a}_m^p \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n^{-1}}{(u_n')^{q-1}} \widetilde{b}_n^q \right\}^{1/q} = \left\{ \sum_{m=m_0}^{\infty} \frac{v_m'}{v_m^{1+\varepsilon}} \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{u_n'}{u_n^{1+\varepsilon}} \right\}^{1/q}. \tag{3.7}$$

On the other hand ($u(x) = \sigma v(y)$ and $v(y) = \tau$),

$$\begin{aligned}
\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u_n^{-\varepsilon/p}u_n' v_m^{-\varepsilon/q}v_m'}{u_n^2+2au_nv_m+v_m^2} &> \int_{m_0}^{\infty}\left(\int_{n_0}^{\infty}\frac{u^{-\varepsilon/p}(x)u'(x)dx}{u^2(x)+2au(x)v(y)+v^2(y)}\right)v(y)^{-\varepsilon/q}v'(y)dy \\
&= \int_{m_0}^{\infty}\left(\int_{u_0/v(y)}^{\infty}\frac{\sigma^{-\varepsilon/p}d\sigma}{\sigma^2+2a\sigma+1}\right)v(y)^{-1-\varepsilon}v'(y)dy \\
&= \int_{v_0}^{\infty}\left(\int_{0}^{\infty}\frac{\sigma^{-\varepsilon/p}d\sigma}{\sigma^2+2a\sigma+1}\right)\tau^{-1-\varepsilon}d\tau \\
&\quad - \int_{v_0}^{\infty}\left(\int_{0}^{u_0/\tau}\frac{\sigma^{-\varepsilon/p}d\sigma}{\sigma^2+2a\sigma+1}\right)\tau^{-1-\varepsilon}d\tau \qquad (3.8)\\
&\geq (K+o(1))\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau - \int_{v_0}^{\infty}\tau^{-1}\int_{0}^{u_0/\tau}\left(\sigma^{-\varepsilon/p}d\sigma\right)d\tau \\
&= (K+o(1))\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau - \frac{u_0^{1-\varepsilon/p}v_0^{-1+\varepsilon/p}}{(1-\varepsilon/p)^2} \\
&= (K+o(1))\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau - O(1).
\end{aligned}$$

By (3.6), (3.7), (3.8), and Lemma 2.3, we have

$$(K+o(1)) - \frac{O(1)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau} < H\left\{\frac{\sum_{m=m_0}^{\infty}\left(v_m'/v_m^{1+\varepsilon}\right)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau}\right\}^{1/p}\left\{\frac{\sum_{n=n_0}^{\infty}\left(u_n'/u_n^{1+\varepsilon}\right)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau}\right\}^{1/q}, \qquad (3.9)$$

$$(K+o(1)) - \frac{O(1)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau} < H\left\{1+\frac{\overline{O}(1)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau}\right\}^{1/p}\left\{1+\frac{\widetilde{O}(1)}{\int_{v_0}^{\infty}\tau^{-1-\varepsilon}d\tau}\right\}^{1/q}. \qquad (3.10)$$

We have $K \leq H$, $(\varepsilon \to 0^+)$. This contracts the fact that $H < K$. □

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