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Research Article

On a Two-Step Algorithm for Hierarchical Fixed Point Problems and Variational Inequalities

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A common method in solving ill-posed problems is to substitute the original problem by a family of well-posed (i.e., with a unique solution) regularized problems. We will use this idea to define and study a two-step algorithm to solve hierarchical fixed point problems under different conditions on involved parameters.

1. Introduction and Preliminar Results

A common method in solving ill-posed problems is to substitute the original problem by a family of well-posed (i.e., with a unique solution) regularized problems. We will use this idea to define and study a two-step algorithm to solve hierarchical fixed point problems under different conditions on involved parameters. We will see that choosing appropriate hypotheses on the parameters, we will obtain convergence to the solution of well-posed problems. Changing these assumptions, we will obtain convergence to one of the solutions of a ill-posed problem. The results are situated on the lines of research of Byrne [1], Yang and Zhao [2], Moudafi [3], and Yao and Liou [4].

In this paper, we consider variational inequalities of the form

$$x^* \in \mathrm{Fix}(T) \text{ such that } \langle (I-S)x^*, x-x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T) \tag{1.1}$$

where $T, S : C \to C$ are nonexpansive mappings such that the fixed points set of T ($\mathrm{Fix}(T)$) is nonempty and C is a nonempty closed convex subset of a Hilbert space H. If we denote with Ω the set of solutions of (1.1), it is evident that $\mathrm{Fix}(S) \subseteq \Omega$.

Variational inequalities of (1.1) cover several topics recently investigated in literature as monotone inclusion ([5] and the references therein), convex optimization [6], quadratic minimization over fixed point set (see, e.g., [5, 7–10] and the references therein).

It is well known that the solutions of (1.1) are the fixed points of the nonexpansive mapping $P_{\mathrm{Fix}(T)}S$.

There are in literature many papers in which iterative methods are defined in order to solve (1.1).

Recently, in [3] Moudafi defined the following explicit iterative algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)T x_n), \tag{1.2}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are two sequences in $(0,1)$, and he proved a weak-convergence's result. In order to obtain a strong-convergence result, Maingé and Moudafi in [11] introduced and studied the following iterative algorithm

$$\begin{aligned} w_{n+1} &= \alpha_n f(w_n) + (1-\alpha_n)z_n, \quad n \geq 1, \\ z_n &= \beta_n S w_n + (1-\beta_n) T w_n, \end{aligned} \tag{1.3}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are two sequences in $(0,1)$.

Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. In this paper, under different conditions on involved parameters, we study the algorithm

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)T y_n, \quad n \geq 1, \\ y_n &= \beta_n S x_n + (1-\beta_n) x_n, \end{aligned} \tag{1.4}$$

and give some conditions which assure that the method converges to a solution which solves some variational inequality.

We will confront the two methods (1.3) and (1.4) later.

We recall some general results of the Hilbert spaces theory and of the monotone operators theory.

Lemma 1.1. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{1.5}$$

If K is closed convex subset of a real Hilbert space H, the metric projection $P_K : H \to K$ is the mapping defined as follows: for each $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf_{y\in K} \|x-y\|. \tag{1.6}$$

Lemma 1.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and let P_K be the metric projection from H onto K. Given $x \in H$ and $z \in K$, $z = P_K x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in K. \tag{1.7}$$

Lemma 1.3 (see [7]). *Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$ and $W : C \to C$ be a nonexpansive mapping. Then, for all $x, y \in C$:*

(a) *the mapping $(I - f)$ is strongly monotone with coefficient $(1-\rho)$, that is,*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1-\rho)\|x - y\|^2, \tag{1.8}$$

(b) *the mapping $(I - W)$ is monotone, that is,*

$$\langle x - y, (I - W)x - (I - W)y \rangle \geq 0. \tag{1.9}$$

Finally, we conclude this section with a lemma due to Xu on real sequences which has a fundamental role in the sequel.

