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Research Article

Analytic Classes on Subframe and Expanded Disk and the $\mathcal{R}^s$ Differential Operator in Polydisk

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We introduce and study new analytic classes on subframe and expanded disk and give complete description of their traces on the unit disk. Sharp embedding theorems and various new estimates concerning differential $\mathcal{R}^s$ operator in polydisk also will be presented. Practically all our results were known or obvious in the unit disk.

1. Introduction and Main Definitions

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_k \in \mathbb{C},\ 1 \le k \le n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$U^n = \{z \in \mathbb{C}^n : |z_k| < 1,\ 1 \le k \le n\}, \tag{1.1}$$

and the distinguished boundary of U^n by

$$T^n = \{z \in \mathbb{C}^n : |z_k| = 1,\ 1 \le k \le n\}. \tag{1.2}$$

We use m_{2n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on T^n. Let $H(U^n)$ be the space of all holomorphic functions on U^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2n} by m_2, m_n by m. We refer to [1, 2] for further details. We denote the expanded disk by

$$U^n_* = \{z = (r_1\xi, \dots, r_n\xi) \in U^n : \xi \in T,\ r_j \in (0,1),\ j = 1, \dots, n\}, \tag{1.3}$$

and the subframe by

$$\widetilde{U^n} = \{z \in U^n : |z_j| = r,\ r \in (0,1],\ j = 1,\dots,n\}. \tag{1.4}$$

The Hardy spaces, denoted by $H^p(U^n)(0 < p \le \infty)$, are defined by

$$H^p(U^n) = \left\{ f \in H(U^n) : \sup_{0 \le r < 1} M_p(f,r) < \infty \right\}, \tag{1.5}$$

where $M_p^p(f,r) = \int_{T^n} |f(r\xi_1,\dots,r\xi_n)|^p dm_n(\xi),\ M_\infty(f,r) = \max_{\xi \in T^n} |f(r\xi_1,\dots,r\xi_n)|,\ r \in (0,1),$ $f \in H(U^n)$.

For $\alpha_j > -1, j = 1,\dots,n\ 0 < p < \infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|^p_{A^p_{\vec{\alpha}}} = \int_{U^n} |f(z_1,\dots,z_n)|^p \prod_{i=1}^n \left(1 - |z_i|^2\right)^{\alpha_i} dm_{2n}(z) < \infty. \tag{1.6}$$

For $\alpha_j > -1,\ j = 1,\dots,n,\ p \in (0,\infty]$, the Bergman class on expanded disk is defined by

$$\begin{aligned} A^p_{\vec{\alpha}}(U^n_*) = \Bigg\{ f \in H(U^n) : \|f\|^p_{A^p_{\vec{\alpha}}(U^n_*)} \\ = \int_0^1 \cdots \int_0^1 \int_T |f(|z_1|\xi,\dots,|z_n|\xi)|^p \prod_{j=1}^n \left(1 - |z_j|^2\right)^{\alpha_j} d|z_j|\, dm(\xi) < \infty \Bigg\}, \end{aligned} \tag{1.7}$$

and similarly the Bergman class on subframe denoted by $A^p_\alpha(\widetilde{U}^n)$ is defined by

$$A^p_\alpha\left(\widetilde{U}^n\right) = \left\{ f \in H(U^n) : \|f\|^p_{A^p_\alpha(\widetilde{U}^n)} = \int_{T^n} \int_0^1 |f(|z|\xi_1,\dots,|z|\xi_n)|^p \left(1 - |z|^2\right)^\alpha dm_n(\xi)\, d|z| < \infty \right\}, \tag{1.8}$$

where $p \in (0,\infty),\ \alpha > -1$.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C such that $(B/C) \le A \le CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

This paper is organized as follows. In first section we collect preliminary assertions. In the second section we present several new results connected with so-called operator of diagonal map in polydisk. Namely, we define two new maps $(\mathrm{Sb})(f)$ and $(\mathrm{Ed})(f)$ from subframe $\widetilde{U}$ and expanded disk U^n_* to unit disk U and, in particular, completely describe traces of Bergman classes $A^p_\alpha(\widetilde{U}^n)$ and $A^p_\alpha(U^n_*)$, $p \in (0,\infty]$ defined on subframe and expanded disk on usual unit disk U on the complex plane. Proofs are based among other things on new projection theorems for these classes.

A separate section will be devoted to the study of $\mathcal{R}^s$ differential operator in polydisk. It is based in particular on results from the recent paper [3]. We will use the dyadic decomposition technique to explore connections between analytic classes on subframe, polydisk, and expanded disk. We also prove new sharp embedding theorems for classes on subframe and expanded disk. Last assertions of the final section generalize some one-dimensional known results to polydisk and to the case of $\mathcal{R}^s$ operators simultaneously.

2. Preliminaries

We need the following assertions.

Lemma A (see [4]). *Let α be a fixed n-tuple of nonnegative numbers and let $\{B_i\}_{i\in\mathbb{Z}}$ be an arbitrary family of α-boxes in $\mathbb{R}^n$ lying in the cube $Q^n = [-2\pi, 2\pi]^n$. There exists a set $J \subset I$ such that $B_i \cap B_{i'} = \emptyset$, $i, i' \in J$ and for all $i \in I$ there exists $j \in J$ such that $B_i \subset 5B_j$.*

The following proposition is heavily based on ideas from [4].

Proposition 2.1. (a) *Let f be a nonnegative summable function on $T^n = T \times \cdots \times T$. Let $f^*(x_1, \dots, x_n) = f(e^{ix_1}, \dots, e^{ix_n})$. Let $\alpha_j \in \mathbb{N}$, $r_j \in (0,1)$, $j = 1, \dots, n, x \in \mathbb{R}^n$, $g(r,x) = \sup_{B\in\widetilde{\mathbf{Z}}}(1/m_n(B))\int_B f^* dm_n, \widetilde{\mathbf{Z}} = \mathbf{Z} \cap \mathbf{Z}_\alpha$, where $\mathbf{Z} = \{B = B(x,y) : y_j \geq 1 - r_j,\ 1 \leq j \leq n\}$, $B(\overrightarrow{\varphi}, \overrightarrow{y}) = \{t \in \mathbb{R}^n : |t_i - \varphi_i| \leq y_i,\ 1 \leq i \leq n\}$, and where $\mathbf{Z}_\alpha$ is a family of all $B(\overrightarrow{\varphi}, \overrightarrow{y})$ boxes such that $(y_1, \dots, y_n)$ is proportional to $(2^{-\alpha_1}, \dots, 2^{-\alpha_n})$, $\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$ for some fixed $\overleftarrow{\alpha}$. Then the following statements hold:*

$$\mu(E_\alpha(a)) = \mu\Big\{z = re^{ix} \in U : g_{\overrightarrow{\alpha}}(r,x) > a\Big\} \leq \frac{C}{a}\int_{T^n} f^*(\varphi)dm_n(\varphi),$$
$$\mu_1\left(E^1_\alpha(a)\right) = \mu_1\left\{z = \left(r_1e^{ix}, \dots, r_ne^{ix}\right) \in U^n_* : g_{\overrightarrow{\alpha}}(r_1, \dots, r_n, x) > a\right\} \leq \frac{C_1}{a}\textstyle\int_{T^n} f^*(\varphi)dm_n(\varphi), \tag{2.1}$$

where μ and μ_1 are any positive Borel measures on U and U^n_* such that

$$\begin{aligned}\mu(\Delta_l(\xi) \cap U) = \mu(\Delta_l(\xi)) \leq Cl^n, &\quad l \in (0,1),\ \xi \in T,\\ \mu_1(\Delta_l(\xi) \cap U^n_*) \leq Cl_1 \cdots l_n, &\quad l_j \in (0,1),\ \xi \in T^n,\end{aligned} \tag{2.2}$$

where $\Delta_l(\xi) = \{z \in U^n : 1 - l_j \leq |z_j| < 1, |\arg(z_j) - \arg(\xi_j)| \leq l_j/2, j = 1, \dots, n\}$. (b) Let $\mu(z) = (1 - |z|)^{n-2}\ dm_2(z), \mu_1(z) = \prod_{k=1}^n (1 - |z_k|)^{-1/n}\ d|z_1| \cdots d|z_n| dm(\xi), n > 1$. Then

$$\mu(E_\alpha(a)) \leq \frac{C}{a}\int_{T^n} f^*(\varphi)dm_n(\varphi), \qquad \mu_1\left(E^1_\alpha(a)\right) \leq \frac{C}{a}\int_{T^n} f^*(\varphi)dm_n(\varphi). \tag{2.3}$$

Proof. Proofs of all parts are similar. The first part of the lemma connected with μ measure and unit disk can be found in [4]. We will give the complete proof of the second part. To prove the second part let $E^0_\alpha(a) = \{z = re^{ix} \in U^n : g_{\overrightarrow{\alpha}}(\overrightarrow{r}, x) > a\}$.

