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Research Article

Generalized (ρ,θ)-η Invariant Monotonicity and Generalized (ρ,θ)-η Invexity of Nondifferentiable Functions

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New concepts of generalized (ρ,θ)-η invex functions for non-differentiable functions and generalized (ρ,θ)-η invariant monotone operators for set-valued mappings are introduced. The relationships between generalized (ρ,θ)-η invexity of functions and generalized (ρ,θ)-η invariant monotonicity of the corresponding Clarke's subdifferentials are studied. Some of our results are extension and improvement of some results obtained in (Jabarootion and Zafarani (2006); Behera et al. (2008)).

1. Introduction

Convexity plays a central role in mathematical economics, engineering, management sciences, and optimization. In recent years several extensions and generalizations have been developed for classical convexity. An important generalization of convex functions is invex functions introduced by Hanson (1981) [1]. He has shown that the Kuhn-Tuker conditions are sufficient for optimality of nonlinear programming problems under invexity conditions. Kaul and Kaur (1985) [2] presented the conpects of pseudoinvex and quasi-invex functions and investigated their applications in nonlinear programming. A concept closely related to the convexity of function is the monotonicity of function; it is worth noting that monotonicity plays a very important role in the study of the existence and sensitivity analysis of solutions for variational inequality problems. An important breakthrough was given by Karamardian and Schaible (1990) [3]. They have proved that the generalized convexity of the function f is equivalent to the generalized monotonicity of its gradient function ∇f. Motivated by the work of Karamardian and Schaible (1990), there has been increasing interest in the study of monotonicity and generalized monotonicity and their relationships to convexity and generalized convexity. Ruiz-Garzón et al. (2003) [4] introduced strongly invex and strongly pseudoinvex functions in R^n and gave the sufficient conditions for (strictly, strongly) invex monotonicity, strictly pseudoinvex monotonicity, and quasi-invex monotonicity. Moreover, in [4] the necessary conditions for strictly pseudoinvex monotonicity and pseudoinvex monotonicity were obtained. The necessary conditions for strongly pseudoinvex monotonicity were given by Yang et al. (2005) [5]. The results on generalized invexity and generalized invex monotonicity obtained in [4–6] are studied in n-dimensional Euclidean space. Several generalizations in real Banach space have been developed for generalized invexity and generalized invex monotonicity. Recently, Fan et al. (2003) [7] have studied the relationships between (strict, strong) convexity, pseudoconvexity, and quasiconvexity of functions and (strict, strong) monotonicity, pseudomonotonicity, and quasimonotonicity of its Clarke's generalized subdifferential mapping, respectively. Jabarootian and Zafarani (2006) [8] generalized convexity to invexity and obtained the relationships between several kinds of generalized invexity of functions and generalized invariant monotonicity of its Clarke's generalized subdifferential mapping.

Behera et al. (2008) [9] introduce the concepts of generalized (ρ,θ)-η invariant monotone operators and generalized (ρ,θ)-η invex functions and discuss the relationships between generalized (ρ,θ)-η invariant monotonicity and generalized (ρ,θ)-η invexity. Some examples are presented by Behera et al. to illustrate the proper generalizations for the corresponding concepts of generalized invariant monotone. However, it is noted that the definition of strictly (ρ,θ)-η quasi-invex is not defined precisely in [9], and Theorem 3.2 of [9] contains some errors. The purpose of this paper is to point out these errors and to suggest appropriate modifications. In real Banach space, we define new concepts of generalized (ρ,θ)-η invexity for non-differentiable functions and generalized (ρ,θ)-η invariant monotonicity for set-valued mappings which are extension and improvement of the corresponding definitions of [8, 9]. In [9], some sufficient conditions for generalized (ρ,θ)-η invariant pseudomonotonicity and generalized (ρ,θ)-η invariant quasimonotonicity were given. We will give some necessary conditions. We also introduce the concept of strongly (ρ,θ)-η invariant pseudomonotone and give its necessary conditions. Some results that obtained in this paper are the improvement of the corresponding results of [8, 9].

2. Preliminaries

In this paper, let X be a real Banach space endowed with a norm $\|\cdot\|$ and X^* its dual space with a norm $\|\cdot\|_*$. We denote by 2^{X^*}, $\langle\cdot,\cdot\rangle$, $[x,y]$, and (x,y) the family of all nonempty subsets of X^*, the dual pair between X and X^*, the line segment for $x,y\in X$, and the interior of $[x,y]$, respectively. Let K be a nonempty open subset of X, $T: X\to 2^{X^*}$ a set-valued mapping, $\eta: X\times X\to X$ a vector-valued function, and $f: X\to R$ a non-differentiable real-valued function. The following concepts and results are taken from [10].

