Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 212915, 13 pages
doi:10.1155/2009/212915

Research Article

Norm Comparison Inequalities for the Composite Operator

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Received 2 August 2008; Accepted 15 January 2009

Recommended by András Rontó

We establish norm comparison inequalities with the Lipschitz norm and the BMO norm for the composition of the homotopy operator and the projection operator applied to differential forms satisfying the A-harmonic equation. Based on these results, we obtain the two-weight estimates for Lipschitz and BMO norms of the composite operator in terms of the L^s-norm.

1. Introduction

The purpose of this paper is to establish the Lipschitz norm and BMO norm inequalities for the composition of the homotopy operator T and the projection operator H applied to differential forms in $\mathbb{R}^n$, $n \geq 2$. The harmonic projection operator H, one of the key operators in the harmonic analysis, plays an important role in the Hodge decomposition theory of differential forms. In the meanwhile, the homotopy operator T is also widely used in the decomposition and the L^p-theory of differential forms. In many situations, we need to estimate the various norms of the operators and their compositions.

We always assume that M is a bounded, convex domain and B is a ball in $\mathbb{R}^n$, $n \geq 2$, throughout this paper. Let σB be the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. For any subset $E \subset \mathbb{R}^n$, we use $|E|$ to denote the Lebesgue measure of E. We call w a weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \ldots, x_n)$ is differentiable, then it is called a differential 0-form. A differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \ldots, n$, that is, $u(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and $\omega_{i_1 i_2 \cdots i_k}(x)$ are differentiable functions. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(M, \wedge^l)$ be the space of all differential l-forms on M

and $L^p(M, \wedge^l)$ be the l-forms $\omega(x) = \sum_I \omega_I(x) dx_I$ on M satisfying $\int_M |\omega_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by $d : D'(M, \wedge^l) \to D'(M, \wedge^{l+1})$ for $l = 0, 1, \ldots, n-1$. The Hodge codifferential operator $d^\star : D'(M, \wedge^{l+1}) \to D'(M, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. We write $\|u\|_{s,M} = (\int_M |u|^s)^{1/s}$ and $\|u\|_{s,M,w} = (\int_M |u|^s w(x) dx)^{1/s}$, where $w(x)$ is a weight. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle and $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M. We set $\mathcal{W}(\wedge^l M) = \{u \in L^1_{\text{loc}}(\wedge^l M) : u$ has generalized gradient$\}$. The harmonic l-fields are defined by $\mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^\star u = 0,\ u \in L^p$ for some $1 < p < \infty\}$. The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : \langle u, h \rangle = 0$ for all $h \in \mathcal{H}\}$. The harmonic projection operator $H : C^\infty(\wedge^l M) \to \mathcal{H}$ is the operator involved in the Poisson's equation $\Delta G(\omega) = \omega - H(\omega)$, where G is the Green's operator. See [1–4] for more propeties of the projection operator and Green's operator.

The differential equation $d^\star A(x, d\omega) = 0$ is called the A-harmonic equation and the nonlinear elliptic partial differential equation

$$d^\star A(x, d\omega) = B(x, d\omega) \tag{1.1}$$

is called the nonhomogeneous A-harmonic equation for differential forms, where $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : M \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1}, \qquad A(x, \xi) \cdot \xi \ge |\xi|^p, \qquad |B(x, \xi)| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in M$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(M, \wedge^{l-1})$ such that $\int_M A(x, d\omega) \cdot d\varphi + B(x, d\omega) \cdot \varphi = 0$ for all $\varphi \in W^{1,p}_{\text{loc}}(M, \wedge^{l-1})$ with compact support. Let $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then A satisfies required conditions and $d^\star A(x, d\omega) = 0$ becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. If u is a function (a 0-form), the above equation reduces to the usual p-harmonic equation

$$\operatorname{div}\left(\nabla u |\nabla u|^{p-2}\right) = 0 \tag{1.3}$$

for functions. Some results have been obtained in recent years about different versons of the A-harmonic equation, see [2–9].