Fix a point z in the $E^0_\alpha(a)$ and associate an α- box B_z such that

$$\frac{1}{|B_z|}\int_{B_z} f^* dm_n > a,\ B_z = B_{z_1}\times\cdots\times B_{z_n},\ |B_z| = |B_{z_1}|\cdots|B_{z_n}|,\ \left|B_{z_j}\right| \geq 1-r_j. \tag{2.4}$$

We use standard covering Lemma A to construct a set $J\subset E^0_\alpha(a)$ such that α- boxes $B_z,\ z\in J$ pairwise disjoint, we have

$$\sum_{z\in J}(|B_z|) \leq \frac{1}{a}\sum_{z\in J}\int_{B_z} f^*(\xi)dm_n(\xi) \leq \int_{T^n} f dm_n. \tag{2.5}$$

So the lemma will be proved if $\mu_1(E^1_\alpha(a)) \leq C\sum_{z\in J}|B_z|$. Let

$$K\left(e^{i\varphi},l\right) = \left\{\xi\in U : \left|\xi - e^{i\varphi}\right| < l\right\},\quad l>0,\quad E^1_\alpha(a) = \left\{z = \overrightarrow{r}e^{ix}\in U^n_* : g_{\overrightarrow{\alpha}}(\overrightarrow{r},x) > a\right\}. \tag{2.6}$$

Let $w\in E^0_\alpha(a)$, then we will show $w\in\bigcup_{z\in J}K(e^{i\varphi_1},l_{z_1})\times\cdots\times K(e^{i\varphi_n},l_{z_n}),\varphi_j = \arg(z_j)$. So $E^0_\alpha(a)\subset\bigcup_{z\in J}K(e^{i\varphi_1},l_{z_1})\times\cdots\times K(e^{i\varphi_n},l_{z_n}) = \bigcup_{z\in J}K^z_1$, where $l_{z_j} = C|B_{z_j}|,\ j=1,\ldots,n$, and constant C will be specified later. Hence we will have $\mu_1(E^0_\alpha(a)\cap U^n_*) = \mu_1(E^1_\alpha(a) \leq C\sum_{z\in J}\mu_1(K^z_1\cap U^n_*)\leq C\sum_{z\in J}|B_z|$.

The last estimate follows from inclusion $K^z_1\cap U^n_*\subset\Delta_{l_z}(\xi)\cap U^n_*$,

$$\Delta_{l_z}(\xi)\cap U^n_* = \left\{z\in U^n_* : \left|z_j\right|\in\left[1-l_{z_j},1\right),\ \left|\arg(\xi_j)-\arg(z)\right|\leq\frac{l_{z_j}}{2}, j=1,\ldots,n\right\}. \tag{2.7}$$

It remains to show the inclusion $E^0_\alpha(a)\subset\bigcup_{z\in J}K^z_1$. To show this inclusion we note if $\xi = (\xi_1,\ldots,\xi_n)\in E^0_\alpha(a)$ then we have $|B_{\xi_i}|\geq 1-|\xi_i|,\ i=1,\ldots,n$. Using covering Lemma A we have $B_{\xi_i}\subset\widetilde{C}B_{z_i}, i=1,\ldots,n,\ z=(z_1,\ldots,z_n)\in J$. Hence

$$\left|\xi_i - e^{i\varphi_i}\right| \leq \left|\xi_i - e^{i\arg(\xi_i)}\right| + \left|e^{i\arg(\xi_i)} - e^{i\varphi_i}\right| \leq (1-|\xi_i|) + \left|\arg(\xi_i)-\varphi_i\right| \leq 2\widetilde{C}|B_{z_i}|. \tag{2.8}$$

It remains to note that we put above $l_{z_i} = |B_{z_i}|2\widetilde{C}$.

(b) Note that for the case of U^n_* we can step by step repeat the same procedure with $E^1_\alpha(a)$ instead of $E^0_\alpha(a)$ and the condition on μ_1 will be replaced by weaker condition

$$\mu_1(\Delta_l(\xi)\cap U^n_*)\leq Cl_1\cdots l_n,\quad l_j\in(0,1),\ \xi\in T. \tag{2.9}$$

Note for $\xi\in T$

$$\Delta_l(\xi)\cap U^n_* = \left\{z\in U^n_* : 1-l_j\leq\left|z_j\right|<1, \left|\arg\left(z_j\right)-\arg(\xi)\right|\leq\frac{\min l_j}{2}, j=1,\ldots,n\right\}. \tag{2.10}$$

Now part (b) can be obtained by direct calculation. □

Remark 2.2. In Proposition 2.1(b) $\prod_{k=1}^{n}(1-|z_k|)^{-1/n}$ can be replaced by $\prod_{k=1}^{n}(1-|z_k|)^{-t_k}$, $t_j \in (0,1)$, $\sum_{j=1}^{n} t_j = 1$.

Lemma 2.3. *Let $z_j \in U$, $j = 1,\ldots,n$, $0 < p \le 1$, $s > (1/p) - 2$, $f \in H(U)$ and $F(z_1,\ldots,z_n) = (C\int_U f(z)(1-|z|)^s dm_2(z))/(\prod_{j=1}^{n}(1-\overline{z}z_j)^{(s+2)/n})$. Then*

$$|F(z_1,\ldots,z_n)|^p \le C\int_U \frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^{sp+2p-2}dm_2(\widetilde{w})}{\prod_{k=1}^{n}|1-\overline{z_k}\widetilde{w}|^{((s+2)/n)p}}. \tag{2.11}$$

Estimate (2.11) for $n = 1$ can be found in [5] and in [1] for general case. The following lemma is well known.

Lemma 2.4. *Let $s > 0$. Then for $I(s), I(s) = \int_s^\infty (t^{\beta-\alpha-2}dt)/(t+1)^{1+\beta}$ one has* (a) $I(s) \sim s^{\beta-\alpha-1}$, $\beta < 1+\alpha, s \to 0$; (b) $I(s) \sim \ln 1/s, 1+\alpha = \beta$, $s \to 0$; (c) $I(s) \le C$, $1+\alpha < \beta$, $s \to 0$.

Lemma 2.5 (see [3]). *Let $w = |w|\xi, w, z \in U^n, 1-w\overline{z} = \prod_{k=1}^{n}(1-w_k\overline{z_k}), s \in \mathbb{N}\cup\{0\}, \beta > 0, p \in (0,\infty)$. Then one has*

$$\int_{T^n}\left|\mathcal{R}^s\frac{1}{(1-\xi|w|z)^\beta}\right|^p dm_n(\xi) \le C\sum_{\alpha_j\ge 0,\ \sum\alpha_j=s}\left(\prod_{k=1}^{n}\frac{1}{(1-|w_k||z_k|)^{p(\alpha_k+\beta)-1}}\right),\ p > \frac{1}{\min_k\ \alpha_k+\beta}. \tag{2.12}$$

Corollary 2.6. *Let $0 < p < \infty, s \in \mathbb{N}\cup\{0\}, l \in (0,\infty), \gamma > 1/p + l, w \in U^n$. Then*

$$\int_{U^n}\left|\mathcal{R}^s\frac{1}{(1-w\overline{z})^\gamma}\right|^p(1-|z|)^{pl-1}dm_{2n}(z) \le \sum_{\alpha_j\ge 0,\sum\alpha_j=s}\prod_{k=1}^{n}\frac{C}{(1-|w_k|)^{(\alpha_k+\gamma)p-pl-1}}. \tag{2.13}$$

We will need the following Theorems A and B.

Theorem A (see [3]). (*a*) *Let $f \in H(U^n)$, $p \in (0,\infty)$, $\alpha_j > -1, j = 1,\ldots,n$, $n \in \mathbb{N}$. Then*

$$\begin{aligned}&\int_U |f(z,\ldots,z)|^p\left(1-\ |z|^2\right)^{\sum_{j=1}^{n}\alpha_j+n-1}dm_2(z)\\ &\quad\le C\int_T\int_{[0,1]^n}|f(|z_1|\xi,\ldots,|z_n|\xi)|^p\prod_{k=1}^{n}\left(1-|z_k|^2\right)^{\alpha_k}d|z_1|\cdots d|z_n|dm(\xi).\end{aligned} \tag{2.14}$$

(*b*) *Let $f \in H(U^n)$, $p \in (0,\infty), \alpha > -1$, $n \in \mathbb{N}$. Then*

$$\int_U |f(z,\ldots,z)|^p\left(1-|z|^2\right)^{\alpha+n-1}dm_2(z) \le C\int_{T^n}\int_0^1 |f(|z|\xi_1,\ldots,|z|\xi_n)|^p\left(1-|z|^2\right)^{\alpha}dm_n(\xi)d|z|. \tag{2.15}$$

Theorem B (see [3]). *(a) Let $f \in H(U^n)$, $p \in (0,\infty)$, $\alpha_j > -1, j = 1,\dots,n$, $n \in \mathbb{N}$. Then*

$$\int_T \int_{[0,1]^n} |f(|z_1|\xi,\dots,|z_n|\xi)|^p \prod_{k=1}^{n} (1-|z_k|)^{\alpha_k+((n-1)/n)}\, d|z_1|\cdots d|z_n| dm(\xi) \leq C\int_{U^n} |f(z_1,\dots,z_n)|^p \prod_{k=1}^{n}\left(1-|z_k|^2\right)^{\alpha_k} dm_{2n}(z). \tag{2.16}$$

(b) Let $f \in H(U^n)$, $p \in (0,\infty)$, $\alpha > -1$, $n \in \mathbb{N}$. Then

$$\int_0^1\int_{T^n} |f(|z|\xi_1,\dots,|z|\xi_n)|^p \left(1-|z|^2\right)^{\alpha} dm_n(\xi) d|z| \leq C\int_{U^n} |f(z_1,\dots,z_n)|^p \left(1-|z_1|^2\right)^{(\alpha/n)-1+(1/n)} \cdots \left(1-|z_n|^2\right)^{(\alpha/n)-1+(1/n)} dm_{2n}(z). \tag{2.17}$$

3. Analytic Classes on Subframe and Expanded Disk

Let us remind the main definition.