Definition 2.1. Let f be locally Lipschitz continuous at a given point $x\in X$ and v any vector in X. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v)=\limsup_{y\to x,t\downarrow 0}\frac{f(y+tv)-f(y)}{t}. \tag{2.1}$$

Definition 2.2. Let f be locally Lipschitz continuous at a given point $x \in X$ and let v be any vector in X. The Clarke's generalized subdifferential of f at x, denoted by $\partial^c f(x)$, is defined as follows:

$$\partial^c f(x) = \{\xi \in X^* : f^0(x;v) \geq \langle \xi, v\rangle, \forall v \in X\}. \tag{2.2}$$

Lemma 2.3 (Mean Value Theorem). *Let x and y be point in X and suppose that f is Lipschitz near each point of a nonempty closed convex set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(x) - f(y) \in \langle \partial^c f(u), x - y\rangle. \tag{2.3}$$

3. (ρ, θ)-η Invariant Monotonicity and (ρ, θ)-η Invexity

Let K be nonempty subset of X and let η and θ be two vector-valued functions from $K \times K$ to X and $\rho \in R$.

Definition 3.1. Let K be nonempty subset of X and let $T : K \to 2^{X^*}$ be a set-valued mapping:

(1) T is said to be (ρ, θ)-η invariant monotone on K with respect to η and θ if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle + \rho[\|\theta(x, y)\| + \|\theta(y, x)\|] \leq 0; \tag{3.1}$$

(2) T is said to be strictly (ρ, θ)-η invariant monotone on K with respect to η and θ if for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle + \rho[\|\theta(x, y)\| + \|\theta(y, x)\|] < 0. \tag{3.2}$$

Remark 3.2. (1) When $T : K \to X^*$, it is [9, Definition 2.3].

(2) When $\rho = 0$, every (strictly) (ρ, θ)-η invariant monotone function is a (strictly) invariant monotone function defined by Jabarootian and Zafarani [8], but the converse is not true. Examples 2.1 and 2.2 of [9] are two counterexamples, where $\theta(x, y)$ is defined as $\sqrt{\theta(x, y)}$.

(3) When $\rho > 0$, $\theta(x, y) = \eta(x, y)$, Definition 3.1(1) is the Definition 3.2(3) in [8], that is, strongly invariant monotone.

Next we introduce (ρ, θ)-η invex functions under non-differentiable condition.

Definition 3.3. Let K be nonempty subset of X, and let $f : K \to R$ be locally Lipschitz continuous on K. Then,

(1) the function f is said to be (ρ, θ)-η invex with respect to η and θ on K if for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$f(y) - f(x) \geq \langle \zeta, \eta(y, x)\rangle + \rho\|\theta(y, x)\|; \tag{3.3}$$

(2) the function f is said to be strictly (ρ,θ)-η invex with respect to η and θ on K if for any $x,y \in K$ with $x \neq y$ and any $\zeta \in \partial^c f(x)$, one has

$$f(y) - f(x) > \langle \zeta, \eta(y,x) \rangle + \rho \|\theta(y,x)\|. \tag{3.4}$$

The following lemma under non-differentiable condition is similar to Lemmas 2.3 and 2.4 under Fréchet differentiable condition in [9].

Lemma 3.4. *Let f be locally Lipschitz on K. If f is (strictly) (ρ,θ)-η invex with respect to η and θ on K, then $\partial^c f$ is (strictly) (ρ,θ)-η invariant monotone with respect to the same η and θ on K.*

4. (ρ,θ)-η Invariant Quasimonotonicity and (ρ,θ)-η Quasi-Invexity

In this section, we will point out some errors in [9].

Definition 4.1 (see [11]). A set K is said to be invex with respect to η if there exists an $\eta : K \times K \to X$ such that, for any $x, y \in K$ and $\lambda \in [0,1]$,

$$y + \lambda \eta(x,y) \in K. \tag{4.1}$$

Definition 4.2 (see [9, Definition 3.2]). A Fréchet differentiable function $f : K \to R$ is said to be (ρ,θ)-η quasi-invex with respect to η and θ on K if for any $x, y \in K$ with $x \neq y$, we have

$$f(y) \leq f(x) \Longrightarrow \langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 \leq 0. \tag{4.2}$$

If strict inequality holds, then it is said to be strictly (ρ,θ)-η quasi-invex, where ∇f is the Fréchet differential of f.