Let $\omega \in L^1_{\text{loc}}(M, \wedge^l)$, $l = 0, 1, \ldots, n$. We write $\omega \in \operatorname{locLip}_k(M, \wedge^l)$, $0 \le k \le 1$, if

$$\|\omega\|_{\operatorname{locLip}_k, M} = \sup_{\sigma Q \subset M} |Q|^{-(n+k)/n} \left\|\omega - \omega_Q\right\|_{1,Q} < \infty \tag{1.4}$$

for some $\sigma \ge 1$. The factor σ here is for convenience and in fact the norm $\|\omega\|_{\operatorname{locLip}_k, M}$ is independent of this expansion factor, see [8]. Further, we write $\operatorname{Lip}_k(M, \wedge^l)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|\omega\|_{\operatorname{Lip}_k, M}$ for this norm. Similarly, for $\omega \in L^1_{\text{loc}}(M, \wedge^l)$, $l = 0, 1, \ldots, n$, we write $\omega \in \operatorname{BMO}(M, \wedge^l)$ if

$$\|\omega\|_{\star, M} = \sup_{\sigma Q \subset M} |Q|^{-1} \left\|\omega - \omega_Q\right\|_{1,Q} < \infty \tag{1.5}$$

for some $\sigma \geq 1$. Again, the factor σ here is for convenience and the norm $\|\omega\|_{\star,M}$ is independent of the expansion factor σ, see [8]. When ω is a 0-form, (1.5) reduces to the classical definition of $\mathrm{BMO}(M)$.

The following operator K_y with the case $y = 0$ was first introduced by Cartan in [10]. Then, it was extended to the following version in [6]. For each point $y \in M$, there is a linear operator $K_y : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ defined by $(K_y\omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})dt$ and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ is defined by $T\omega = \int_M \varphi(y)K_y\omega\, dy$, averaging K_y over all points y in M, where $\varphi \in C_0^\infty(M)$ is normalized by $\int_M \varphi(y)dy = 1$ and the decomposition

$$\omega = d(T\omega) + T(d\omega) \tag{1.6}$$

holds for any differential form ω. The l-form $\omega_M \in D'(M, \wedge^l)$ is defined by

$$\omega_M = |M|^{-1}\int_M \omega(y)dy, \quad l = 0, \qquad \omega_M = d(T\omega), \quad l = 1, 2, \ldots, n \tag{1.7}$$

for all $\omega \in L^p(M, \wedge^l)$, $1 \leq p < \infty$. From [6], we know that for any differential form $u \in L^s_{\mathrm{loc}}(B, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, we have

$$\|\nabla(Tu)\|_{s,B} \leq C|B|\|u\|_{s,B}, \tag{1.8}$$

$$\|Tu\|_{s,B} \leq C|B|\mathrm{diam}(B)\|u\|_{s,B}. \tag{1.9}$$

2. Lipschitz Norm Estimates

The following Hölder inequality will be used in the proofs of main theorems.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then $\|fg\|_{s,E} \leq \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E}$ for any $E \subset \mathbb{R}^n$.*

Lemma 2.2 (see [1]). *Let $u \in C^\infty(\wedge^l M)$ and $l = 1, 2, \ldots, n$, $1 < s < \infty$. Then, there exists a positive constant C, independent of u, such that*

$$\|dd^*G(u)\|_{s,M} + \|d^*dG(u)\|_{s,M} + \|dG(u)\|_{s,M} + \|d^*G(u)\|_{s,M} + \|G(u)\|_{s,M} \leq C\|u\|_{s,M}. \tag{2.1}$$

We first prove the following Poincaré-type inequality for the composition of the homotopy operator and the projection operator.

Theorem 2.3. *Let $u \in L^s_{\mathrm{loc}}(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth differential form in a bounded, convex domain M, H be the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u)) - (T(H(u)))_B\|_{s,B} \leq C|B|\mathrm{diam}(B)\|u\|_{s,B} \tag{2.2}$$

for all balls B with $B \subset M$.