Definition 3.1. Let $\mathcal{X} \subset H(U), \mathcal{Y} \subset H(U^n)$ be subspaces of $H(U)$ and $H(U^n)$. We say that the diagonal of $\mathcal{Y}$ coincides with $\mathcal{X}$ if for any function $f, f \in \mathcal{Y}$, $f(z,\dots,z) \in \mathcal{X}$, and the reverse is also true for every function g from $\mathcal{X}$ there exists an expansion $f(z_1,\dots,z_n) \in \mathcal{Y}$, such that $f(z,\dots,z) = g(z)$. Then we write $\operatorname{Diag}(\mathcal{Y}) = \mathcal{X}$.

Note when $\operatorname{Diag}(\mathcal{Y}) = \mathcal{X}$, then

$$\|f\|_{\mathcal{X}} \asymp \inf_{\Phi} \|\Phi(f)\|_{\mathcal{Y}}, \tag{3.1}$$

where $\Phi(f)$ is an arbitrary analytic expansion of f from diagonal of polydisk to polydisk.

The problem of study of diagonal map and its applications for the first time was also suggested by Rudin in [2]. Later several papers appeared where complete solutions were given for classical holomorphic spaces such as Hardy, Bergman classes; see [1, 4, 6, 7] and references there. Recently the complete answer was given for so-called mixed norm spaces in [8]. Partially the goal of this paper is to add some new results in this direction. Theorems on diagonal map have numerous applications in the theory of holomorphic functions (see, e.g., [9, 10]).

In this section we concentrate on the study of two maps closely connected with diagonal mapping Sb from subframe into disk $\mathrm{Sb} : f(z_1,\dots,z_n) \to f(z,\dots,z)$ where all $|z_j| = |z| \in (0,1)$ and another map $\mathrm{Ed} : f(z_1,\dots,z_n) \to f(z,\dots,z)$ from expanded disk into disk where $z_1 = |z_1|\xi,\dots,z_n = |z_n|\xi$, $\xi \in T$, and f function is from a functional class on subframe $\widetilde{U}^n$ or expanded disk U_*^n.

Note that the study of maps which are close to diagonal mapping was suggested by Rudin in [2] and previously in [11] Clark studied such a map.

Theorem 3.2. *Let* $0 < p < \infty$. *If*

(a) $0 < p \leq 1$ *and* $\beta > -1$, *or*

(b) $p > 1$ *and* $\beta > \max((n/p) - 1, (n-1)(p-1) - 1)$, *then for every function* $g \in A^p_\beta(\widetilde{U^n})$, $g(|z|\xi, \ldots, |z|\xi) \in A^p_{\beta+n-1}(U)$ *and for every function* $f \in A^p_{\beta+n-1}(U)$ *there exist* $g \in A^p_\beta(\widetilde{U^n})$ *such that* $g(|z|\xi, \ldots, |z|\xi) = f(z)$.

Proof. Note that one part of the theorem was proved in [3] and follows from Theorem A. Let us show the reverse. Let first $p \leq 1$. Consider the following g function:

$$g(r\xi_1, \ldots, r\xi_n) = C_\alpha \int_U \frac{f(w)(1-|w|)^\alpha dm_2(w)}{\prod_{k=1}^n (1 - r\xi_k w)^{(\alpha+2)/n}}, \quad z = (r\xi_1, \ldots, r\xi_n), \tag{3.2}$$

where $\alpha > 0$ can be large enough, C_α is a constant of Bergman from representation formula (see [1]), obviously $g(r\xi, \ldots, r\xi) = f(r\xi)$ by Bergman representation formula in the unit disk. It remains to note that the following estimate hold by Lemma 2.3:

$$\begin{aligned} &\int_{T^n}\int_0^1 \left|g(|z|\xi_1, \ldots, |z|\xi_n)\right|^p (1-|z|)^\beta |z| d|z| dm_n(\xi) \\ &\quad \lesssim C \int_0^1 \int_{T^n}\int_U \frac{|f(w)|^p (1-|w|)^{p\alpha+2p-2}(1-r)^\beta dr dm_2(w) dm_n(\xi)}{|1 - r\xi_1\overline{w}|^{((\alpha+2)p)/n} \cdots |1 - r\xi_n\overline{w}|^{((\alpha+2)p)/n}}, \end{aligned} \tag{3.3}$$

where α can be large enough. Using Fubini's theorem and calculating the inner integral we get what we need.

We consider now $1 < p < \infty$ case. Let $f \in A^p_{\beta+n-1}(U)$, $p > 1$. Then

$$f(z) = C_\alpha \int_U \frac{f(w)(1-|w|)^\alpha}{(1-\overline{w}z)^{\alpha+2}} dm_2(w), \quad z \in U, \tag{3.4}$$

by Bergman representation formula. Obviously $(\mathrm{Sb})(g) = f$, as

$$g(r\xi_1, \ldots, r\xi_n) = C_\alpha \int_U \frac{f(w)(1-|w|)^\alpha}{(1 - r\xi_1\overline{w})^{(\alpha+2)/n} \cdots (1 - r\xi_n\overline{w})^{(\alpha+2)/n}} dm_2(w), \tag{3.5}$$

where $(\mathrm{Sb})(g) = g(r\xi, \ldots, r\xi)$, $r\xi \in U$ is a map from subframe $\widetilde{U} = \{z \in U^n : |z_j| = |z|\}$ to diagonal. Using duality arguments and Fubini's theorem we have

$$\|g\|_{A^p_\beta(\tilde{U}^n)} = C_\alpha \int_U (1-|w|)^\alpha |f(w)| \left| \int_0^1 \int_{T^n} \frac{\overline{h(r\xi_1, \ldots, r\xi_n)}(1-r)^\beta dr\, dm_n(\xi)}{\prod_{k=1}^n (1 - r\xi_k w)^{(\alpha+2)/n}} \right| dm_2(w), \tag{3.6}$$

where $\alpha = \beta + n - 1$, $\alpha > -1$, $h \in L^{p'}_\beta(\tilde{U}^n)$, $(1/p) + (1/p') = 1$.

We will need the following assertion. Let $h \in L^{p'}_{\beta}(\widetilde{U}^n)$, $\beta \in (-1,\infty)$, $p' \in (1,\infty)$. Then let

$$g(w_1,\ldots,w_n) = \int_0^1 \int_{T^n} \frac{h(r\xi_1,\ldots,r\xi_n)(1-|z|)^{\beta} dr\, dm_n(\xi)}{\prod_{j=1}^n \left(1-r\xi_j\overline{w}_j\right)^{(\beta+1+n)/n}}, \tag{3.7}$$

where $w_j \in U$, $w_j = |w|\varphi_j$, $j = 1,\ldots,n$. We assert that g belongs to $A^{p'}_{\beta}(\widetilde{U}^n)$.

Indeed using Hölder's inequality we get

$$\begin{aligned} &\int_0^1 \int_{T^n} \left|g(w_1,\ldots,w_n)\right|^{p'} (1-|w|)^{\beta} dm_n(\varphi) d|w| \\ &\quad \leq C \int_0^1 \int_0^1 \int_{T^n} \int_{T^n} \frac{|h(r\xi_1,\ldots,r\xi_n)|^{p'}(1-r)^{\beta p'}(1-|w|)^{\beta+(p'/p)(n(\varepsilon p+1))}}{\prod_{k=1}^n |1-r\xi_k\overline{w}_k|^{((\beta+n+1)/n-2-\varepsilon)p'+2}} \\ &\quad \times dr\, dm_n(\xi) dm_n(\varphi) d|w| \leq C\|h\|_{L^{p'}_{\beta}(\widetilde{U}^n)}, \quad 1 < p' < \infty, \end{aligned} \tag{3.8}$$

where $\varepsilon > 0$. We used above the estimate

$$\int_{T^n} \int_0^1 \frac{(1-|w|)^{\beta+(p'/p)\left(n(\varepsilon p+1)\right)} dm_n(\varphi) d|w|}{\prod_{k=1}^n |1-r\xi_k\overline{w}_k|^{\,(((\beta+1))/n)\,-\varepsilon-1)p'+2}} \leq C(1-r)^{\beta-\beta p'}, \tag{3.9}$$

$r \in (0,1)$, $\beta > (n-1)(p/p') - 1$, $\beta > (n/p) - 1$.

Returning to estimate for $\|g\|_{A^p_{\beta}}$ we have by Hölder's inequality

$$\begin{aligned} &\|g\|_{A^p_{\beta}}\left(\widetilde{U}^n\right) C\left(\int_U |f(w)|^p (1-|w|)^{\beta+n-1} dm_2(w)\right)^{1/p} \\ &\quad \times \left(\int_U |\Phi(w,\ldots,w)|^{p'} (1-|w|)^{\beta+n-1} dm_2(w)\right)^{1/p'} \lesssim C\|f\|_{A^p_{\beta+n-1}}. \end{aligned} \tag{3.10}$$

We used the fact that for all $\Phi \in A^{p'}_{\beta}, 1 < p' < \infty$, $(\mathrm{Sb})\Phi \in A^{p'}_{\beta+n-1}$ proved before. □

The complete analogue of Theorem 3.2 is true for Bergman classes on expanded disk we defined previously.