Lemma 1.4 (see [9]). *Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 0, \tag{1.10}$$

where $(\gamma_n)_n$ is a sequence in $(0,1)$, and $(\delta_n)_n$ is a sequence in $\mathbb{R}$ such that,

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2. Convergence of the Two-Step Iterative Algorithm

Let us consider the scheme

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \quad n \geq 1, \\ y_n &= \beta_n S x_n + (1-\beta_n)x_n. \end{aligned} \tag{2.1}$$

As we will see the convergence of the scheme depends on the choice of the parameters $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1]$ and $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$. We list some possible hypotheses on them:

(H1) there exists $\gamma > 0$ such that $\beta_n \leq \gamma\alpha_n$;

(H2) $\lim_{n\to\infty} \beta_n/\alpha_n = \tau \in [0, +\infty]$;

(H3) $\alpha_n \to 0$ as $n \to \infty$ and $\sum_{n\in\mathbb{N}} \alpha_n = \infty$;

(H4) $\sum_{n\in\mathbb{N}} |\alpha_n - \alpha_{n-1}| < \infty$;

(H5) $\sum_{n\in\mathbb{N}} |\beta_n - \beta_{n-1}| < \infty$;

(H6) $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\alpha_n = 0$;

(H7) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\beta_n = 0$;

(H8) $\lim_{n\to\infty} (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)/\alpha_n\beta_n = 0$;

(H9) there exists $K > 0$ such that $(1/\alpha_n)|(1/\beta_n) - (1/\beta_{n-1})| \leq K$.

Proposition 2.1. *Assume that (H1) holds. Then $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded.*

Proof. Let $z \in \mathrm{Fix}(T)$. Then,

$$\begin{aligned} \|x_{n+1}-z\| &\le \alpha_n \|f(x_n)-f(z)\| + \alpha_n\|f(z)-z\| + (1-\alpha_n)\|Ty_n - z\| \\ &\le \alpha_n\rho\|x_n-z\| + \alpha_n\|f(z)-z\| + (1-\alpha_n)\beta_n\|Sx_n-z\| + (1-\alpha_n)(1-\beta_n)\|x_n-z\| \\ &\le \alpha_n\rho\|x_n-z\| + \alpha_n\|f(z)-z\| + (1-\alpha_n)\beta_n\|Sz-z\| + (1-\alpha_n)\|x_n-z\| \\ &= (1-(1-\rho)\alpha_n)\|x_n-z\| + \alpha_n\left[\|f(z)-z\| + \gamma\|Sz-z\|\right]. \end{aligned} \tag{2.2}$$

So, by induction, one can see that

$$\|x_n - z\| \le \max\left\{\|x_0-z\|, \frac{1}{1-\rho}\left[\|f(z)-z\| + \gamma\|Sz-z\|\right]\right\}. \tag{2.3}$$

Of course $(y_n)_{n\in\mathbb{N}}$ is bounded too. □

Proposition 2.2. *Suppose that (H1), (H3) hold. Also, assume that either (H4) and (H5) hold, or (H6) and (H7) hold. Then*

(1) *$(x_n)_{n\in\mathbb{N}}$ is asymptotically regular, that is,*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0, \tag{2.4}$$

(2) *the weak cluster points set $\omega_w(x_n) \subset \mathrm{Fix}(T)$.*

Proof. Observing that

$$\begin{aligned} x_{n+1}-x_n &= \alpha_n f(x_n) + (1-\alpha_n)Ty_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})Ty_{n-1} \\ &= \alpha_n\left(f(x_n)-f(x_{n-1})\right) + \left(f(x_{n-1}) - Ty_{n-1}\right)(\alpha_n-\alpha_{n-1}) + (1-\alpha_n)\left(Ty_n - Ty_{n-1}\right), \end{aligned} \tag{2.5}$$

then, passing to the norm we have

$$\begin{aligned} \|x_{n+1}-x_n\| &\le \alpha_n\|f(x_n)-f(x_{n-1})\| + \|f(x_{n-1})-Ty_{n-1}\||\alpha_n-\alpha_{n-1}| + (1-\alpha_n)\|Ty_n-Ty_{n-1}\| \\ &\le \alpha_n\rho\|x_n-x_{n-1}\| + \|f(x_{n-1})-Ty_{n-1}\||\alpha_n-\alpha_{n-1}| + (1-\alpha_n)\|y_n-y_{n-1}\|. \end{aligned} \tag{2.6}$$