Proof. Let H be the projection operator and T be the homotopy operator. For any differential form u, we know that

$$\|u_B\|_{s,B} \le C_1\|u\|_{s,B}. \tag{2.3}$$

Replacing u by $H(u)$ in (2.3) yields

$$\|(H(u))_B\|_{s,B} \le C_1\|H(u)\|_{s,B}. \tag{2.4}$$

Since $H(u) = u - \Delta G(u)$ and $\Delta = d^\star d + dd^\star$, by Lemma 2.2, we have

$$\begin{aligned}\|H(u)\|_{s,B} &= \|u - \Delta G(u)\|_{s,B}\\ &\le \|u\|_{s,B} + \|\Delta G(u)\|_{s,B}\\ &\le \|u\|_{s,B} + C_2\|u\|_{s,B}\\ &\le C_3\|u\|_{s,B}.\end{aligned} \tag{2.5}$$

Using (1.9), (2.4), and (2.5), we find that

$$\begin{aligned}\|T(H(u)) - (T(H(u)))_B\|_{s,B} &= \|Td(T(H(u)))\|_{s,B}\\ &\le C_4|B|\text{diam}(B)\|d(T(H(u)))\|_{s,B}\\ &= C_4|B|\text{diam}(B)\|(H(u))_B\|_{s,B}\\ &\le C_5|B|\text{diam}(B)\|H(u)\|_{s,B}\\ &\le C_6|B|\text{diam}(B)\|u\|_{s,B}.\end{aligned} \tag{2.6}$$

The proof of Theorem 2.3 has been completed. □

Using Theorem 2.3, we estimate the following Lipschitz norm of the composite operator $T \circ H$.

Theorem 2.4. *Let $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth differential form in a bounded, convex domain M, H be the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u))\|_{\text{locLip}_k, M} \le C\|u\|_{s,M}, \tag{2.7}$$

where k is a constant with $0 \le k \le 1$.

Proof. From Theorem 2.3, we have

$$\|T(H(u)) - (T(H(u)))_B\|_{s,B} \le C_1|B|\text{diam}(B)\|u\|_{s,B} \tag{2.8}$$

for all balls B with $B \subset M$. Using the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}
\|T(H(u)) - (T(H(u)))_B\|_{1,B} &= \int_B |T(H(u)) - (T(H(u)))_B| dx \\
&\le \left(\int_B |T(H(u)) - (T(H(u)))_B|^s dx\right)^{1/s} \left(\int_B 1^{s/(s-1)} dx\right)^{(s-1)/s} \\
&= |B|^{(s-1)/s} \|T(H(u)) - (T(H(u)))_B\|_{s,B} \\
&= |B|^{1-1/s} \|T(H(u)) - (T(H(u)))_B\|_{s,B} \\
&\le |B|^{1-1/s} (C_1 |B| \text{diam}(B) \|u\|_{s,B}) \\
&\le C_2 |B|^{2-1/s+1/n} \|u\|_{s,B},
\end{aligned} \tag{2.9}$$

where we have used $\text{diam}(B) = C|B|^{1/n}$. Now, from the definition of Lipschitz norm, (2.9) and $2 - 1/s + 1/n - 1 - k/n = 1 - 1/s + 1/n - k/n > 0$, we obtain

$$\begin{aligned}
\|T(H(u))\|_{\text{locLip}_k, M} &= \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \|T(H(u)) - (T(H(u)))_B\|_{1,B} \\
&= \sup_{\sigma B \subset M} |B|^{-1-k/n} \|T(H(u)) - (T(H(u)))_B\|_{1,B} \\
&\le \sup_{\sigma B \subset M} |B|^{-1-k/n} C_2 |B|^{2-1/s+1/n} \|u\|_{s,B} \\
&= \sup_{\sigma B \subset M} C_2 |B|^{1-1/s+1/n-k/n} \|u\|_{s,B} \\
&\le \sup_{\sigma B \subset M} C_2 |M|^{1-1/s+1/n-k/n} \|u\|_{s,B} \\
&\le C_3 \sup_{\sigma B \subset M} \|u\|_{s,B} \\
&\le C_3 \|u\|_{s,M}.
\end{aligned} \tag{2.10}$$

The proof of Theorem 2.4 has been completed. □

In order to prove Theorem 2.6, we extend [11, Lemma 8.2.2] into the following version for differential forms.