Definition 4.3 (see [9, Definition 4.2]). A Fréchet differentiable function $f : K \to R$ is said to be (ρ,θ)-η pseudoinvex with respect to η and θ on K if for any $x, y \in K$ with $x \neq y$, we have

$$\langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 \geq 0 \Longrightarrow f(y) \geq f(x). \tag{4.3}$$

Definition 4.4 (see [9, Definition 5.2]). A Fréchet differentiable function $f : K \to R$ is said to be strictly (ρ,θ)-η pseudoinvex with respect to η and θ on K if for any $x, y \in K$ with $x \neq y$, we have

$$\langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 \geq 0 \Longrightarrow f(y) > f(x). \tag{4.4}$$

Remark 4.5. In Definition 4.2, the definition of strictly (ρ,θ)-η quasi-invex is not defined precisely, as can be seen below. If it holds that

$$f(y) < f(x) \Longrightarrow \langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 < 0, \tag{4.5}$$

then, by Definition 4.3, f is (ρ,θ)-η pseudoinvex function with respect to η and θ on K. On the other hand, if the following is OK:

$$f(y) \le f(x) \Longrightarrow \langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 < 0, \tag{4.6}$$

then, by Definition 4.4, f is strictly (ρ,θ)-η pseudoinvex function with respect to η and θ on K.

Thus, we must conclude that the following implication holds

$$f(y) < f(x) \Longrightarrow \langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 \le 0, \tag{4.7}$$

that is, the "strict inequality" for strictly (ρ,θ)-η quasi-invex functions in Definition 4.2 means (4.7).

Definition 4.6 (see [9, Definition 3.1]). A function $f : K \to X^*$ is said to be (ρ,θ)-η invariant quasimonotone (strictly) with respect to η and θ on K if for any $x, y \in K$ with $x \neq y$, we have

$$\langle F(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 > 0 \Longrightarrow \langle F(y), \eta(x,y) \rangle + \rho \|\theta(x,y)\|^2 \le 0 (< 0). \tag{4.8}$$

Definition 4.7 (see [9, Definition 4.1]). A function $f : K \to X^*$ is said to be (ρ,θ)-η invariant pseudomonotone with respect to η and θ on K if for any $x, y \in K$,we have

$$\langle F(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 \ge 0 \Longrightarrow \langle F(y), \eta(x,y) \rangle + \rho \|\theta(x,y)\|^2 \le 0. \tag{4.9}$$

The following Theorem is due to Behera et al. in [9].

Theorem A (see [9, Theorem 3.2]). *Let K be an invex set with respect to η, and let $f : K \to R$ be Fréchet differentiable on K. If f is strictly (ρ,θ)-η quasi-invex with respect to η and θ on K, then ∇f is a strictly (ρ,θ)-η invariant quasimonotone function with respect to the same η and θ on K.*

Remark 4.8. By Remark 4.5, Theorem A is not true as can be seen from the following example.

Example 4.9. Let f, η, and θ be functions defined by

$$\begin{gathered} f(x) = e^x, \quad x \in R, \\ \eta(x,y) = e^x - e^y, \\ \theta(y,x) = \begin{cases} \sqrt{e^x(e^x - e^y)}, & x \ge y, \\ 0, & x < y. \end{cases} \end{gathered} \tag{4.10}$$

Take $\rho = 1$.

When $x < y$, we have

$$\langle \nabla f(x), \eta(y,x) \rangle + \rho \|\theta(y,x)\|^2 = \langle e^x, e^y - e^x \rangle > 0. \tag{4.11}$$

Then, $f(x) = e^x < e^y = f(y)$. Clearly, by (4.7), f is strictly (ρ, θ)-η quasi-invex with respect to η and θ on R, but

$$\langle \nabla f(y), \eta(x, y) \rangle + \rho \| \theta(x, y) \|^2 = \langle e^y, e^x - e^y \rangle + e^y (e^y - e^x) = 0. \tag{4.12}$$

Thus, ∇f is not strictly (ρ, θ)-η invariant quasimonotone.

On the other hand, when $x > y$,

$$\langle \nabla f(x), \eta(y, x) \rangle + \rho \| \theta(y, x) \|^2 = \langle e^x, e^y - e^x \rangle + e^x (e^x - e^y) = 0, \tag{4.13}$$

but

$$\langle \nabla f(y), \eta(x, y) \rangle + \rho \| \theta(x, y) \|^2 = \langle e^y, e^x - e^y \rangle > 0. \tag{4.14}$$

Thus, ∇f is not (ρ, θ)-η invariant pseudomonotone.

Furthermore, ∇f is (ρ, θ)-η invariant quasimonotone. Hence, Example 4.9 also illustrates that (ρ, θ)-η invariant quasimonotone is not necessarily strictly (ρ, θ)-η invariant quasimonotone or (ρ, θ)-η invariant pseudomonotone.

In [9], sufficient conditions for (ρ, θ)-η invariant quasimonotone for Fréchet differentiable functions were given. However, necessary conditions have been missing. In what follows, we will give the necessary conditions for (ρ, θ)-η invariant quasimonotonity and study the relationship between (ρ, θ)-η invariant quasimonotonicity and (ρ, θ)-η quasi-invexity.

