

Research Article

On the Identities of Symmetry for the Generalized Bernoulli Polynomials Attached to χ of Higher Order

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We give some interesting relationships between the power sums and the generalized Bernoulli numbers attached to χ of higher order using multivariate p -adic invariant integral on \mathbb{Z}_p .

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1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols \mathbb{Z} , \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p denote the ring of rational integers, the ring of p -adic integers, the field of p -adic rational numbers, and the completion of algebraic closure of \mathbb{Q}_p , respectively. Let \mathbb{N} be the set of natural numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-v_p(p)} = p^{-1}$ (see [1–24]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on \mathbb{Z}_p . Let d be a fixed positive integer. For $n \in \mathbb{N}$, let

$$\begin{aligned} X &= X_d = \lim_{\leftarrow N} \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, & X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For $f \in \text{UD}(X)$, the p -adic invariant integral on X is defined as

$$I(f) = \int_X f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{dp^N} \sum_{x=0}^{dp^N-1} f(x) \quad (1.2)$$

(see [11–19]). From (1.2), we note that

$$I(f_1) = I(f) + f'(0), \quad (1.3)$$

where $f'(0) = (df(x)/dx)|_{x=0}$ and $f_1(x) = f(x+1)$. Let $f_n(x) = f(x+n)$ ($n \in \mathbb{N}$). Then we can derive the following equation from (1.3):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \quad (1.4)$$

(see [1–11]). Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to χ are defined as

$$\sum_{a=0}^{d-1} \frac{\chi(a)e^{at}}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad (1.5)$$

and the generalized Bernoulli numbers attached to χ , $B_{n,\chi}$, are defined as $B_{n,\chi} = B_{n,\chi}(0)$ (see [1–20, 25]). The purpose of this paper is to derive some identities of symmetry for the generalized Bernoulli polynomials attached to χ of higher order.

2. Symmetric Properties for the Generalized Bernoulli Polynomials of Higher Order

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then we note that

$$\int_X \chi(x) e^{xt} dx = \frac{t \sum_{i=0}^{d-1} \chi(i) e^{it}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad (2.1)$$

where $B_{n,\chi}$ are the n th generalized Bernoulli numbers attached to χ (see [7, 9, 15, 25]). Now we also see that the generalized Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y) e^{(x+y)t} dy = \frac{t \sum_{i=0}^{d-1} \chi(i) e^{it}}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \quad (2.2)$$

By (2.1) and (2.2), we have

$$\int_X \chi(x) x^n dx = B_{n,\chi} \quad (2.3)$$

(see [15, 25]), and

$$\int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x) \quad (2.4)$$

(see [1–19, 25]). For $n \in \mathbb{N}$, we obtain that

$$\int_X f(x+n)dx = \int_X f(x)dx + \sum_{i=0}^{n-1} f'(i), \quad (2.5)$$

where $f'(i) = (df(x)/dx)|_{x=i}$. Thus, we have

$$\frac{1}{t} \left(\int_X \chi(x)e^{(nd+x)t} dx - \int_X \chi(x)e^{xt} dx \right) = \frac{nd \int_X \chi(x)e^{xt} dx}{\int_X e^{ndxt} dx} = \frac{e^{ndt} - 1}{e^{dt} - 1} \left(\sum_{i=0}^{d-1} \chi(i)e^{it} \right). \quad (2.6)$$

Then

$$\frac{1}{t} \left(\int_X \chi(x)e^{(nd+x)t} dx - \int_X \chi(x)e^{xt} dx \right) = \sum_{l=0}^{nd-1} \chi(l)e^{lt} = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{nd-1} \chi(l)l^k \right) \frac{t^k}{k!}. \quad (2.7)$$

Let us define the p -adic function $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{l=0}^n \chi(l)l^k \quad (2.8)$$

(see [25]). By (2.7) and (2.8), we see that

$$\frac{1}{t} \left(\int_X \chi(x)e^{(nd+x)t} dx - \int_X \chi(x)e^{xt} dx \right) = \sum_{k=0}^{\infty} T_k(\chi, nd-1) \frac{t^k}{k!} \quad (2.9)$$

(see [25]). Thus, we have

$$\int_X \chi(x)(nd+x)^k dx - \int_X \chi(x)x^k dx = kT_{k-1}(\chi, nd-1), \quad k, n, d \in \mathbb{N}. \quad (2.10)$$

This means that

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd-1), \quad k, n, d \in \mathbb{N} \quad (2.11)$$

(see [25]).