By definition of y_n one obtain that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\beta_n(Sx_n - Sx_{n-1}) + (Sx_{n-1} - x_{n-1})(\beta_n - \beta_{n-1}) + (1-\beta_n)(x_n - x_{n-1})\| \\ &\le \beta_n\|x_n - x_{n-1}\| + \|Sx_{n-1} - x_{n-1}\||\beta_n - \beta_{n-1}| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &= \|x_n - x_{n-1}\| + \|Sx_{n-1} - x_{n-1}\||\beta_n - \beta_{n-1}|,\end{aligned} \tag{2.7}$$

so, substituting (2.7) in (2.6) we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| \le \alpha_n\rho\|x_n - x_{n-1}\| &+ |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - Ty_{n-1}\| \\ &+ (1-\alpha_n)\left[\|x_n - x_{n-1}\| + \|Sx_{n-1} - x_{n-1}\||\beta_n - \beta_{n-1}|\right].\end{aligned} \tag{2.8}$$

By Proposition 2.1, we call $M := \max\{\sup_{n\in\mathbb{N}}\|f(x_{n-1}) - Ty_{n-1}\|, \sup_{n\in\mathbb{N}}\|Sx_{n-1} - x_{n-1}\|\}$ so we have

$$\|x_{n+1} - x_n\| \le [1 - \alpha_n(1-\rho)]\|x_n - x_{n-1}\| + M\left[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right]. \tag{2.9}$$

So, if (H4) and (H5) hold, we obtain the asymptotic regularity by Lemma 1.4.

If, instead, (H6) and (H7) hold, from (H1) we can write

$$\begin{aligned}\|x_{n+1} - x_n\| &\le [1 - \alpha_n(1-\rho)]\|x_n - x_{n-1}\| + M\alpha_n\left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}\right] \\ &\le [1 - \alpha_n(1-\rho)]\|x_n - x_{n-1}\| + M\alpha_n\left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \gamma\frac{|\beta_n - \beta_{n-1}|}{\beta_n}\right],\end{aligned} \tag{2.10}$$

so, the asymptotic regularity follows by Lemma 1.4 also.

In order to prove (2), we can observe that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + (1-\alpha_n)\|Ty_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + (1-\alpha_n)\|y_n - x_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + (1-\alpha_n)\beta_n\|Sx_n - x_n\|.\end{aligned} \tag{2.11}$$

By (H1), and (H3) it follows that $\beta_n \to 0$, as $n \to \infty$, so that $\|x_n - Tx_n\| \to 0$ since $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular. By demiclosedness principle we obtain the thesis. □

Corollary 2.3. *Suppose that the hypotheses of Proposition 2.2 hold. Then*

(i) $\lim_{n\to\infty}\|x_n - Ty_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n - y_n\| = 0$;

(iii) $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$.

Proof. To prove (i), we can observe that

$$\left\|x_n - Ty_n\right\| \leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - Ty_n\right\| = \|x_n - x_{n+1}\| + \alpha_n\left\|\gamma f(x_n) - Ty_n\right\|. \tag{2.12}$$

The asymptotical regularity of $(x_n)_{n\in\mathbb{N}}$ gives the claim.

Moreover, noting that

$$\left\|y_n - x_n\right\| = \beta_n\|Sx_n - x_n\|, \tag{2.13}$$

since $\beta_n \to 0$ as $n \to \infty$ we obtain (ii). In the end (iii) follows easily by (i) and (ii). □

Theorem 2.4. *Suppose (H2) with* $\tau = 0$ *and (H3). Moreover Suppose that either (H4) and (H5) hold, or (H6) and (H7) hold. If one denote by* $z \in C$ *the unique element in* $\mathrm{Fix}(T)$ *such that* $z = P_{\mathrm{Fix}(T)}fz$*, then*

(1)

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0, \tag{2.14}$$

(2) $x_n \to z$ *as* $n \to \infty$.