Theorem 3.3. *Let $0 \leq p < \infty$. If*

(a) $p \in [0,1]$, $\overrightarrow{\beta} = (\beta_1,\ldots,\beta_n)$, $\beta_k > -1, k = 1,\ldots,n$, $|\beta| = \sum_{k=1}^n \beta_k$, *or*

(b) $p > 1$, $\overrightarrow{\beta} = (\beta,\ldots,\beta), \beta > 0$, $\beta > ((1/n)-1)(p/p') - 1$, *then the following assertion holds. For every function f, $f \in A^p_{\overrightarrow{\beta}}(U^n_*)$, $(\mathrm{Ed} f)(z) = f(z,\ldots,z)$, $z \in U$ belongs to $A^p_{|\beta|+n-1}$ and the reverse is also true, for any function f from $A^p_{|\beta|+n-1}$ there exists a function $g \in A^p_{\overrightarrow{\beta}}(U^n_*)$ such that $g(z,\ldots,z) = f(z)$, for all $z \in U$.*

Proof. We give a short sketch of proof of Theorem 3.3 and omit details. Note that the half of the theorem the inclusion $\operatorname{Ed}A^p_{\vec{\beta}}(U^n_*) \subset A^p_{|\beta|+n-1}$ was proved in [3] and follows directly from Theorem A.

For $p \le 1$ we have to use again Lemma 2.3 and Fubini's theorem. For $p > 1$ we first prove $g(w_1,\dots,w_n) \in A^p_{\vec{\beta}}(U^n_*)$ if $h \in A^p_{\vec{\beta}}(U^n_*)$ and if $|\beta|p/n > \sup_k \beta_k > \min_k \beta_k > ((1/n) - 1)(p/p') - 1$

$$g(w_1,\dots,w_n) = C\int_T\int_0^1\cdots\int_0^1 \frac{h(z_1,\dots,z_n)\prod_{k=1}^n (1-|z_k|)^{|\beta|/n} d|z_k| d\xi}{\prod_{k=1}^n |1-\overline{z_k}w_k|^{(|\beta|+1+n)/n}}. \tag{3.11}$$

Indeed by Hölder's inequality we have

$$\begin{aligned} |g(w_1,\dots,w_n)|^p &\lesssim C\left(\int_T\int_0^1\cdots\int_0^1 \frac{|h(z_1,\dots,z_n)|^p \prod_{k=1}^n (1-|z_k|)^{|\beta|p/n} d|z_k| d\xi}{\prod_{k=1}^n |1-\overline{z_k}w_k|^{((|\beta|+1+n)/n)p-(2+\varepsilon)p+2}}\right) \\ &\quad\times\left(\int_T\int_0^1\cdots\int_0^1 \frac{d|z_k|d\xi}{\prod_{k=1}^n |1-\overline{z_k}w_k|^{2+\varepsilon p'}}\right)^{p/p'} \\ &\lesssim A(h)\left(\int_T \frac{d\xi}{\prod_{k=1}^n \left|1-\xi|w_k\overline{\varphi}|\right|^{1+\varepsilon p'}}\right)^{p/p'} \\ &\lesssim A(h)\left(\prod_{k=1}^n \frac{1}{(1-|w_k|)^{(1+\varepsilon p'-(1/n))(p/p')}}\right), \quad \text{where } \varepsilon > 0. \end{aligned} \tag{3.12}$$

Hence calculating integrals we finally have $\|g\|_{A^p_{\vec{\beta}}(U^n_*)} \lesssim C\|h\|_{A^p_{\vec{\beta}}(U^n_*)}$.

We used the estimate

$$\int_T\int_0^1\cdots\int_0^1 \frac{\prod_{k=1}^n (1-|w_k|)^{\beta_k+((1/n)-(1+\varepsilon p'))p/p'} d|w_k|d\xi}{\prod_{k=1}^n |1-z_k\overline{w_k}|^{(|\beta|+1+n)/np-(2+\varepsilon)p+2}} \le C\prod_{k=1}^n (1-|z_k|)^{(-|\beta|p/n)+\beta_k}, \tag{3.13}$$

which is true under the conditions on indexes we have in formulation of theorem and can be obtained by using Hölder's inequality for n functions. Using this projection theorem and repeating arguments of proof of the previous theorem we will complete the proof of Theorem 3.3. □

Remark 3.4. Note that Theorems 3.2 and 3.3 are obvious for $n = 1$.

Remark 3.5. Note that estimates between expanded disk, unit disk, and polydisk can be also obtained directly from Liuville's formula

$$\int_0^1 \cdots \int_0^1 \varphi(v_1 \cdots v_n) \prod_{k=1}^{n} (1 - v_k)^{p_k - 1} v_2^{p_1} v_3^{p_1+p_2} \cdots v_n^{p_1+\cdots p_{n-1}} dv_1 \cdots dv_n$$
$$= \frac{\prod_{k=1}^{n} \Gamma(p_k)}{\Gamma(p_1 \cdots p_n)} \int_0^1 \varphi(u)(1-u)^{\sum_{i=1}^{n} p_i - 1} du, \quad p_i > 0,\ \varphi \text{ is continuous function on } (0,1). \tag{3.14}$$

Remark 3.6. The complete description of traces of classes with $\|f\|_{A_\alpha^\infty(U_*^n)}$ and $\|f\|_{A_\alpha^\infty(\widetilde{U}^n)}$ quasinorms on the unit disk can be obtained similarly by small modification of the proof of Theorem 3.2,

$$\|f\|^p_{A_\alpha^\infty(\widetilde{U}^n)} = \int_0^1 (M_\infty \cdots M_\infty(f,r))^p (1-r)^\alpha dr, \quad \alpha > -1,\ p \in (0,\infty),$$
$$\|f\|^p_{A_\alpha^\infty(U_*^n)} = \int_0^1 \int_0^1 (M_\infty(f, r_1, \ldots, r_n))^p (1-r_1)^\alpha \cdots (1-r_n)^\alpha dr_1 \cdots dr_n, \quad \alpha > -1,\ p \in (0,\infty) \tag{3.15}$$

Let

$$\Lambda_\alpha(U_*^n) = \left\{ f \in H(U^n) : \sup_{r_j \in (0,1), \xi \in T} \left| f(r_1\xi, \ldots, r_n\xi) \right| \prod_{k=1}^{n} (1-r_k)^\alpha < \infty \right\}, \quad \alpha > 0,$$
$$\Lambda_\alpha\left(\widetilde{U}^n\right) = \left\{ f \in H(U^n) \sup_{r \in (0,1), \xi_i \in T} \left| f(r\xi_1, \ldots, r\xi_n) \right| (1-r)^\alpha < \infty \right\}, \quad \alpha > 0. \tag{3.16}$$

We formulate complete analogues of Theorems 3.2 and 3.3 for classes $\Lambda_\alpha(U_*^n)$ *and* $\Lambda_\alpha(\widetilde{U}^n)$.

Theorem 3.7. *Let $\alpha > 0$ and $f \in \Lambda_\alpha(\widetilde{U}^n)$. Then* $(\mathrm{Sb})(f) = f(z, \ldots, z)$ *is in $\Lambda_\alpha(U)$ and any $g \in \Lambda_\alpha(U)$ can be expanded to f, $f \in \Lambda_\alpha(\widetilde{U}^n)$ such that $f(z, \ldots, z) = g(z)$. The same statement is true for pairs* $(\mathrm{Ed}, \Lambda_\alpha(U_*^n)), \mathrm{Ed}(\Lambda_\alpha(U_*^n) = \Lambda_{n\alpha}(U)$.

Note that one part of statement is obvious. If, for example, $f \in \Lambda_\alpha(U_*^n)$, then $\mathrm{Ed} f \in \Lambda_{n\alpha}(U)$. On the other side, let $g \in \Lambda_{n\alpha}(U)$. Then define as above that

$$f(r_1\xi, \ldots, r_n\xi) = C_\beta \int_U \frac{g(z)(1-|z|)^\beta}{\prod_{k=1}^{n} (1 - \bar{z} r_k \xi)^{(\beta+2)/n}} dm_2(z), \tag{3.17}$$

β is big enough, C_β is a Bergman constant of Bergman representation formula.

Obviously $f(r\xi, \ldots, r\xi) = g(r\xi)$, $z = r\xi$, $z \in U$. Using Hölder's inequality for n functions we get $\|f\|_{\Lambda_\alpha(U_*^n)} \le C_1 \|g\|_{\Lambda_{n\alpha}(U)}$. Similarly $\|f\|_{\Lambda_\alpha(\widetilde{U}^n)} \le C_2 \|g\|_{\Lambda_\alpha(U)}$.

It is natural to question about discrete analogues of operators we considered previously.

Let $C_k > 0$, $r_k \searrow 1$, $k \to \infty$, $\beta \geq 0$. Let

$$B_{n,C_k,r_k,\beta} = \Bigg\{ f \in H(U_*^n) : f(z_1, \dots, z_n) = \sum_{k=1}^{\infty} C_k \prod_{j=1}^{n} \frac{1}{(r_k - z_j)^{1+\beta/n}} < \infty, \quad |z_j| = |z| \in (0,1),\ z_j \in U, j = 1, \dots, n \Bigg\}. \tag{3.18}$$

We have for such a function

$$\begin{aligned} \int_0^1 (1-r)^{1+\alpha} (M_\infty(f,r))dr &= \int_0^1 (1-r)^{1+\alpha} \sup_{|z_1|=r,\dots,|z_n|=r} |f(z)| dr \\ &= \sum_{k=1}^{\infty} C_k \int_0^1 \frac{(1-r)^{1+\alpha}}{(r_k - r)^{1+\beta}} dr = \sum_{k=1}^{\infty} C_k (r_k - 1)^{\alpha-\beta+1} \int_{r_k-1}^{\infty} \frac{t^{\beta-\alpha-2}}{(t+1)^{1+\beta}} dt. \end{aligned} \tag{3.19}$$

As a consequence of these arguments and using Lemma 2.4 we have the following proposition, a discrete copy of assertions we proved above.