Definition 4.10. Let K be a nonempty subset of X and $T : K \to 2^{X^*}$ is said to be (ρ, θ)-η invariant quasimonotone on K with respect to η and θ if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u, \eta(y, x) \rangle + \rho \| \theta(y, x) \| > 0 \Longrightarrow \langle v, \eta(x, y) \rangle + \rho \| \theta(x, y) \| \le 0. \tag{4.15}$$

Remark 4.11. When $\rho = 0$, every (ρ, θ)-η invariant quasimonotone function is an invariant quasimonotone function defined by Jabarootian and Zafarani [8], but the converse is not true. See [9, Example 3.1], for a counterexample, where

$$\theta(y, x) = \begin{cases} (\sin x - \sin y)^2, & x > y, \\ 0, & x \le y. \end{cases} \tag{4.16}$$

Definition 4.12. Let K be a nonempty subset of X, and let $f : K \to R$ be locally Lipschitz continuous on K. Then, the function f is said to be (ρ, θ)-η quasi-invex with respect to η and θ on K if for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$f(y) \le f(x) \Rightarrow \langle \zeta, \eta(y, x) \rangle + \rho \| \theta(y, x) \| \le 0. \tag{4.17}$$

The following theorem under non-differentiable condition is similar to Theorem 3.1 of Behera et al. in [9].

Theorem 4.13. *Let f be locally Lipschitz continuous on K. If f is (ρ,θ)-η quasi-invex with respect to η and θ on K, then $\partial^c f$ is a (ρ,θ)-η invariant quasimonotone mapping with respect to the same η and θ on K.*

Proof. Suppose f is (ρ,θ)-η quasi-invex with respect to η and θ on K. Let $x, y \in K$, $u \in \partial^c f(x)$, and $v \in \partial^c f(y)$ be such that

$$\langle u, \eta(y,x)\rangle + \rho\|\theta(y,x)\| > 0. \tag{4.18}$$

By (ρ,θ)-η quasi-invexity of f, we have

$$f(y) > f(x). \tag{4.19}$$

Note that (ρ,θ)-η quasi-invexity of f implies

$$\langle v, \eta(x,y)\rangle + \rho\|\theta(x,y)\| \leq 0. \tag{4.20}$$

Therefore, $\partial^c f$ is a (ρ,θ)-η invariant quasimonotone mapping with respect to the same η and θ on K. □

Condition C (see [12]). Let $\eta : X \times X \to X$. Then, for any $x, y \in K$ and for any $\lambda \in [0,1]$,

$$\begin{aligned} \eta(y, y+\lambda\eta(x,y)) &= -\lambda\eta(x,y), \\ \eta(x, y+\lambda\eta(x,y)) &= (1-\lambda)\eta(x,y). \end{aligned} \tag{4.21}$$

Definition 4.14 (see [4]). A function $\eta : K \times K \to X$ is said to be a skew function if $\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in K \subseteq X$.

Now we will give the sufficient conditions for (ρ,θ)-η quasi-invexity.

Theorem 4.15. *Let $K \subseteq X$ be an open invex set with respect to η, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *$\partial^c f$ is a (ρ,θ)-η invariant quasimonotone mapping with respect to η and θ on K;*

(2) *η satisfies Condition C;*

(3) *for each $x \neq y$ and $f(x) \leq f(y)$, there exist $\overline{\lambda} \in (0,1)$ and $\omega \in \partial^c f(y+\overline{\lambda}\eta(x,y))$, such that*

$$\langle \omega, \eta(x,y)\rangle - \rho\|\theta(x,y)\| < 0; \tag{4.22}$$

(4) $\|\theta(y, y+\overline{\lambda}\eta(x,y))\| = \|\theta(y+\overline{\lambda}\eta(x,y), y)\| = \overline{\lambda}\|\theta(x,y)\|$.

Then, f is a (ρ,θ)-η quasi-invex function with respect to the same η and θ on K.

Proof. Suppose that f is not a (ρ,θ)-η quasi-invex function with respect to η and θ on K. Then, there exist $x, y \in K$ and $\xi \in \partial^c f(y)$, such that

$$f(x) \leq f(y), \tag{4.23}$$

but

$$\langle \xi, \eta(x,y)\rangle + \rho\|\theta(x,y)\| > 0. \tag{4.24}$$

By hypothesis 3 and (4.23) , there exist $\overline{\lambda} \in (0,1)$, $\omega \in \partial^c f(y+\overline{\lambda}\eta(x,y))$ such that

$$\langle \omega, \eta(x,y)\rangle - \rho\|\theta(x,y)\| < 0. \tag{4.25}$$

It follows from Condition C, (4.25), and hypothesis 4 that

$$\langle \omega, \eta(y, y+\overline{\lambda}\eta(x,y))\rangle + \rho\|\theta(y, y+\overline{\lambda}\eta(x,y))\| > 0. \tag{4.26}$$

Since $\partial^c f$ is a (ρ,θ)-η invariant quasimonotone mapping with respect to η and θ on K, (4.26) implies

$$\langle \xi, \eta(y+\overline{\lambda}\eta(x,y), y)\rangle + \rho\|\theta(y+\overline{\lambda}\eta(x,y), y)\| \leq 0. \tag{4.27}$$

From Condition C, hypothesis 4, and (4.27), we obtain

$$\langle \xi, \eta(x,y)\rangle + \rho\|\theta(x,y)\| \leq 0, \tag{4.28}$$

which contradicts (4.24). Hence, f is a (ρ,θ)-η quasi-invex function with respect to the same η and θ on K. □

Similar to proof for Theorem 4.15, we can obtain the following theorem.