The generalized Bernoulli polynomials attached to χ of order k , which is denoted by $B_{n,\chi}^{(k)}(x)$, are defined as

$$\left(\frac{t \sum_{i=0}^{d-1} \chi(i) e^{it}}{e^{dt} - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)}(x) \frac{t^n}{n!}. \quad (2.12)$$

Then the values of $B_{n,\chi}^{(k)}(x)$ at $x = 0$ are called the generalized Bernoulli numbers attached to χ of order k . When $k = 1$, the polynomials of numbers are called the generalized Bernoulli polynomials or numbers attached to χ . Let $w_1, w_2 \in \mathbb{N}$. Then we set

$$\begin{aligned} & K(m, \chi; w_1, w_2) \\ &= \frac{d \left(\int_{X^m} \prod_{i=1}^m \chi(x_i) e^{(\sum_{i=1}^m x_i + w_2 x) w_1 t} \prod_{i=1}^m dx_i \right) \left(\int_{X^m} \prod_{i=1}^m \chi(x_i) e^{(\sum_{i=1}^m x_i + w_1 y) w_2 t} \prod_{i=1}^m dx_i \right)}{\int_X e^{d w_1 w_2 x t} dx}, \end{aligned} \quad (2.13)$$

where

$$\int_{X^m} f(x_1, \dots, x_m) dx_1 \cdots dx_m = \int_X \cdots \int_X f(x_1, \dots, x_m) dx_1 \cdots dx_m. \quad (2.14)$$

In (2.13), we note that $K(m, \chi; w_1, w_2)$ is symmetric in w_1, w_2 . From (2.13), we derive

$$\begin{aligned} & K(m, \chi; w_1, w_2) \\ &= \left(\int_{X^m} \prod_{i=1}^m \chi(x_i) e^{(\sum_{i=1}^m x_i) w_1 t} dx_1 \cdots dx_m \right) e^{w_1 w_2 x t} \left(\frac{d \int_X \chi(x_m) e^{w_2 x_m t} dx_m}{\int_X e^{d w_1 w_2 x t} dx} \right) \\ &\quad \times \left(\int_{X^{m-1}} \prod_{i=1}^{m-1} \chi(x_i) e^{(\sum_{i=1}^{m-1} x_i) w_2 t} dx_1 \cdots dx_{m-1} \right) e^{w_1 w_2 y t}. \end{aligned} \quad (2.15)$$

It is easy to see that

$$\begin{aligned} & \frac{w_1 d \int_X \chi(x) e^{xt} dx}{\int_X e^{d w_1 x t} dx} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{w_1 d - 1} \chi(i) i^k \right) \frac{t^k}{k!} = \sum_{k=0}^{\infty} T_k(\chi, w_1 d - 1) \frac{t^k}{k!}, \\ & e^{w_1 w_2 x t} \int_{X^m} \prod_{i=1}^m \chi(x_i) e^{(\sum_{i=1}^m x_i) w_1 t} dx_1 \cdots dx_m \\ &= e^{w_1 w_2 x t} \left(\frac{w_1 t}{e^{d w_1 t} - 1} \sum_{a=0}^{d-1} \chi(a) e^{w_1 a t} \right)^m = \sum_{n=0}^{\infty} B_{n,\chi}^{(m)}(w_2 x) w_1^n \frac{t^n}{n!}. \end{aligned} \quad (2.16)$$