Proof. First of all, $P_{\mathrm{Fix}(T)}f$ is a contraction, so there exists a unique $z \in \mathrm{Fix}(T)$ such that $P_{\mathrm{Fix}(T)}f(z) = z$. Moreover, from Lemma 1.2, z is characterized by the fact that

$$\left\langle f(z) - z, y - z\right\rangle \leq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{2.15}$$

Since (H2) implies (H1), thus $(x_n)_{n\in\mathbb{N}}$ is bounded. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\left\langle f(z) - z, x_n - z\right\rangle = \lim_{k\to\infty}\left\langle f(z) - z, x_{n_k} - z\right\rangle, \tag{2.16}$$

and $x_{n_k} \rightharpoonup x'$. Thanks to either ((H4) and (H5)) or ((H6) and (H7)), by Proposition 2.2 it follows that $x' \in \mathrm{Fix}(T)$. Then

$$\lim_{k\to\infty}\left\langle f(z) - z, x_{n_k} - z\right\rangle = \left\langle f(z) - z, x' - z\right\rangle \leq 0. \tag{2.17}$$

Now we observe that, by Lemma 1.1

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\alpha_n(f(x_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(Ty_n - z)\right\|^2 \\
&\le \left\|\alpha_n(f(x_n) - f(z)) + (1-\alpha_n)(Ty_n - z)\right\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le \alpha_n\rho\|x_n - z\|^2 + (1-\alpha_n)\left\|y_n - z\right\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le \alpha_n\rho\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + (1-\alpha_n)\left\|\beta_n(Sx_n - Sz) + \beta_n(Sz - z) + (1-\beta_n)(x_n - z)\right\|^2 \\
&\le \alpha_n\rho\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + (1-\alpha_n)\|x_n - z\|^2 + 2(1-\alpha_n)\beta_n\langle Sz - z, y_n - z\rangle \\
&= (1-(1-\rho)\alpha_n)\|x_n - z\|^2 + 2(1-\alpha_n)\beta_n\langle Sz - z, y_n - z\rangle + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned} \tag{2.18}$$

Since $\tau = 0$, then

$$\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}\langle Sz - z, y_n - z\rangle = 0. \tag{2.19}$$

Thus, by Lemma 1.4, $x_n \to z$ as $n \to \infty$. □

Theorem 2.5. *Suppose that (H2) with $0 < \tau < \infty$, (H3), (H8), (H9) hold. Then $x_n \to \tilde{x}$, as $n \to \infty$, where $\tilde{x} \in Fix(T)$ is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)\tilde{x} + (I - S)\tilde{x}, \tilde{x} - y\right\rangle \le 0, \quad \forall y \in \mathrm{Fix}(T). \tag{2.20}$$

Proof. First of all, we show that (2.20) cannot have more than one solution. Indeed, let $\overline{x}$ and $\tilde{x}$ be two solutions. Then, since $\tilde{x}$ is solution, for $y = \overline{x}$ one has

$$\langle (I - f)\tilde{x}, \tilde{x} - \overline{x}\rangle \le \tau\langle (I - S)\tilde{x}, \overline{x} - \tilde{x}\rangle. \tag{2.21}$$

Analogously

$$\langle (I - f)\overline{x}, \overline{x} - \tilde{x}\rangle \le \tau\langle (I - S)\overline{x}, \tilde{x} - \overline{x}\rangle. \tag{2.22}$$

Adding (2.21) and (2.22), we obtain

$$(1-\rho)\|\overline{x} - \tilde{x}\|^2 \le \langle (I-f)\tilde{x} - (I-f)\overline{x}, \tilde{x} - \overline{x}\rangle \le -\tau\langle (I - S)\tilde{x} - (I - S)\overline{x}, \tilde{x} - \overline{x}\rangle \le 0 \tag{2.23}$$

so $\tilde{x} = \overline{x}$. Also now the condition (H2) with $0 < \tau < \infty$ implies (H1) so the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Moreover, since (H8) implies (H6) and (H7), then $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular.