Lemma 2.5. *Let φ be a strictly increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$, and D be a bounded domain in $\mathbb{R}^n$. Assume that u is a smooth differential form in D such that $\varphi(k(|u| + |u_D|)) \in L^1(D; \mu)$ for any real number $k > 0$ and $\mu(\{x \in D : |u - u_D| > 0\}) > 0$, where μ is a Radon measure defined by $d\mu = w(x)dx$ for a weight $w(x)$. Then, for any positive constant a, we have*

$$\int_D \varphi(a|u|) d\mu \le C \int_D \varphi(2a|u - u_D|) d\mu, \tag{2.11}$$

where C is a positive constant.

Proof. Let $C_1 = \int_D \varphi(2a|u_D|)d\mu$. Note that $\mu(\{x \in D : 2a|u - u_D| > 0\}) = \mu(\{x \in D : |u - u_D| > 0\}) > 0$. Then, there exists a constant C_2 such that $C_1 \le C_2 \int_D \varphi(2a|u - u_D|)d\mu$, that is

$$\int_D \varphi(2a|u_D|)d\mu \le C_2 \int_D \varphi(2a|u - u_D|)d\mu. \tag{2.12}$$

Since φ is an increasing convex function, we obtain

$$\begin{aligned}
\int_D \varphi(a|u|)d\mu &\le \int_D \varphi\left(\frac{1}{2}(2a|u - u_D|) + \frac{1}{2}(2a|u_D|)\right)d\mu \\
&\le \frac{1}{2}\int_D \varphi(2a|u - u_D|)d\mu + \frac{1}{2}\int_D \varphi(2a|u_D|)d\mu \\
&\le \frac{1}{2}\int_D \varphi(2a|u - u_D|)d\mu + \frac{C_2}{2}\int_D \varphi(2a|u - u_D|)d\mu \\
&\le C_3 \int_D \varphi(2a|u - u_D|)d\mu.
\end{aligned} \tag{2.13}$$

The proof of Lemma 2.5 is completed. □

Theorem 2.6. *Let $u \in L^s_{\text{loc}}(M, \wedge^1)$, $1 < s < \infty$, be a smooth differential form satisfying the nonhomogeneous A-harmonic equation in a bounded, convex domain M and the Lebesgue $|\{x \in B : |u - u_B| > 0\}| > 0$ for any ball $B \subset M$. Assume that H is the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ is the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u))\|_{\text{locLip}_k, M} \le C\|u\|_{\star, M}, \tag{2.14}$$

where k is a constant with $0 \le k \le 1$.

Proof. Using Lemma 2.5 with $\varphi(t) = t^s$ and the weight $w(x) = 1$ over the ball B, we have

$$\|u\|_{s,B} \le C_1 \|u - u_B\|_{s,B}. \tag{2.15}$$

From Theorem 2.3 and (2.15), we obtain

$$\begin{aligned}
\|T(H(u)) - (T(H(u)))_B\|_{s,B} &\le C_2 |B| \text{diam}(B) \|u\|_{s,B} \\
&\le C_3 |B| \text{diam}(B) \|u - u_B\|_{s,B}.
\end{aligned} \tag{2.16}$$

From the definition of the Lipschitz norm, the Hölder inequality with $1 = 1/s + (s-1)/s$ and (2.16), for any ball B with $B \subset M$, we find that

$$
\begin{aligned}
\|T(H(u)) - (T(H(u)))_B\|_{1,B} &= \int_B |T(H(u)) - (T(H(u)))_B| dx \\
&\leq \left(\int_B |T(H(u)) - (T(H(u)))_B|^s dx\right)^{1/s} \left(\int_B 1^{s/(s-1)} dx\right)^{(s-1)/s} \\
&= |B|^{(s-1)/s} \|T(H(u)) - (T(H(u)))_B\|_{s,B} \\
&= |B|^{1-1/s} \|T(H(u)) - (T(H(u)))_B\|_{s,B} \\
&\leq C_4 |B|^{2-1/s+1/n} \|u - u_B\|_{s,B}.
\end{aligned}
\tag{2.17}
$$