Theorem 4.16. *Let K be an open convex subset of X, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *$\partial^c f$ is a (ρ,θ)-η invariant quasimonotone mapping with respect to η and θ on K;*

(2) *η and θ are affine in the first argument and skew;*

(3) *for each $x, y \in K$, $x \neq y$ and $f(x) \leq f(y)$, there exist $\overline{\lambda} \in (0,1)$, $\omega \in \partial^c f(\overline{x})$, where $\overline{x} = \overline{\lambda}x + (1-\overline{\lambda})y$, such that*

$$\langle \omega, \eta(y,\overline{x})\rangle + \rho\|\theta(y,\overline{x})\| > 0. \tag{4.29}$$

Then, f is a (ρ,θ)-η quasi-invex function with respect to the same η and θ on K.

5. (ρ, θ)-η Invariant Pseudomonotonicity and (ρ, θ)-η Pseudoinvexity

Definition 5.1. Let K be a nonempty subset of X, and let $T : K \to 2^{X^*}$ be a set-valued mapping. Then,

(1) T is said to be (ρ, θ)-η invariant pseudomonotone on K with respect to η and θ if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0 \Longrightarrow \langle v, \eta(x,y)\rangle + \rho\|\theta(x,y)\| \leq 0; \tag{5.1}$$

(2) T is said to be strictly (ρ, θ)-η invariant pseudomonotone on K with respect to η and θ if for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0 \Longrightarrow \langle v, \eta(x,y)\rangle + \rho\|\theta(x,y)\| < 0. \tag{5.2}$$

Remark 5.2. When $\rho = 0$, every (strictly) (ρ, θ)-η invariant pseudomonotone function is (strictly) invariant pseudomonotone function [8] on K with respect to the same η, but the converse is not ture. Examples 4.1 and 5.1 of [9] are two counterexamples, where $\theta(x, y)$ is defined as $\sqrt{\theta(x,y)}$.

Definition 5.3. Let K be a nonempty subset of X, and let $f : K \to R$ be locally Lipschitz continuous on K. Then,

(1) the function f is said to be (ρ, θ)-η pseudoinvex with respect to η and θ on K if for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0 \Longrightarrow f(y) \geq f(x); \tag{5.3}$$

(2) the function f is said to be strictly (ρ, θ)-η pseudoinvex with respect to η and θ on K if for any $x, y \in K$ with $x \neq y$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0 \Longrightarrow f(y) > f(x). \tag{5.4}$$

In this section we will give sufficient conditions and necessary conditions for (ρ, θ)-η invariant pseudomonotonicity.

Theorem 5.4. *Let $K \subseteq X$ be an open invex set with respect to η, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *f is a (ρ, θ)-η pseudoinvex function with respect to η and θ on K;*

(2) *η satisfies Condition C;*

(3) *for any $x, y \in K$ and $f(x) \leq f(y)$, there exists $\overline{\lambda} \in (0,1)$ such that $f(y + \overline{\lambda}\eta(x,y)) < f(y)$;*

(4) $\|\theta(y + \overline{\lambda}\eta(x,y), y)\| = \overline{\lambda}\|\theta(x,y)\|$.

Then, $\partial^c f$ is a (ρ, θ)-η invariant pseudomonotone mapping with respect to the same η and θ on K.

Proof. Suppose that f is a (ρ,θ)-η pseudoinvex function with respect to η and θ on K. Let $\rho \in R$, $x, y \in K$. If $\forall \xi \in \partial^c f(y)$,

$$\langle \xi, \eta(x,y)\rangle + \rho\|\theta(x,y)\| \geq 0. \tag{5.5}$$

We need to show

$$\langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \leq 0, \quad \forall \zeta \in \partial^c f(x). \tag{5.6}$$

Suppose, on the contrary , $\exists \overline{\zeta} \in \partial^c f(x)$, such that

$$\langle \overline{\zeta}, \eta(y,x)\rangle + \rho\|\theta(y,x)\| > 0. \tag{5.7}$$