Similarly, by Proposition 2.2, the weak cluster points set of x_n, $\omega_w(x_n)$, is a subset of $\mathrm{Fix}(T)$. Now we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n\left(f(x_n) - x_n\right) + (1-\alpha_n)\left(Ty_n - x_n\right) \\ &= \alpha_n\left(f(x_n) - x_n\right) + (1-\alpha_n)\left(Ty_n - y_n\right) + (1-\alpha_n)\left(y_n - x_n\right) \\ &= \alpha_n\left(f - I\right)x_n + (1-\alpha_n)(T-I)y_n + (1-\alpha_n)\beta_n(S-I)x_n, \end{aligned} \tag{2.24}$$

so that

$$\frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n} = (I-S)x_n + \frac{1}{\beta_n}(I-T)y_n + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(I-f)x_n, \tag{2.25}$$

and denoting by $v_n := (x_n - x_{n+1})/(1-\alpha_n)\beta_n$, we have

$$v_n = (I-S)x_n + \frac{1}{\beta_n}(I-T)y_n + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(I-f)x_n. \tag{2.26}$$

Dividing by β_n in (2.9), one observe that

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \left[1 - \alpha_n(1-\rho)\right]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \left[1 - \alpha_n(1-\rho)\right]\|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\ &\quad + M\left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{\left|\beta_n - \beta_{n-1}\right|}{\beta_n}\right] \\ &\leq \left[1 - \alpha_n(1-\rho)\right]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_{n+1} - x_n\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\ &\quad + M\left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{\left|\beta_n - \beta_{n-1}\right|}{\beta_n}\right], \\ \text{by (H9)} &\leq \left[1 - \alpha_n(1-\rho)\right]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_{n+1} - x_n\| \\ &\quad + M\left[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{\left|\beta_n - \beta_{n-1}\right|}{\beta_n}\right]. \end{aligned} \tag{2.27}$$

By Lemma 1.4, we have

$$\lim_{n\to\infty}\frac{\|x_{n+1} - x_n\|}{\beta_n} = 0, \tag{2.28}$$

so, also $v_n := (x_n - x_{n+1})/(1-\alpha_n)\beta_n$ is a null sequence as $n \to \infty$. Fixing $z \in \mathrm{Fix}(T)$, by (2.26) it results

$$\begin{aligned}
\langle v_n, x_n - z\rangle &= \langle (I-S)x_n, x_n - z\rangle + \frac{1}{\beta_n}\langle (I-T)y_n, x_n - z\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle \\
&= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T)y_n - (I-T)z, x_n - z\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n - (I-f)z, x_n - z\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle.
\end{aligned} \tag{2.29}$$

By Lemma 1.3, we obtain that

$$\begin{aligned}
\langle v_n, x_n - z\rangle &\geq \langle (I-S)z, x_n - z\rangle + \frac{1}{\beta_n}\langle (I-T)y_n - (I-T)z, x_n - y_n\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T)y_n - (I-T)z, y_n - z\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2 \\
&\geq \langle (I-S)z, x_n - z\rangle + \langle (I-T)y_n - (I-T)z, x_n - Sx_n\rangle \\
&\quad + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2 + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle.
\end{aligned} \tag{2.30}$$

Now, we observe that

$$\begin{aligned}
\|x_n - z\|^2 &\leq \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\left[\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle - \langle (I-T)y_n, x_n - Sx_n\rangle\right] \\
&\quad - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle,
\end{aligned} \tag{2.31}$$

so, since $v_n \to 0$ and $(I-T)y_n \to 0$, as $n \to \infty$, then every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is also a strong cluster point.