Next, from the weak reverse Hölder inequality for solutions of the nonhomogeneous A-harmonic equation, we have

$$
\|u - u_B\|_{s,B} \leq C_5 |B|^{(1-s)/s} \|u - u_B\|_{1,\sigma_1 B} \tag{2.18}
$$

for some constant $\sigma_1 > 1$. Combination of (2.17) and (2.18) gives

$$
\begin{aligned}
\|T(H(u)) - (T(H(u)))_B\|_{1,B} &\leq C_4 |B|^{2-1/s+1/n} \|u - u_B\|_{s,B} \\
&\leq C_6 |B|^{1+1/n} \|u - u_B\|_{1,\sigma_1 B}.
\end{aligned}
\tag{2.19}
$$

Hence, we obtain

$$
\begin{aligned}
|B|^{-(n+k)/n} \|T(H(u)) - (T(H(u)))_B\|_{1,B} &\leq C_6 |B|^{1/n-k/n} \|u - u_B\|_{1,\sigma_1 B} \\
&= C_6 |B|^{1+1/n-k/n} |B|^{-1} \|u - u_B\|_{1,\sigma_1 B} \\
&\leq C_7 |B|^{1+1/n-k/n} |\sigma_1 B|^{-1} \|u - u_B\|_{1,\sigma_1 B} \\
&\leq C_7 |M|^{1+1/n-k/n} |\sigma_1 B|^{-1} \|u - u_B\|_{1,\sigma_1 B} \\
&\leq C_8 |\sigma_1 B|^{-1} \|u - u_B\|_{1,\sigma_1 B}.
\end{aligned}
\tag{2.20}
$$

Thus, taking the supremum on both sides of (2.20) over all balls $\sigma_2 B \subset M$ with $\sigma_2 > \sigma_1$ and using the definitions of the Lipschitz and BMO norms, we find that

$$
\begin{aligned}
\|T(H(u))\|_{\mathrm{locLip}_k, M} &= \sup_{\sigma_2 B \subset M} |B|^{-(n+k)/n} \|T(H(u)) - (T(H(u)))_B\|_{1,B} \\
&\leq C_7 \sup_{\sigma_2 B \subset M} |\sigma_1 B|^{-1} \|u - u_B\|_{1,\sigma_1 B} \\
&\leq C_7 \|u\|_{\star,M},
\end{aligned}
\tag{2.21}
$$

that is,

$$\|T(H(u))\|_{\text{locLip}_k,M} \leq C\|u\|_{\star,M}. \tag{2.22}$$

The proof of Theorem 2.6 has been completed. □

Note that inequality (2.14) implies that the norm $\|T(H(u))\|_{\text{locLip}_k,M}$ of $T(H(u))$ can be controlled by the norm $\|u\|_{\star,M}$ when u is a 1-form.

Theorem 2.7. *Let $u \in L^s_{\text{loc}}(M,\wedge^l)$, $l = 1,2,\ldots,n$, $1 < s < \infty$, be a smooth differential form in a bounded, convex domain M, H be the projection operator and $T : C^\infty(M,\wedge^l) \to C^\infty(M,\wedge^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u))\|_{\star,M} \leq C\|T(H(u))\|_{\text{locLip}_k,M}. \tag{2.23}$$

Proof. From the definitions of the Lipschitz and BMO norms, we obtain

$$\begin{aligned}
\|T(H(u))\|_{\star,M} &= \sup_{\sigma B\subset M} |B|^{-1}\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B} \\
&= \sup_{\sigma B\subset M} |B|^{k/n}|B|^{-(n+k)/n}\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B} \\
&\leq \sup_{\sigma B\subset M} |M|^{k/n}|B|^{-(n+k)/n}\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B} \\
&\leq |M|^{k/n}\sup_{\sigma B\subset M} |B|^{-(n+k)/n}\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B} \\
&\leq C_1\sup_{\sigma B\subset M} |B|^{-(n+k)/n}\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B} \\
&\leq C_1\left\|T(H(u))\right\|_{\text{locLip}_k,M},
\end{aligned} \tag{2.24}$$

that is

$$\|T(H(u))\|_{\star,M} \leq C_1\|T(H(u))\|_{\text{locLip}_k,M}, \tag{2.25}$$

where C_1 and k are constants with $0 \leq k \leq 1$. We have completed the proof of Theorem 2.7. □

3. BMO Norm Estimates

We have developed some estimates for the Lipschitz norm $\|\cdot\|_{\text{locLip}_k,M}$ in last section. Now, we estimates the BMO norm $\|\cdot\|_{\star,M}$. We first prove the following inequality between the BMO norm and the Lipschitz norm for the composite operator.