Since f is a (ρ,θ)-η pseudoinvex function with respect to η and θ on K, (5.7) implies

$$f(x) \leq f(y). \tag{5.8}$$

From hypothesis 3 and (5.8), $\exists \overline{\lambda} \in (0,1)$ such that

$$f(y + \overline{\lambda}\eta(x,y) < f(y)). \tag{5.9}$$

Hence, (ρ,θ)-η pseudoinvexity of f implies

$$\langle \xi, \eta(y + \overline{\lambda}\eta(x,y), y)\rangle + \rho\|\theta(y + \overline{\lambda}\eta(x,y), y)\| < 0. \tag{5.10}$$

From Condition C, hypothesis 4, and (5.10), we have

$$\langle \xi, \eta(x,y)\rangle + \rho\|\theta(x,y)\| < 0, \tag{5.11}$$

which contradicts (5.5). Hence, $\partial^c f$ is a (ρ,θ)-η invariant pseudomonotone mapping with respect to the same η and θ on K. □

Corollary 5.5. *Let $K \subseteq X$ be an open invex set with respect to η, and let f be locally Lipschitz continuous on K, and let η satisfy Condition C. If, for any $x, y \in K$,*

(1) $f(x) \leq f(y) \Rightarrow \exists \overline{\lambda} \in (0,1)$ *such that* $f(y + \overline{\lambda}\eta(x,y)) < f(y)$;

(2) f *is a pseudoinvex function with respect to η on K,*

then, $\partial^c f$ is an invariant pseudomonotone mapping with respect to the same η on K.

Theorem 5.6. *Let $K \subseteq X$ be an open invex set with respect to η and let f be locally Lipschitz continuous on K. Suppose that*

(1) $\partial^c f$ *is a (ρ,θ)-η invariant pseudomonotone mapping with respect to η and θ on K;*

(2) *η satisfies Condition C;*

(3) *for any $x, y \in K$ and $f(x) < f(y)$, there exist $\bar{\lambda} \in (0,1)$, $\omega \in \partial^c f(y + \bar{\lambda}\eta(x,y))$, such that*

$$\langle \omega, \eta(x,y) \rangle - \rho \|\theta(x,y)\| < 0; \tag{5.12}$$

(4) $\|\theta(y + \bar{\lambda}\eta(x,y), y)\| = \|\theta(y, y + \bar{\lambda}\eta(x,y))\| = \bar{\lambda}\|\theta(x,y)\|$.

Then, f is a (ρ,θ)-η pseudoinvex function with respect to the same η and θ on K.

Proof. Suppose that $\partial^c f$ is a (ρ,θ)-η invariant pseudomonotone mapping with respect to η and θ on K. Let $\rho \in R$, $x, y \in K$, for all $\xi \in \partial^c f(y)$, be such that

$$\langle \xi, \eta(x,y) \rangle + \rho \|\theta(x,y)\| \geq 0. \tag{5.13}$$

We need to show

$$f(y) \leq f(x). \tag{5.14}$$

Assume, on the contrary,

$$f(y) > f(x). \tag{5.15}$$

By hypothesis 3, $\exists \bar{\lambda} \in (0,1)$, $\exists \omega \in \partial^c f(y + \bar{\lambda}\eta(x,y))$ such that

$$\langle \omega, \eta(x,y) \rangle - \rho \|\theta(x,y)\| < 0. \tag{5.16}$$

From Condition C, hypothesis 4, and (5.16), we have

$$\langle \omega, \eta(y, y + \bar{\lambda}\eta(x,y)) \rangle + \rho \|\theta(y, y + \bar{\lambda}\eta(x,y))\| > 0. \tag{5.17}$$

Since $\partial^c f$ is an invariant pseudomonotone mapping with respect to η and θ, (5.17) implies

$$\langle \xi, \eta(y + \bar{\lambda}\eta(x,y), y) \rangle + \rho \|\theta(y + \bar{\lambda}\eta(x,y), y)\| < 0, \quad \forall \xi \in \partial^c f(y). \tag{5.18}$$

From Condition C, hypothesis 4, and (5.18), we obtain

$$\langle \xi, \eta(x,y) \rangle + \rho \|\theta(x,y)\| < 0, \quad \forall \xi \in \partial^c f(y), \tag{5.19}$$

which contradicts (5.13). Hence, f is a (ρ,θ)-η pseudoinvex function with respect to the same η and θ on K. □

Corollary 5.7. *Let $K \subseteq X$ be an open invex set with respect to η, let f be locally Lipschitz continuous on K, and let η satisfy Condition C. For any $x, y \in K$, $\exists \overline{\lambda} \in (0,1)$, $\exists \omega \in \partial^c f(y + \overline{\lambda}\eta(x,y))$ such that*

$$f(x) < f(y) \Longrightarrow \langle \omega, \eta(x,y) \rangle < 0. \tag{5.20}$$

If $\partial^c f$ is an invariant pseudomonotone mapping with respect to η on K, then f is a pseudoinvex function with respect to the same η on K.