By Proposition 2.2, $(x_n)_{n\in\mathbb{N}}$ is bounded, thus there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to x'. For all $z \in \mathrm{Fix}(T)$ by (2.26)

$$\begin{aligned}\langle (I-f)x_{n_k}, x_{n_k}-z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)x_{n_k}, x_{n_k}-z\rangle \\ &\quad - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\langle (I-T)y_{n_k}, x_{n_k}-z\rangle \\ \text{by Lemma 1.3} &\le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k}-z\rangle \\ &\quad - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\langle (I-T)y_{n_k}-(I-T)z, x_{n_k}-y_{n_k}\rangle \\ &\le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k}-z\rangle \\ &\quad - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\beta_{n_k}\langle (I-T)y_{n_k}, x_{n_k}-Sx_{n_k}\rangle.\end{aligned} \tag{2.32}$$

Passing to $k \to \infty$, we obtain

$$\langle (I-f)x', x'-z\rangle \le -\tau\langle (I-S)z, x'-z\rangle, \quad \forall z \in \mathrm{Fix}(T), \tag{2.33}$$

which (2.20). Thus, since the (2.20) cannot have more than one solution, it follows that $\omega_w(x_n) = \omega_s(x_n) = \{\tilde{x}\}$ and this, of course, ensures that $x_n \to \tilde{x}$, as $n \to \infty$. □

Proposition 2.6. *Suppose that (H2) holds with* $\tau = +\infty$*. Suppose that (H3), (H8) and (H9) hold. Moreover let* $(x_n)_{n\in\mathbb{N}}$ *be bounded and* $(\beta_n)_{n\in\mathbb{N}}$ *be a null sequence. Then every* $v \in \omega_w(x_n)$ *is solution of the variational inequality*

$$\langle (I-S)v, v-x\rangle \le 0, \quad \forall x \in \mathrm{Fix}(T), \tag{2.34}$$

that is, $v \in \Omega$*.*

Proof. Since (H8) implies (H6) and (H7), by boundedness of $(x_n)_{n\in\mathbb{N}}$, we can obtain its asymptotical regularity as in proof of Proposition 2.2. Moreover, since $\beta_n \to 0$, as in proof of Proposition 2.2, $\omega_w(x_n) \subset \mathrm{Fix}(T)$. With the same notation in proof of Theorem 2.5 we have that

$$\begin{aligned}\langle v_n, x_n-z\rangle &\ge \langle (\mathrm{I}-S)z, x_n-z\rangle + \langle (I-T)y_n, x_n-Sx_n\rangle \\ &\quad + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n-z\|^2 + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n-z\rangle \\ &\ge \langle (I-S)z, x_n-z\rangle + \langle (I-T)y_n, x_n-Sx_n\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n-z\rangle\end{aligned} \tag{2.35}$$

holds for all $z \in \mathrm{Fix}(T)$. So, if $v \in \omega_w(x_n)$ and $x_{n_k} \rightharpoonup v$, by (H2), the boundedness of $(x_n)_{n\in\mathbb{N}}$, $v_n \to 0$ and (iii) of Corollary 2.3 we have

$$\begin{aligned}\langle (I-S)z, w-z\rangle &= \lim_k \langle (I-S)z, x_{n_k}-z\rangle \\ &\le \lim_k \Bigg[\langle v_{n_k}, x_{n_k}-z\rangle + \langle (I-T)y_{n_k}, Sx_{n_k}-x_{n_k}\rangle \\ &\quad + \frac{\alpha_{n_k}}{(1-\alpha_{n_k})\beta_{n_k}}\langle (I-f)z, z-x_{n_k}\rangle\Bigg] \le 0, \quad \forall z \in \mathrm{Fix}(T).\end{aligned} \tag{2.36}$$

If we change z with $v+\mu(z-v)$, $\mu \in (0,1)$, we have

$$\langle (I-S)\big(v+\mu(z-v)\big), v-z\rangle \le 0. \tag{2.37}$$

Letting $\mu \to 0$ finally

$$\langle (I-S)v, v-z\rangle \le 0, \quad \forall z \in \mathrm{Fix}(T). \tag{2.38}$$

□

Remark 2.7. If we choose $\alpha_n = n^{-\xi}$ and $\beta_n = n^{-\gamma}$ (with $\xi, \gamma > 0$), since $|\alpha_n - \alpha_{n-1}| \approx n^{-\xi}$ and $|\alpha_n - \alpha_{n-1}| \approx n^{-\gamma}$ it is not difficult to prove that (H8) is satisfied for $0 < \xi, \gamma < 1$ and (H9) is satisfied if $\xi + \gamma \le 1$.