Proposition 3.8. *Let $f \in B_{n,C_k,r_k,\beta}$ and $\alpha > -1$. Then $\int_0^1 (1-r)^{1+\alpha} (M_\infty(f,r))dr < \infty$ if and only if $\sum_{k=1}^{\infty} C_k < \infty$ if $1+\alpha > \beta$; $\sum_{k=1}^{\infty} C_k \ln 1/(r_k - 1) < \infty$ if $1+\alpha = \beta$; $\sum_{k=1}^{\infty} C_k/(r_k-1)^{\beta+1} < \infty$ if $1+\alpha < \beta$.*

4. Sharp Embeddings for Analytic Spaces in Polydisk with $\mathcal{R}^s$ Operators and Inequalities Connecting Classes on Polydisk, Subframe, and Expanded Disk

The goal of this section is to present various generalizations of well-known one-dimensional results providing at the same time new connections between standard classes of analytic functions with quazinorms on polydisk and $\mathcal{R}^s$ differential operator with corresponding classes on subframe and expanded disk.

In this section we also study another two maps connected with the diagonal mapping from polydisk to subframe and expanded disk using, in particular, estimates for maximal functions from Lemma 2.3 which are of independent interest. Note that for the first time the study of such mappings which are close to diagonal mapping was suggested by Rudin in [2]. Later Clark studied such a map in [11].

In this section we also introduce the $\mathcal{R}^s$ differential operator as follows (see [3, 12, 13]). $\mathcal{R}^s f = \sum_{k_1,\dots,k_n \geq 0} (k_1 + \cdots + k_n + 1)^s a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$, $s \in \mathbb{R}$, where $f(z) = \sum_{k_1,\dots,k_n \geq 0} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n} \in H(U^n)$.

Note it is easy to check that $\mathcal{R}^s$ acts from $H(U^n)$ into $H(U^n)$.

In the case of the unit ball an analogue of $\mathcal{R}^s$ operator is a well-known radial derivative which is well studied. We note that in polydisk the following fractional derivative is well studied (see [1]):

$$(\mathfrak{D}^\alpha f)(z) = \sum_{k_1,\ldots,k_n \geq 0} (k_1+1)^\alpha \cdots (k_n+1)^\alpha a_{k_1,\ldots,k_n} z_1^{k_1} \cdots z_n^{k_n}, \tag{4.1}$$

where $\alpha \in \mathbb{R}$, $f \in H(U^n)$, and $\mathfrak{D} : H(U^n) \to H(U^n)$. Apparently the $\mathcal{R}^s$ operator was studied in [12] for the first time. Then in [13], the second author studied some properties of this operator. In this section we also continue to study the $\mathcal{R}^s$ operator. We need the following simple but vital formula which can be checked by easy calculation:

$$f(\tau\xi_1,\ldots,\tau\xi_n) = C_s \int_0^1 \mathcal{R}^s f(\tau\xi_1\rho,\ldots,\tau\xi_n\rho)\left(\log\frac{1}{\rho}\right)^{s-1} d\rho, \tag{4.2}$$

where $s > 0, \tau \in (0,1), C_s > 0, \xi_j \in T, j = 1,\ldots,n$. This simple integral representation of holomorphic $f(z), z \in U^n$ functions in polydisk will allow us to consider them in close connection with functional spaces on subframe $\widetilde{U^n}$.

The following dyadic decomposition of subframe and polydisk was introduced in [1] and will be also used by us:

$$\begin{aligned}
\widetilde{U}_{k,l_1,\ldots,l_n} &= \widetilde{U}_{k,l_1} \times \cdots \times \widetilde{U}_{k,l_n} \\
&= \left\{(\tau\xi_1,\ldots,\tau\xi_n) : \tau \in \left(1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}\right],\right. \\
&\qquad \left. k = 0,1,2,\ldots;\ \frac{\pi l_j}{2^k} < \xi_j \leq \frac{\pi(l_j+1)}{2^k}, l_j = -2^k,\ldots,2^k-1, j = 1,\ldots,n\right\}, \\
m\left(\widetilde{I}_{k,l_j}\right) &= m\left(\xi \in T : \frac{\pi l_j}{2^k} < \xi_j \leq \frac{\pi(l_j+1)}{2^k}\right) \asymp 2^{-k}, m_{2n}\left(\widetilde{U}_{k,l_1,\ldots,l_n}\right) \asymp 2^{-2kn}, \\
U_{k_1,\ldots,k_n,l_1,\ldots,l_n} &= U_{k_1,l_1} \times \cdots \times U_{k_n,l_n} \\
&= \left\{(\tau_1\xi_1,\ldots,\tau_n\xi_n) : \tau_j \in \left(1-\frac{1}{2^{k_j}}, 1-\frac{1}{2^{k_j+1}}\right],\right. \\
&\qquad \left. k_j = 0,1,2,\ldots;\ \frac{\pi l_j}{2^{k_j}} < \xi_j \leq \frac{\pi(l_j+1)}{2^{k_j}}, l_j = -2^{k_j},\ldots,2^{k_j}-1, j = 1,\ldots,n\right\},
\end{aligned} \tag{4.3}$$

$M_p(f,r)$, $0 < p \leq \infty$ averages in analytic spaces in polydisk can obviously have a mixed form, for example, $M_p M_\infty(f,r_1,r_2)$, $r_j \in (0,1)$, $j = 1,2$, $p < \infty$. In [8] Ren and Shi described the diagonal of mixed norm spaces, but the above mentioned mixed case was omitted there. Our approach is also different. It is based on dyadic decomposition we introduced previously.

Theorem 4.1. *Let* $p \in (0,\infty)$ *and* $H^{p,\infty}_{\alpha,n} = \{f \in H(U^n) : \int_0^1 \cdots \int_0^1 \int_T (M_\infty \cdots M_\infty(f, r))^p d\xi \prod_{k=1}^n (1-r_k)^{(\alpha+1/n)-1}\, dr_1 \cdots dr_n < \infty,\ \alpha > -1\}$. *Then* $\operatorname{Diag} H^{p,\infty}_{\alpha,n} = A^p_\alpha(U)$.

Proof. Using diadic decomposition of polydisk we have

$$
\begin{aligned}
M &= \int_U \left| f(z,\ldots,z)\right|^p (1-|z|)^\alpha dm_2(z) = \sum_{j,k}\int_{U_{j,k}} \left| f(z,\ldots,z)\right|^p (1-|z|)^\alpha dm_2(z) \\
&\le C\sum_{j,k}\max_{U_{j,k}}\left| f(z,\ldots,z)\right|^p 2^{-2k}2^{-\alpha k} \\
&\le C\sum_{k_1\ge 0}\cdots\sum_{k_n\ge 0}\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}\max_{U_{k_1,\ldots,k_n,j}}\left| f(z_1,\ldots,z_n)\right|^p 2^{-(\alpha+2)k_1/n}\ldots 2^{-(\alpha+2)k_n/n} \\
&\lesssim \sum_{k_1\ge 0}\cdots\sum_{k_n\ge 0}\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}\left(\int_{1-2^{-(k_1-1)}}^{1-2^{-(k_1+2)}}\cdots\int_{1-2^{-(k_n-1)}}^{1-2^{-(k_n+2)}}\right. \\
&\quad \left.\times\int_{I_{j,k_1,\ldots,k_n}}\left| f(z_1,\ldots,z_n)\right|^p dr_1\cdots dr_n d\xi\right)2^{k_1\tau}\cdots 2^{k_n\tau}\left(2^{\min k_j}\right)^n, \quad \tau = -\frac{\alpha+2}{n}+1.
\end{aligned}
\tag{4.4}
$$

We used above the following estimate which can be found, for example, in [1]

$$
\sup_{z\in U_{k_1,\ldots,k_n,j}}\left| f(z_1,\ldots,z_n)\right|^p\left(2^{\min k_j}\right)^n 2^{k_1+\cdots k_n}\int_{U^*_{k_1,\ldots,k_n,j}}\left| f(z_1,\ldots,z_n)\right|^p dm_{2n}(z), \quad 0<p<\infty,
\tag{4.5}
$$

where $U^*_{k_1,\ldots,k_n,j}$ are enlarged dyadic cubes (see [1]) and $0<p<\infty$, $f\in H(U^n)$, and

$$
\begin{aligned}
I_{j,k_1,\ldots,k_n} &= \left\{\xi : z = r\xi \in U_{k_1,\ldots,k_n,j}, \frac{\pi j}{2^{\min k_j}} \le \xi < \frac{\pi(j+1)}{2^{\min k_j}}\right\}, \\
I^*_{j,k_1,\ldots,k_n} &= \left\{\xi : z = r\xi \in U^*_{k_1,\ldots,k_n,j}, \frac{\pi(j-(1/2))}{2^{\min k_j}} \le \xi < \frac{\pi(j+(1/2))}{2^{\min k_j}}\right\}.
\end{aligned}
\tag{4.6}
$$

Note further since $m_n(I_{j,k_1,\ldots,k_n}) = (2^{-n})^{\min k_j}$,

$$
\begin{gathered}
\int_{I^*_{j,k_1,\ldots,k_n}}\left| f(z_1,\ldots,z_n)\right|^p d\xi \lesssim C\int_{I^1_{j,k_1,\ldots,k_n}}(M_\infty\cdots M_\infty(f,r))^p d\xi\left(2^{-(n-1)\min k_j}\right), \\
I^*_{j,k_1,\ldots,k_n} = I^1_{j,k_1,\ldots,k_n}\times\cdots\times I^n_{j,k_1,\ldots,k_n}, \quad 0<p<\infty.
\end{gathered}
\tag{4.7}
$$

We have

$$
\begin{gathered}
M \lesssim C\sum_{k_1\ge 0}\cdots_{k_n\ge 0}\sum\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}\int_{1-2^{-(k_1-1)}}^{1-2^{-(k_1+2)}}\cdots\int_{1-2^{-(k_n-1)}}^{1-2^{-(k_n+2)}}\int_{I^1_{j,k_1,\ldots,k_n}} \\
(M_\infty\cdots M_\infty(f,r))^p d\xi 2^{-k_1((\alpha+1)/n)-1)}\cdots 2^{-k_n(((\alpha+1)/n)-1/n)}.
\end{gathered}
\tag{4.8}
$$

We used the fact that $2^{\min k_j} < 2^{(k_1+\cdots+k_n)\ /n}$.