Similar to proof for Theorems 5.4 and 5.6, we can obtain the following two theorems.

Theorem 5.8. *Let K be an open convex subset of X, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *f is a (ρ, θ)-η pseudoinvex function with respect to η and θ on K;*

(2) *η and θ are affine in the first argument and skew;*

(3) *for any $x, y \in K$ and $f(x) \leq f(y)$, there exists $\overline{\lambda} \in (0,1)$ such that*

$$f\left(\overline{\lambda}x + (1-\overline{\lambda})y\right) < f(y). \tag{5.21}$$

Then, $\partial^c f$ is a (ρ, θ)-η invariant pseudomonotone mapping with respect to the same η and θ on K.

Theorem 5.9. *Let K be an open convex subset of X, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *$\partial^c f$ is a (ρ, θ)-η invariant pseudomonotone mapping with respect to η and θ on K;*

(2) *η and θ are affine in the first argument and skew;*

(3) *for each $x, y \in K$ and $f(x) < f(y)$, there exist $\overline{\lambda} \in (0,1)$, $\omega \in \partial^c f(\overline{x})$, where $\overline{x} = \overline{\lambda}x + (1-\overline{\lambda})y$ such that*

$$\langle \omega, \eta(y, \overline{x}) \rangle + \rho \| \theta(y, \overline{x}) \| > 0. \tag{5.22}$$

Then, f is a (ρ, θ)-η pseudoinvex function with respect to the same η and θ on K.

6. Strongly (ρ, θ)-η Invariant Pseudomonotonicity and Strongly (ρ, θ)-η Pseudoinvexity

In this section, we introduce the concepts of strongly (ρ, θ)-η invariant pseudomonotonicity and strongly (ρ, θ)-η pseudoinvexity. We will give a necessary condition for strongly (ρ, θ)-η invariant pseudomonotonicity.

Definition 6.1. Let K be a nonempty subset of X. Then, $T : K \to 2^{X^*}$ is said to be strongly (ρ, θ)-η invariant pseudomonotone on K with respect to η and θ if there exists a constant $\beta > 0$, such that for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u, \eta(y,x) \rangle + \rho \| \theta(y,x) \| \geq 0 \Longrightarrow \langle v, \eta(x,y) \rangle + \rho \| \theta(x,y) \| \leq -\beta \| \eta(y,x) \|. \tag{6.1}$$

Definition 6.2. Let K be a nonempty subset of X, and let $f : K \to R$ be locally Lipschitz continuous on K. Then, the function f is said to be strongly (ρ,θ)-η pseudoinvex with respect to η and θ on K, if there exists a constant $\beta > 0$, such that for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0 \Longrightarrow f(y) \geq f(x) + \beta\|\eta(y,x)\|. \tag{6.2}$$

Theorem 6.3. *Let $K \subseteq X$ be an open invex set with respect to η, and let f be locally Lipschitz continuous on K. Suppose that*

(1) *$\partial^c f$ is a strongly (ρ,θ)-η invariant pseudomonotone with respect to η and θ on K;*

(2) *η satisfies Condition C;*

(3) *$f(y + \eta(x,y)) \leq f(x) + \rho\|\theta(x,y)\|$, for any $x, y \in K$;*

(4) *$\|\theta(x + \lambda\eta(y,x), x)\| = \|\theta(x, x + \lambda\eta(y,x))\| = \lambda\|\theta(y,x)\|$, $\forall\lambda \in (0,1)$.*

Then, f is a strongly (ρ,θ)-η pseudoinvex function with respect to the same η and θ on K.

Proof. Suppose that $\partial^c f$ is strongly (ρ,θ)-η invariant pseudomonotone with respect to η and θ on K. Let $\rho \in R$, $x, y \in K$, and for all $\zeta \in \partial^c f(x)$ be such that

$$\langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \geq 0. \tag{6.3}$$

Let $z = x + (1/2)\eta(y,x)$. By the Mean Value Theorem, there exist $\lambda_1, \lambda_2 \in (0,1)$, $0 < \lambda_2 < 1/2 < \lambda_1 < 1$, and $\sigma \in \partial^c f(u)$, where $u = x + \lambda_2\eta(y,x)$, such that

$$f(z) - f(x) = \langle \sigma, z - x\rangle = \frac{1}{2}\langle \sigma, \eta(y,x)\rangle. \tag{6.4}$$

Hence, by Condition C, we have

$$f(z) - f(x) = -\frac{1}{2\lambda_2}\langle \sigma, \eta(x,u)\rangle, \tag{6.5}$$

and there exists $\tau \in \partial^c f(v)$, where $v = x + \lambda_1\eta(y,x)$, such that

$$f(x + \eta(y,x)) - f(z) = \langle \tau, x + \eta(y,x) - z\rangle = \frac{1}{2}\langle \tau, \eta(y,x)\rangle. \tag{6.6}$$