Theorem 3.1. *Let $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth differential form in a bounded, convex domain M, H be the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u))\|_{\star,M} \le C\|u\|_{s,M}. \tag{3.1}$$

Proof. From Theorems 2.4 and 2.7, we have

$$\|T(H(u))\|_{\mathrm{locLip}_k,M} \le C_1\|u\|_{s,M}, \tag{3.2}$$

$$\|T(H(u))\|_{\star,M} \le C_2\|T(H(u))\|_{\mathrm{locLip}_k,M}, \tag{3.3}$$

respectively. Combination of (3.2) and (3.3) yields

$$\|T(H(u))\|_{\star,M} \le C_3\|u\|_{s,M}. \tag{3.4}$$

The proof of Theorem 3.1 has been completed. □

Based on the above results, we discuss the weighted Lipschitz and BMO norms. For $\omega \in L^1_{\mathrm{loc}}(M, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $\omega \in \mathrm{locLip}_k(M, \wedge^l, w^\alpha)$, $0 \le k \le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k,M,w^\alpha} = \sup_{\sigma Q \subset M} (\mu(Q))^{-(n+k)/n} \|\omega - \omega_Q\|_{1,Q,w^\alpha} < \infty \tag{3.5}$$

for some $\sigma > 1$, where M is a bounded domain, the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight and α is a real number. For convenience, we will write the following simple notation $\mathrm{locLip}_k(M, \wedge^l)$ for $\mathrm{locLip}_k(M, \wedge^l, w^\alpha)$. Similarly, for $\omega \in L^1_{\mathrm{loc}}(M, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we will write $\omega \in \mathrm{BMO}(M, \wedge^l, w^\alpha)$ if

$$\|\omega\|_{\star,M,w^\alpha} = \sup_{\sigma Q \subset M} (\mu(Q))^{-1} \|\omega - \omega_Q\|_{1,Q,w^\alpha} < \infty \tag{3.6}$$

for some $\sigma > 1$, where the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight and α is a real number. Again, the factor σ in the definitions of the norms $\|\omega\|_{\mathrm{locLip}_k,M,w^\alpha}$ and $\|\omega\|_{\star,M,w^\alpha}$ is for convenience and in fact these norms are independent of this expansion factor. We also write $\mathrm{BMO}(M, \wedge^l)$ to replace $\mathrm{BMO}(M, \wedge^l, w^\alpha)$ when it is clear that the integral is weighted.

Definition 3.2. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(E)$-condition in a set $E \subset \mathbb{R}^n$, write $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$, for some $\lambda \ge 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$ if

$$\sup_{B \subset E} \left(\frac{1}{|B|} \int_B (w_1)^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B w_2^{-\lambda r'/r} dx \right)^{1/\lambda r'} < \infty. \tag{3.7}$$

Lemma 3.3 (see [8]). *Let u be a smooth differential form satisfying the nonhomogeneous A-harmonic equation in M, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{3.8}$$

for all balls or cubes B with $\sigma B \subset M$.

Using the reverse Hölder inequality (Lemma 3.3) and Theorem 2.3, one obtains the following weighted version:

$$\left\|T(H(u)) - (T(H(u)))_B\right\|_{s,B,w_1^\alpha} \le C|B|\operatorname{diam}(B)\|u\|_{s,\sigma B,w_2^\alpha} \tag{3.9}$$

for all balls B with $\sigma B \subset M$, where $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$, and r, s, α, λ and σ are constants with $1 < r < \infty$, $s > 1$, $0 < \alpha \le 1$, $\lambda \ge 1$ and $\sigma > 1$.