Hence using the fact that $I^1_{j,k_1,\dots,k_n}$ is a finite covering of T, finally we have for all $0 < p < \infty$ $M \lesssim C\|f\|_{H^{p,\infty}_{\alpha,n}}$. One part of theorem is proved.

To get the reverse statement we use the estimate from Lemma 2.3. Then we have

$$|F(z_1,\dots,z_n)| = C\left|\int_U \frac{f(z)(1-|z|)^{\alpha}dm_2(z)}{\prod_{j=1}^n (1-\overline{z}z_j)^{((\alpha+2)/n)p}}\right|, \quad z_j \in U, \tag{4.9}$$

for any $\alpha > -1$. Hence $F(z,\dots,z) = f(z)$ and for $p < 1$ by Lemma 2.3

$$\begin{aligned}
&\|F\|_{H^{p,\infty}_{\alpha,n}} \int_0^1\!\!\int_0^1\!\!\int_T (M_\infty \cdots M_\infty)\int_U \frac{|f(z)|^p(1-|z|)^{\alpha p+2p-2}dm_2(z)}{\prod_{j=1}^n |1-\overline{z}z_j|^{((\alpha+2)/n)p}} \\
&\quad\times \prod_{k=1}^n (1-r_k)^{((\alpha+1)/n)-1}dr_1\cdots dr_n d\xi_1\cdots d\xi_n \\
&\lesssim \int_U |f(z)|^p(1-|z|)^{\alpha p+2p-2}\int_0^1\!\!\int_0^1 \frac{\prod_{k=1}^n (1-r_k)^{((\alpha+1)/n)-1}dr_1\cdots dr_n}{\prod_{k=1}^{n-1}(1-|z|r_k)^{((\alpha+2)/n)p}(1-r_n|z|)^{((\alpha+2)/n)p-1}}dm_2(z) \\
&\lesssim \int_U |f(z)|^p(1-|z|)^{\alpha p+2p-2}(1-|z|)^{((-(\alpha+2)/n)\,p+((\alpha+1)/n))(n-1)} \\
&\quad\times(1-|z|)^{((-(\alpha+2)/n)p+((\alpha+1)/n))+1}dm_2(z) \\
&\lesssim \int_U |f(z)|^p(1-|z|)^{\alpha}dm_2(z),\ \alpha > -1,\ 0 < p \le 1,\ f \in A^p_\alpha.
\end{aligned} \tag{4.10}$$

Let $1 < p < \infty$. Then we may assume that again s is large enough. Let $1/p + 1/q = 1, \gamma_i > 0,\ i = 1,2, \gamma_1+\gamma_2 = (s+1)/n$. Then

$$\begin{aligned}
&\left(\int_U \frac{|f(w)|\left(1-|w|^2\right)^{s-1}dm_2(w)}{\prod_{j=1}^n |1-z_j\overline{w}|^{(s+1)/n}}\right)^p \\
&\quad\le C\int_U \frac{|f(w)|^p\left(1-|w|^2\right)^{s-1}dm_2(w)}{\prod_{j=1}^n |1-z_j\overline{w}|^{\gamma_1 p}}\prod_{j=1}^n (1-|z_j|)^{\gamma_1 p-((s+1)/n)}.
\end{aligned} \tag{4.11}$$

We used estimate

$$\left(\int_U \frac{\left(1-|w|^2\right)^{s-1}dm_2(w)}{\prod_{j=1}^n |1-z_j\overline{w}|^{\gamma_2 q}}\right)^{p/q} \le C\prod_{j=1}^n (1-|z_j|)^{\gamma_1 p-((s+1)/n)}. \tag{4.12}$$

Choosing appropriate $\gamma_1,\ \gamma_2$ we repeat now arguments that we presented for $p \le 1$ above to get what we need. The proof is complete. □

Remark 4.2. The case of $k,\ 1 < k \le n,\quad M_p$ averages can be considered similarly. Note in Theorem 4.1$k = 1$. Thus our Theorem 4.1 extends known description of diagonal of classical Bergman classes (see [1, 7]).

In [14] Carleson as Rudin and Clark showed that in the case of the polydisk one cannot expect so simple description of Carleson measures as one has for measures defined in the disk. We would like to study embeddings of the type

$$\left(\int_{\widetilde{U}^n} |f(z)|^p d\mu(z)\right)^{1/p} \le C\|\mathcal{R}^s f\|_X, \quad s \ge 0, \tag{4.13}$$

where μ is a positive Borel measure on $\widetilde{U}^n$ and X is a Bergman class on polydisk or subframe.

Theorem 4.3. *Let $\gamma_1 + \beta_1 = (s-1)p + (n+1)(p-1),\ \gamma_1 > 0, \beta_1 > -1,\ 0 < p \le 1,\ s > (((\beta_1 + 1)/n) + n - pn)/p,\ s, n \in \mathbb{N}$, and μ is a positive Borel measure on $\widetilde{U}^n$. If*

$$\int_{\widetilde{U}^n} |f(z)|^p (1-|z|)^t d\mu(z) \|\mathcal{R}^s f\|_{A^p_{\beta_1}(\widetilde{U}^n)}, \quad t > -1, \tag{4.14}$$

then

$$\|\mu\|_{\widetilde{U}^n} = \sup_{\widetilde{w}\in\widetilde{U}^n} \int_{T^n}\int_0^1 \frac{(1-|w|)^t d\mu(w)}{\prod_{k=1}^n |1-\widetilde{w}_k \overline{w}_k|^p} (1-|\widetilde{w}|)^{\gamma_1} < \infty, \tag{4.15}$$

and conversely if

$$\|\mu\|_{\widetilde{U}^n} = \sup_{\widetilde{w}\in\widetilde{U}^n} \int_{T^n}\int_0^1 \frac{(1-|w|)^t d\mu(w)}{\prod_{k=1}^n |1-\widetilde{w}_k \overline{w}_k|^{p+\varepsilon}} (1-|\widetilde{w}|)^{\gamma_1+\varepsilon} < \infty \tag{4.16}$$

holds for some $\varepsilon > 0$ where $\widetilde{w} = (|\widetilde{w}|\xi_1, \dots, |\widetilde{w}|\xi_n),\ |\widetilde{w}| \in (0,1),\ \xi_j \in T,\ j = 1, \dots, n$ then

$$\int_{\widetilde{U}^n} |f(z)|^p (1-|z|)^t d\mu(z) \|\mathcal{R}^s f\|_{A^p_{\beta_1}(\widetilde{U}^n)}, \quad t > -1. \tag{4.17}$$

Remark 4.4. With another "ε-sharp" embedding theorem, the complete analogue of Theorem 4.3 is true also when we replace the left side by $\int_{U^n} |f(z)|^p \prod_{k=1}^n (1-|z_k|)^{t_k} d\mu(z), t_k > -1,\ k = 1, \dots, n$. The proof needs small modification of arguments we present in the proof of Theorem 4.3.

Remark 4.5. For $n = 1$ and $s = 1$ Theorem 4.3 is known (see [15]).

Proof of Theorem 4.3. It can be checked by direct calculations based on formula (4.2) that the following integral representation holds:

$$f(z_1,\dots,z_n)=C_s\int_{T^n}\int_0^1 \mathcal{R}^s f(\rho\xi_1,\dots,\rho\xi_n)\left(\log\frac{1}{\rho}\right)^{s-1}\frac{d\rho dm_n(\xi)}{\prod_{k=1}^n(1-\rho\varphi_k r\xi_k)},\tag{4.18}$$

$s>0,\ z_k=\varphi_k r,\ k=1,\dots,n,\ r\in(0,1),(z_1,\dots,z_n)\in\widetilde{U}^n$. Using the fact that $M_p(f,r)$ in increasing by $r,\ r\in(0,1)$ and $M_1(f,\tau^2)\le C(1-\tau)^{n(1-(1/p))}M_p(f,\tau),\tau\in(0,1)$ (see [1]) we get from above by using the estimate

$$\left(\int_0^1(f(\rho))(1-\rho)^{\alpha}d\rho\right)^p\lesssim\int_0^1(f(\rho))^p(1-\rho)^{\alpha p+p-1}d\rho,\tag{4.19}$$

where f is growing measurable, $p\le 1,\ \alpha>-1$,

$$\int_{U^n}|f(w)|^p(1-|w|)^t d\mu(w)\le C\|\mu\|_{\widetilde{U}^n}\int_0^1\int_{T^n}|\mathcal{R}^s f(w)|^p(1-|w|)^{\beta_1}dm_n(\xi)d|w|.\tag{4.20}$$

To obtain the reverse implication we use standard test function $g_z(w)=1/\prod_{k=1}^n(1-w_k\overline{z_k})^{\gamma}$, $z_k,w_k\in U,\ k=1,\dots,n,z,w\in\widetilde{U}^n,\ \gamma>1$ and Lemma 2.5

$$\|\mathcal{R}^s g_z(w)\|_{A^p_{\beta_1}(\widetilde{U}^n)}\int_0^1(1-|w|)^{\beta_1}\sum_{\alpha_j\ge 0,\sum\alpha_j=s}G(z,w)d|w|,\tag{4.21}$$

where $G(z,w)=\prod_{k=1}^n 1/(1-|w_k||z_k|)^{p(\alpha_k+\gamma)-1},\ \gamma>1,\ p>1/\gamma$.