From (2.16), we note that

$$\begin{aligned}
 &K(m, \chi; w_1, w_2) \\
 &= \left(\sum_{l=0}^{\infty} B_{l, \chi}^{(m)}(w_2 x) \frac{w_1^l t^l}{l!} \right) \left(\sum_{k=0}^{\infty} T_k(\chi, w_1 d - 1) \frac{w_2^k t^k}{k!} \right) \left(\sum_{i=0}^{\infty} B_{i, \chi}^{(m-1)}(w_1 y) \frac{w_2^i t^i}{i!} \right) \left(\frac{1}{w_1} \right) \\
 &= \sum_{n=0}^{\infty} \left[\sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi}^{(m)}(w_2 x) \sum_{k=0}^j T_k(\chi, w_1 d - 1) \binom{j}{k} B_{j-k, \chi}^{(m-1)}(w_1 y) \right] \frac{t^n}{n!}.
 \end{aligned} \tag{2.17}$$

By the symmetry of $K(m, \chi; w_1, w_2)$ in w_1 and w_2 , we see that

$$\begin{aligned}
 &K(m, \chi; w_1, w_2) \\
 &= \sum_{n=0}^{\infty} \left[\sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi}^{(m)}(w_1 x) \sum_{k=0}^j T_k(\chi, w_2 d - 1) \binom{j}{k} B_{j-k, \chi}^{(m-1)}(w_2 y) \right] \frac{t^n}{n!}.
 \end{aligned} \tag{2.18}$$

By comparing the coefficients on the both sides of (2.17) and (2.18), we see the following theorem.

Theorem 2.1. For $d, w_1, w_2 \in \mathbb{N}$, $n \geq 0$, $m \geq 1$, one has

$$\begin{aligned}
 &\sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi}^{(m)}(w_2 x) \sum_{k=0}^j T_k(\chi, w_1 d - 1) \binom{j}{k} B_{j-k, \chi}^{(m-1)}(w_1 y) \\
 &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi}^{(m)}(w_1 x) \sum_{k=0}^j T_k(\chi, w_2 d - 1) \binom{j}{k} B_{j-k, \chi}^{(m-1)}(w_2 y).
 \end{aligned} \tag{2.19}$$

Remark 2.2. Let $y = 0$ and $m = 1$ in (1.4). Then we have

$$\begin{aligned}
 &\sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi}(w_2 x) T_j(\chi, w_1 d - 1) \\
 &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi}(w_1 x) T_j(\chi, w_2 d - 1)
 \end{aligned} \tag{2.20}$$

(see [25]).

We also calculate that

$$K(m, \chi; w_1, w_2) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi}^{(m-1)}(w_1 y) \sum_{i=0}^{dw_1-1} B_{k, \chi}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \right] \frac{t^n}{n!}. \quad (2.21)$$

From the symmetric property of $K(m, \chi; w_1, w_2)$ in w_1 and w_2 , we derive

$$K(m, \chi; w_1, w_2) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi}^{(m-1)}(w_2 y) \sum_{i=0}^{dw_2-1} B_{k, \chi}^{(m)} \left(w_1 x + \frac{w_1}{w_2} i \right) \right] \frac{t^n}{n!}. \quad (2.22)$$

By comparing the coefficients on the both sides of (2.21) and (2.22), we obtain the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$, one has

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi}^{(m-1)}(w_1 y) \sum_{i=0}^{dw_1-1} B_{k, \chi}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \\ &= \sum_{k=0}^n \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi}^{(m-1)}(w_2 y) \sum_{i=0}^{dw_2-1} B_{k, \chi}^{(m)} \left(w_1 x + \frac{w_1}{w_2} i \right). \end{aligned} \quad (2.23)$$

Remark 2.4. Let $y = 0$ and $m = 1$ in (2.23). We have

$$w_1^{n-1} \sum_{i=0}^{dw_1-1} B_{n, \chi} \left(w_2 x + \frac{w_2}{w_1} i \right) = w_2^{n-1} \sum_{i=0}^{dw_2-1} B_{n, \chi} \left(w_1 x + \frac{w_1}{w_2} i \right) \quad (2.24)$$

(see [25]).

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