Theorem 3.4. *Let $u \in L^s(M, \wedge^l, \nu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded, convex domain M, H be the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator, where the measure μ and ν are defined by $d\mu = w_1^\alpha dx$, $d\nu = w_2^\alpha dx$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $w_1(x) \ge \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u))\|_{\mathrm{locLip}_k, M, w_1^\alpha} \le C\|u\|_{s,M,w_2^\alpha}, \tag{3.10}$$

where k and α are constants with $0 \le k \le 1$ and $0 < \alpha \le 1$.

Proof. Since $\mu(B) = \int_B w_1^\alpha dx \ge \int_B \varepsilon^\alpha dx = C_1|B|$, we have

$$\frac{1}{\mu(B)} \le \frac{C_2}{|B|} \tag{3.11}$$

for any ball B. Using (3.9) and the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}
\left\|T(H(u)) - (T(H(u)))_B\right\|_{1,B,w_1^\alpha} &= \int_B \left|T(H(u)) - (T(H(u)))_B\right| d\mu \\
&\le \left(\int_B \left|T(H(u)) - (T(H(u)))_B\right|^s d\mu\right)^{1/s} \left(\int_B 1^{s/(s-1)} d\mu\right)^{(s-1)/s} \\
&= (\mu(B))^{(s-1)/s} \left\|T(H(u)) - (T(H(u)))_B\right\|_{s,B,w_1^\alpha} \\
&= (\mu(B))^{1-1/s} \left\|T(H(u)) - (T(H(u)))_B\right\|_{s,B,w_1^\alpha} \\
&\le (\mu(B))^{1-1/s} \left(C_3|B|\operatorname{diam}(B)\|u\|_{s,\sigma B,w_2^\alpha}\right) \\
&\le C_4(\mu(B))^{1-1/s}|B|^{1+1/n}\|u\|_{s,\sigma B,w_2^\alpha}.
\end{aligned} \tag{3.12}$$

Notice that $-1/s - k/n + 1 + 1/n > 0$ and $|M| < \infty$, from (3.5), (3.11), and (3.12), we have

$$
\begin{aligned}
\|T(H(u))\|_{\mathrm{locLip}_k, M, w_1^\alpha} &= \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \left\| T(H(u)) - (T(H(u)))_B \right\|_{1,B,w_1^\alpha} \\
&= \sup_{\sigma B \subset M} (\mu(B))^{-1-k/n} \left\| T(H(u)) - (T(H(u)))_B \right\|_{1,B,w_1^\alpha} \\
&\le C_5 \sup_{\sigma B \subset M} (\mu(B))^{-1/s-k/n} |B|^{1+1/n} \|u\|_{s,\sigma B,w_2^\alpha} \\
&\le C_6 \sup_{\sigma B \subset M} |B|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w_2^\alpha} \\
&\le C_6 \sup_{\sigma B \subset M} |M|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w_2^\alpha} \\
&\le C_6 |M|^{-1/s-k/n+1+1/n} \sup_{\sigma B \subset M} \|u\|_{s,\sigma B,w_2^\alpha} \\
&\le C_7 \|u\|_{s,M,w_2^\alpha}.
\end{aligned}
\tag{3.13}
$$

We have completed the proof of Theorem 3.4. □

We now estimate the $\|\cdot\|_{\star,M,w_1^\alpha}$ norm in terms of the L^s norm.

Theorem 3.5. *Let $u \in L^s(M, \wedge^l, \nu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded domain M H be the projection operator and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator, where the measure μ and ν are defined by $d\mu = w_1^\alpha dx$, $d\nu = w_2^\alpha dx$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $w_1(x) \ge \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$
\|T(H(u))\|_{\star,M,w_1^\alpha} \le C \|u\|_{s,M,w_2^\alpha}, \tag{3.14}
$$

where α is a constant with $0 < \alpha \le 1$.