The rest is clear. The proof is complete. □

Below we continue to study connections between standard classes in polydisk and corresponding spaces on subframe and expanded disk.

Let

$$\begin{aligned}(\mathcal{R}A)^p_{s,v}&=\left\{f\in H(U^n):\mathcal{R}^s f\in A^p_v\left(\widetilde{U}^n\right)\right\},\\ (\mathfrak{D}A)^p_{\vec{\gamma},\alpha}&=\left\{f\in H(U^n):\widetilde{\mathfrak{D}}^{\gamma}f\in A^p_{\alpha}(U^n)\right\},\quad \widetilde{\mathfrak{D}}^{\gamma}=\mathfrak{D}^{\gamma_1}_{z_1}\cdots\mathfrak{D}^{\gamma_n}_{z_n},\sum_{j=1}^n\gamma_j=\gamma,\ \gamma_j\ge 0.\end{aligned}\tag{4.22}$$

Let us note that Theorems 3.2 and 3.7 of [3] show that under some restrictions on $\gamma,\ \alpha,\ s,\ v$ the following assertion is true.

For every function $f,\ f\in(\mathfrak{D}A)^p_{\gamma,\alpha},\ 0<p<\infty,\ f(|z|\xi_1,\dots,|z|\xi_n)$ belongs to $(\mathcal{R}A)^p_{s,v},\ 0<p<\infty$, and the reverse is also true, for any function $f,\ f\in(\mathcal{R}A)^p_{s,v}$ there exists "an extension" F such that

$$F(|z|\xi_1,\dots,|z|\xi_n)=f(|z|\xi_1,\dots,|z|\xi_n),\quad F\in(\mathfrak{D}A)^p_{\gamma,\alpha}.\tag{4.23}$$

The following Theorem 4.6 gives an answer for the same map from polydisk to expanded disk

$$f(z_1,\dots,z_n)\longrightarrow f(|z_1|\xi,\dots,|z_n|\xi),\quad \xi\in T,\ |z_j|\in(0,1),\ j=1,\dots,n, \tag{4.24}$$

for f functions from $H^p(U^n)$, $1<p<\infty$.

We will develop ideas from [4] to get the following sharp embedding theorem for classes on expanded disk.

Theorem 4.6. *Let $n>1$, $1<p<\infty$, μ be a positive Borel measure on U_*^n. Then*

$$\int_0^1\cdots\int_0^1\int_T |f(r_1\xi,\dots,r_n\xi)|^p d\mu(\vec{r}\xi)\le \|f\|_{H^p(U^n)} \tag{4.25}$$

if and only if $\int_{1-l_1}^1\cdots\int_{1-l_n}^1\int_{-(\min_j l_j)/2}^{(\min_j l_j)/2} d\mu(\vec{r}\xi)\le Cl_1\cdots l_n$, $l_j>0$, $j=1,\dots,n$.

Proof. Obviously if $K=\int_{U_*^n}|f(z)|^p d\mu z\le C\|f\|_{H^p}^p$, then by putting

$$f=\prod_{j=1}^n f_j,\ f_j(z)=\left(\frac{1-\tilde{l}_j}{\left(1-z_j\tilde{l}_j\right)^2}\right)^{1/p},\quad \tilde{l}_j=1-l_j,\ \tilde{l}_j\in(0,1),\ j=1,\dots,n, \tag{4.26}$$

we have $\|f\|_{H^p}\asymp$ const and $K\ge 1/(l_1\cdots l_n)\int_{1-l_1}^1\cdots\int_{1-l_n}^1\int_{-\tau}^{\tau} d\mu(r_1\xi,\dots,r_n\xi)$, $\tau=\min_j l_j$, $|1-\overline{z}_j\tilde{l}_j|\asymp 1-|z_j|$, $|z_j|\in(1-l_j,1)$, $\arg(z_j)\in(-\tau,\tau)$. Hence we get what we need. Now we will show the sufficiency of the condition.

Let $N>0:\quad 2^N(1-r_j)\le\pi<2^{N+1}(1-r_j)$, $j=1,\dots,n$.

Let $z=\vec{r}e^{i\varphi}\in U_*^n$, $z=(r_1e^{i\varphi},\dots,r_ne^{i\varphi})$. Let also $B_z^k=\{t=(t_1,\dots,t_n):|t_j-\varphi|<2^{k_j}(1-r_j),1\le j\le n\}$ and $W_{k_1,\dots,k_n}=I_{k_1}\times\cdots\times I_{k_n}$, where

$$\begin{gathered} I_{k_j}=\left\{t:2^{k_j}(1-r_j)\le|t-\varphi|<2^{k_j+1}(1-r_j)\right\},\quad 0\le k_j\le N-1,\\ I_N=\left\{t:2^N(1-r_j)\le|t-\varphi|<2^{N+1}(1-r_j),\ |t|<\pi\right\},\quad I_{-1}=\{t:|t|<1-r_j\}. \end{gathered} \tag{4.27}$$

In what follows we will use notations of Proposition 2.1. Consider the Poisson integral of a function f, $f\in H^1(T)$, $u(z)=C\int_{T^n}P(z,\xi)f(\xi)dm_n(\xi)$, $z\in U^n$ (see [2]).

Let $E(a)=\{z\in U_*^n:(\mathrm{Ed})u(z)>a\}$. We will show now as in [4] that

$$\mu(E(a))\le\frac{C}{a}\int_{T^n}f(t)dm_n(t). \tag{4.28}$$

Indeed, this will be enough, since the operator $f \to (\mathrm{Ed})\int_{T^n} P(z,\xi) f(\xi) dm_n(\xi)$ is (L^∞, L^∞) operator we can apply the Marcinkiewicz interpolation theorem (see [16, Chapter 1]) to assert that

$$\mathcal{J} = \int_0^1 \cdots \int_0^1 \int_T |f(z)|^p d\mu(z) \le C \|f\|_{H^p(U^n)}, \quad 1 < p < \infty. \tag{4.29}$$

We have as in [4] for $z \in U_*^n$

$$\begin{aligned}(\mathrm{Ed})u(z) &= \int_{T^n} P(z,w) f(w) dm_n(w) = \sum_{k_1,\ldots,k_n=-1}^{N} \int_{W_{k_1,\ldots,k_n}} P(z,w) f(w) dm_n(w) \\ &\le C_0 \sum_{k_1,\ldots,k_n=0}^{N} \frac{1}{4^{k_1+\cdots+k_n} \prod_{k=1}^n (1-r_k)} \int_{B_z^k} f(t) dt \le C \sum_{k_1,\ldots,k_n=0}^{N} \frac{1}{2^{|k|}} g_k(r_1,\ldots,r_n,\varphi) \\ &\le \sum_{k_1,\ldots,k_n=0}^{N} 2^{-|k|} g_k(\overrightarrow{r},\varphi) \le C_1 \sup_k 2^{|k|/2} g_k(\overrightarrow{r},\varphi),\end{aligned} \tag{4.30}$$

where we used the standard partition of Poisson integral. Hence $E(a) \subset \bigcup_{\alpha \in \mathbb{Z}_+^n} E_\alpha(C_2 2^{|\alpha|/2} a)$, and so using Proposition 2.1 we finally get $\mu(E(a)) \le C/a \int_{T^n} f(t) d(t)$.

Indeed by Proposition 2.1 we have

$$\begin{aligned}\mu(E(a)) &\le \sum_{\alpha \in \mathbb{Z}_+^n} \left(\mu\left(E_\alpha\left(C_2 2^{|\alpha|/2} a\right)\right)\right) \\ &\le C_3 \sum_{\alpha \in \mathbb{Z}_+^n} \frac{C_2^{-1}}{2^{|\alpha|/2} a} \int_{T^n} f(t) d(t) \le \frac{C_4}{a} \int_{T^n} f(t) d(t).\end{aligned} \tag{4.31}$$