Thus, by Condition C, we have

$$f(x + \eta(y,x)) - f(z) = -\frac{1}{2\lambda_1}\langle \tau, \eta(x,v)\rangle. \tag{6.7}$$

On the other hand, it follows from hypothesis 4, Condition C, and (6.3) that

$$\begin{aligned} 0 &\le \langle \zeta, \eta(y,x)\rangle + \rho\|\theta(y,x)\| \\ &= \frac{1}{\lambda_1}(\langle \zeta, \eta(v,x)\rangle + \rho\|\theta(v,x)\|) \\ &= \frac{1}{\lambda_2}(\langle \zeta, \eta(u,x)\rangle + \rho\|\theta(u,x)\|). \end{aligned} \tag{6.8}$$

By strongly (ρ,θ)-η invariant pseudomonotonicity of $\partial^c f$, we obtain

$$\langle \sigma, \eta(x,u)\rangle + \rho\|\theta(x,u)\| \le -\beta\|\eta(u,x)\| = -\beta\lambda_2\|\eta(y,x)\|; \tag{6.9}$$

$$\langle \tau, \eta(x,v)\rangle + \rho\|\theta(x,v)\| \le -\beta\|\eta(v,x)\| = -\beta\lambda_1\|\eta(y,x)\|. \tag{6.10}$$

Then, by (6.5), (6.9), and hypothesis 4, we have

$$f(z) - f(x) \ge \frac{\beta}{2}\|\eta(y,x)\| + \frac{\rho}{2\lambda_2}\|\theta(x,u)\| = \frac{\beta}{2}\|\eta(y,x)\| + \frac{\rho}{2}\|\theta(y,x)\|. \tag{6.11}$$

Furthermore, by (6.7), (6.10), and hypothesis 4, we obtain

$$f(x+\eta(y,x)) - f(z) \ge \frac{\beta}{2}\|\eta(y,x)\| + \frac{\rho}{2\lambda_1}\|\theta(x,v)\| = \frac{\beta}{2}\|\eta(y,x)\| + \frac{\rho}{2}\|\theta(y,x)\|. \tag{6.12}$$

Adding inequalities (6.11) and (6.12), we have

$$f(x+\eta(y,x)) - f(x) \ge \beta\|\eta(y,x)\| + \rho\|\theta(y,x)\|. \tag{6.13}$$

By hypothesis 3, it is clear that

$$f(y) - f(x) \ge \beta\|\eta(y,x)\|. \tag{6.14}$$

Thus, f is a strongly (ρ,θ)-η pseudoinvex function with respect to the same η and θ on K. □

7. Conclusions

In this paper, we introduced various concepts of generalized (ρ,θ)-η invariant monotonicity and established their relations with generalized (ρ,θ)-η invexity. Diagram (7.1) summarizes these relations, where $\Rightarrow$ means that the implication relation holds and $\not\Leftarrow$ means that the implication relation does not hold:

$$\begin{array}{ccccc}
(\rho,\theta)\text{-}\eta \text{ IM} & \overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}} & (\rho,\theta)\text{-}\eta \text{ IPM} & \overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}} & (\rho,\theta)\text{-}\eta \text{ IQM} \\
\Uparrow \; \not\Downarrow & & \Uparrow \; \not\Downarrow & & \Uparrow \; \not\Downarrow \\
\text{strictly } (\rho,\theta)\text{-}\eta \text{ IM} & \overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}} & \text{strictly } (\rho,\theta)\text{-}\eta \text{ IPM} & \overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}} & \text{strictly } (\rho,\theta)\text{-}\eta \text{ IQM}
\end{array} \tag{7.1}$$

where IM stands for invariant monotonicity, IPM stands for invariant pseudomonotonicity, and IQM stands for invariant quasimonotonicity.

For the generalized (ρ,θ)-η invexity of a real-valued function, the relations in Diagram (7.2) hold:

$$\begin{array}{ccccc}
\text{strictly } (\rho,\theta)\text{-}\eta \text{ invexity} & \Longrightarrow & (\rho,\theta)\text{-}\eta \text{ invexity} & \Longrightarrow & (\rho,\theta)\text{-}\eta \text{ quasiinvexity} \\
\Downarrow & & \Downarrow & & \\
\text{strictly } (\rho,\theta)\text{-}\eta \text{ pseudoinvexity} & \Longrightarrow & (\rho,\theta)\text{-}\eta \text{ pseudoinvexity} & &
\end{array} \tag{7.2}$$

Under certain conditions, we obtain the relationships between generalized (ρ,θ)-η invexity in Diagram (7.2) and the corresponding generalized (ρ,θ)-η invariant monotonicity in Diagram (7.1).

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