Proof. From the definitions of the weighted Lipschitz and the weighted BMO norms, we have

$$
\begin{aligned}
\|u\|_{\star,M,w_1^\alpha} &= \sup_{\sigma B \subset M} (\mu(B))^{-1} \left\| u - u_B \right\|_{1,B,w_1^\alpha} \\
&= \sup_{\sigma B \subset M} (\mu(B))^{k/n} (\mu(B))^{-(n+k)/n} \left\| u - u_B \right\|_{1,B,w_1^\alpha} \\
&\le \sup_{\sigma B \subset M} (\mu(M))^{k/n} (\mu(B))^{-(n+k)/n} \left\| u - u_B \right\|_{1,B,w_1^\alpha} \\
&\le (\mu(M))^{k/n} \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \left\| u - u_B \right\|_{1,B,w_1^\alpha} \\
&\le C_1 \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \left\| u - u_B \right\|_{1,B,w_1^\alpha} \\
&\le C_1 \|u\|_{\mathrm{locLip}_k, M, w_1^\alpha},
\end{aligned}
\tag{3.15}
$$

where C_1 is a positive constant. Replacing u by $T(H(u))$ in (3.15), we obtain

$$\|T(H(u))\|_{\star,M,w_1^{\alpha}} \le C_1\|T(H(u))\|_{\mathrm{locLip}_k,M,w_1^{\alpha}}, \tag{3.16}$$

where k is a constant with $0 \le k \le 1$. Now, from Theorem 3.4, we find that

$$\|T(H(u))\|_{\mathrm{locLip}_k,M,w_1^{\alpha}} \le C_2\|u\|_{s,M,w_2^{\alpha}}. \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain

$$\|T(H(u))\|_{\star,M,w_1^{\alpha}} \le C_3\|u\|_{s,M,w_2^{\alpha}}. \tag{3.18}$$

The proof of Theorem 3.5 has been completed. □

Theorem 3.6. *Let $u \in D'(M,\wedge^l)$ and $du \in L^s(M,\wedge^{l+1})$, $l = 0,1,\dots,n-1$, $1 < s < \infty$, be a smooth differential form in a bounded and convex domain M. Then, there exists a constant C, independent of u, such that*

$$\left\|u - u_M\right\|_{\star,M} \le C|M|^{1/n}\|du\|_{s,M}. \tag{3.19}$$

Proof. From the decomposition (1.6), we have

$$\left\|u - u_B\right\|_{s,B} = \|Tdu\|_{s,B} \le C_1|B|\mathrm{diam}(B)\|du\|_{s,B} \le C_2|B||B|^{1/n}\|du\|_{s,B}. \tag{3.20}$$

Using (1.5), (3.20) and the Hölder inequality, it follows that

$$\begin{aligned}
\left\|u - u_M\right\|_{\star,M} &= \sup_{\sigma B\subset M}|B|^{-1}\int_B\left|u - u_M - \left(u - u_M\right)_B\right|dx\\
&= \sup_{\sigma B\subset M}|B|^{-1}\int_B\left|u - u_M - u_B + u_M\right|dx\\
&= \sup_{\sigma B\subset M}|B|^{-1}\int_B\left|u - u_B\right|dx\\
&\le \sup_{\sigma B\subset M}|B|^{-1}\left(\int_B\left|u - u_B\right|^s dx\right)^{1/s}\left(\int_B 1^{s/(s-1)}dx\right)^{(s-1)/s}\\
&\le \sup_{\sigma B\subset M}|B|^{-1/s}\left(\int_B\left|u - u_B\right|^s dx\right)^{1/s}\\
&\le \sup_{\sigma B\subset M}|B|^{-1/s}C_2|B||B|^{1/n}\|du\|_{s,B}\\
&\le \sup_{\sigma B\subset M}C_2|B|^{1/n}\|du\|_{s,B}\\
&\le C_2|M|^{1/n}\|du\|_{s,M}.
\end{aligned} \tag{3.21}$$

This ends the proof of Theorem 3.6. □

Acknowledgments

The authors thank the referee and the editor, Professor András Rontó, for their precious and thoughtful suggestions on this paper. The first author was supported by Science Research Foundation in Harbin Institute of Technology (HITC200709) and Development Program for Outstanding Young Teachers in HIT.

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