Theorem 4.6 is proved. □

Theorem 4.7. *Let $f \in H(U^n)$. Then*

$$\begin{aligned}&\sup_{|z_j|<1} |\mathfrak{D}^\alpha f(z_1,\ldots,z_n)| \prod_{k=1}^n (1-|z_k|)^{\alpha+1} \\ &\quad \le C \int_{T^n} \int_0^1 |\mathcal{R}^s f(\rho\xi_1,\ldots,\rho\xi_n)|^p (1-\rho)^{sp+n(p-1)-1} dm_n(\xi) d\rho,\end{aligned} \tag{4.32}$$

where $s > 0$, $\alpha > 0$, $p \le 1$, $sp + n(p-1) > 0$;

$$\sup_{z \in U^n} \left|\mathfrak{D}^\beta f(z_1,\ldots,z_n)\right| \left(\sum_{\alpha_k \ge 0, \sum \alpha_k = s} \prod_{k=1}^n \frac{1}{(1-|z_k|)^{\beta+\alpha_k-(2s-\alpha)/n}} \right)^{-1} \le C \sup_{z \in \tilde{U}^n} \left|\mathcal{R}^s f(z)\right| (1-|z|)^\alpha, \tag{4.33}$$

where $2s - \alpha > 0,\ \beta > 1,\ \alpha > 0,\ s > 0,$

$$\begin{aligned}&\sup_{|z_k|<1}\int_{T^n}\left|\mathcal{R}^s f(|z_1|\varphi_1,\dots,|z_n|\varphi_n)\right|^p dm_n(\varphi)\sum_{\alpha_j\geq 0,\sum\alpha_j=2s}\left(\prod_{k=1}^{n}\frac{1}{(1-|z_k|)^{p(\alpha_k+s+1)-1}}\right)^{-1}\\&\quad\leq C\int_{T^n}\int_0^1\left|\mathfrak{D}^{-s}f(w)\right|^p(1-|w|)^t d|w|dm_n(\varphi),\end{aligned}\tag{4.34}$$

where $s > 0,\ 1/(s+1) < p \leq 1,\ t = p(s-1)+(n+1)(p-1) > -1.$

Proof. We use systematically the integral representation (4.18), Lemma 2.5, and it is corollary. The proof of the estimate (4.32) follows from equality

$$\mathfrak{D}^{\alpha} f(z_1,\dots,z_n) = C_s\int_{T^n}\int_0^1\mathcal{R}^s f(\rho\xi_1,\dots,\rho\xi_n)\left(\log\frac{1}{\rho}\right)^{s-1}\prod_{k=1}^{n}\frac{1}{(1-\xi\varphi_k\widetilde{\rho}_k)^{\alpha+1}}d\rho\, dm_n(\xi),\tag{4.35}$$

$z_k = \varphi_k\widetilde{\rho}_k, k = 1,\dots,n,\ s>0,\alpha>0,\ f\in H(U^n)$, and estimate

$$\begin{aligned}&\left(\int_{T^n}\int_0^1\left|\mathcal{R}^s f(\rho\xi_1,\dots,\rho\xi_n)\right|(1-\rho)^{\alpha}d\rho\, dm_n(\xi)\right)^p\\&\quad\leq C\int_{T^n}\int_0^1\left|\mathcal{R}^s f(\rho\xi_1,\dots,\rho\xi_n)\right|^p(1-\rho)^{\alpha p+(n+1)(p-1)}d\rho\, dm_n(\xi),\end{aligned}\tag{4.36}$$

$p\leq 1,\ \alpha>0,\ s>0,\ f\in H(U^n)$, obtained during the proof of Theorem 4.3.

The proof of the estimate (4.33) follows from Lemma 2.5 and its corollary and integral representation (4.35).

The proof of the estimate (4.34) follows from equality ($z_j = \widetilde{\rho}_j\varphi_j,\ j=1,\dots,n$)

$$\begin{aligned}&\left|\mathcal{R}^s f(z_1,\dots,z_n)\right|\\&\quad=\left|C_s\int_{T^n}\int_0^1(\mathfrak{D}^{-s}f)(\rho\xi_1,\dots,\rho\xi_n)\left(\log\frac{1}{\rho}\right)^{s-1}\mathcal{R}^{2s}\prod_{k=1}^{n}\frac{1}{\left(1-\bar{\xi}\varphi_k\rho\widetilde{\rho_k}\right)^{s+1}}d\rho\, dm_n(\xi)\right|.\end{aligned}\tag{4.37}$$

Indeed using (4.36) integrating both sides of (4.37) by T^n and using Lemma 2.5 we arrive at (4.34). □

Remark 4.8. All estimates in Theorem 4.7 for $n = 1$ are well known (see [1, Chapter1]).

We present below a complete analogue of Theorems 3.2 and 3.3 for a map from polydisk to expanded disk. Note the continuation of f function is done again from diagonal $(z,\dots,z)$. Let $z_j \in U,\ j = 1,\dots,n$ and

$$f(z_1,\dots,z_n) = C_s\int_U \frac{f(z,\dots,z)(1-|z|)^s}{\prod_{k=1}^n (1-z_k\overline{z})^{(s+2)/n}} dm_2(z), \tag{4.38}$$

where C_s is a constant of Bergman representation formula (see [1]).

Proposition 4.9. (1) (*a*) *Let* $n \in \mathbb{N},\ p \in (0,\infty),\ \alpha > -1,\ f \in H(U^n)$. *Then*

$$\begin{aligned}&\int_T\int_0^1\cdots\int_0^1 |f(|z_1|\xi,\dots,|z_n|\xi)|^p \prod_{k=1}^n (1-|z_k|)^{\alpha+((n-1)/n)} d|z_1|\cdots d|z_n| dm(\xi)\\ &\quad\le C\int_{U^n} |f(z)|^p \prod_{k=1}^n \left(1-|z_k|^2\right)^\alpha dm_{2n}(z).\end{aligned} \tag{4.39}$$

And reverse is also true:

(*b*) *Let* $0 < p < \infty,\ \alpha > -1$ *and*

$$\int_T\int_0^1\cdots\int_0^1 |f(|z_1|\xi,\dots,|z_n|\xi)|^p \prod_{k=1}^n (1-|z_k|)^{\alpha+((n-1)/n)} d|z_1|\cdots d|z_n| dm(\xi) < \infty, \tag{4.40}$$

then for all functions f *such that condition* (4.38) *holds for* $s > max(((\alpha+2)n)/p) - 2, -1)$ *we have* $f \in A^p_\alpha(U^n)$.

(2) (a) *Let* $f \in H^p(U^n),\ 1 < p < \infty, n > 1$. *Then*

$$\|f\|_{A^p_{-1/n}(U^n_*)} = \int_T\int_0^1\int_0^1 |f(|z_1|\xi,\dots,|z_n|\xi)|^p \prod_{k=1}^n (1-|z_k|)^{-1/n} dm(\xi) d|z_1|\cdots d|z_m| < \infty, \tag{4.41}$$

and the reverse is also true:

(*b*) *For any function* f *with a finite quasinorm* $\|f\|_{A^p_{-1/n}(U^n_*)}$ *such that condition* (4.38) *holds for* $s = 2n-2$ *one has* $f \in H^p(U^n), p > 1$.

Proof. (1) (a) Proof of estimate (4.39) follows directly from Theorem B.

(b) Indeed from (4.38) and results of [1] on diagonal map in Bergman classes we have $\|f\|_{A^p_\alpha(U^n)} \le C\|\mathrm{Diag}(f)\|_{A^p_t(U)}, t = \alpha n + 2n - 2$. It remains to apply Theorem A.

(2) For the proof of (a) we use Theorem 4.6 and get the result we need.

For the proof of (b) we use the same argument as in the proof of part (1). Namely first from (4.38) and from a Diagonal map theorem on H^p classes from [1] we get $\|f\|_{H^p(U^n)} \le C\|f\|_{A^p_{n-2}(U)}$. It remains to apply Theorem A. □

Remark 4.10. Note Proposition 4.9 is obvious for $n = 1$.

We give only a sketch of the proof of the following result. It is based completely on a technique we developed above.

Proposition 4.11. *(a) Let $\alpha > 0$, $n \in \mathbb{N}$, $n > 1$, $s > n\alpha$ then the following assertions are true: If $\mathfrak{D}^\alpha f \in H^p(U^n)$ then*

$$\mathcal{Q}_{p,\alpha,s}(f) = \int_{T^n}\int_0^1 \left|\mathcal{R}^s f(|z|\xi_1,\ldots,|z|\xi_n)\right|^p (1-|z|)^{sp-np\alpha-1} d|z| dm_n(\xi) < \infty,\ p \geq 2. \tag{4.42}$$

If $\mathcal{Q}_{p,\alpha,s}(f) < \infty$ then $\mathfrak{D}^\alpha f \in H^p(U^n)$, $1/(\alpha+1) < p \leq 1$.

(b) Let $\mathfrak{D}^\alpha f \in H^p(U^n)$, $p \geq 2$. If $s > n\alpha$, $\alpha > 0$, then $\mathcal{Q}_{p,\alpha,s}(f) < \infty$. Moreover the reverse is also true if condition (4.38) holds for $s = 2n-2$ then $\|\mathfrak{D}^\alpha f\|_{H^p}^p \leq C\mathcal{Q}_{p,\alpha,s}(f)$.

The proof of first part of Proposition 4.11 follows from (4.35), (4.36) and Lemma 2.5 directly. The reverse assertion follows from Theorem B (b) and estimate

$$\int_{U^n} \left|\mathfrak{D}^{\alpha_1,\ldots,\alpha_n} f(z)\right|^p \prod_{k=1}^{n}(1-|z_k|)^{\alpha_k p-1} dm_{2n}(z) \leq C\left\|f\right\|_{H^p}, \alpha_k > 0, k = 1,\ldots,n,\ p \geq 2, \tag{4.43}$$

which can obtained from one dimensional result by induction.

The proof of second part of Proposition 4.11 can be obtained from Theorem A and results on diagonal map on Hardy classes $H^p(U^n)$ from [1] similarly as the proof of Proposition 4.9. For $n = 1$ part (a) is well known (e.g., see [1]) and (b) follows from (a) since for $n = 1$ condition (4.38) vanishes by Bergman representation formula.

Remark 4.12. Theorem 4.6, Propositions 4.9 and 4.11 give an answer to a problem of Rudin (see [2]) to find traces of $H^p(U^n)$ Hardy classes on subvarieties other than diagonal $(z,\ldots,z)$. Note that in [11] Clark solved this problem for subvarietes of U^n based on finite Blaschke products.

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