Remark 2.8. It is clear that our algorithm (1.4) is different from (1.3). At the same time, our algorithm (1.4) includes some algorithms in the literature as special cases. For instance, if we take $S = I$ in (1.4), then we get $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n$ which is well-known as the viscosity method studied by Moudafi [8] and Xu [10].

Remark 2.9. We do not know the rate of convergence of our method. Nevertheless, the rates of convergence of our method (1.4) that generates the sequence x_n and the Mainge-Moudafi method (1.3), seem not comparable. To see this, we consider three examples. In such examples we take $H = R$, $C = [-1,1]$, $\alpha_n = (n+1)^{-1/4}$, $\beta_n = (n+1)^{-1/6}$, $f = (1/2)I$, $x_0 = 1$.

In all three examples all the assumptions (that are the same of the Mainge-Moudafi method) are satisfied and the point at which both the sequences x_n and w_n converge is $\tilde{x} = (P_\Omega f)\tilde{x} = 0$.

Example 2.10. Take $S = I$ and $T = -I$. Then

$$x_{n+1} = -\left(1 - \frac{3}{2(n+1)^{1/4}}\right)x_n \tag{2.39}$$

while

$$w_{n+1} = \left[-\left(1 - \frac{3}{2(n+1)^{1/4}}\right) + \frac{2}{(n+1)^{1/6}}\left(1 - \frac{1}{(n+1)^{1/4}}\right)\right]\mathrm{w}_n. \tag{2.40}$$

Now $x_1 = w_1 = 0.5$, while, for $1 < n \leq 63$, it results $|x_n| < |w_n|$. For instance, we report here some value

$$\begin{aligned}
x_2 &= 0.130672, & w_2 &= 0.272417\\
x_{10} &= -2.52698e-11, & w_{10} &= 0.00086297\\
x_{20} &= -1.40916e-17, & w_{20} &= 6.2157e-08\\
x_{30} &= -2.19772e-22, & w_{30} &= 5.95813e-13\\
x_{40} &= -1.55804e-26, & w_{40} &= 1.0547e-18\\
x_{50} &= -2.79005e-30, & w_{50} &= 4.10857e-25\\
x_{60} &= -9.53183e-34, & w_{60} &= 3.8945e-32\\
x_{63} &= 9.61011e-35, & w_{63} &= 2.33246e-34.
\end{aligned} \tag{2.41}$$

However from the 64th iteration onward, w_n becomes quickly very exiguous with respect to x_n. For instance, $w_{259} = -1.4822e-323$ while $x_n = 7.18026e-83$.

Example 2.11. Take $S = T = -I$. Then

$$x_{n+1} = \left[-\left(1 - \frac{3}{2(n+1)^{1/4}}\right) + \frac{2}{(n+1)^{1/6}}\left(1 - \frac{1}{(n+1)^{1/4}}\right)\right]x_n, \tag{2.42}$$

while

$$w_{n+1} = -\left(1 - \frac{3}{2(n+1)^{1/4}}\right)w_n, \tag{2.43}$$

that is the sequences x_n and w_n are interchanged with respect to the previous example. So this time $|x_n| > |w_n|$ for $1 < n < 64$ and $|x_n| < |w_n|$ for $n \geq 64$.

Example 2.12. Take $S = -I, T = P_{[-1/2,1/2]}$. Then

$$\begin{aligned}
x_{n+1} &= \frac{1}{2(n+1)^{1/4}}x_n + \left(1 - \frac{1}{(n+1)^{1/4}}\right)P_{[-1/2,1/2]}\left[\left(1 - \frac{2}{(n+1)^{1/6}}\right)x_n\right],\\
w_{n+1} &= \frac{1}{2(n+1)^{1/4}}w_n + \left(1 - \frac{1}{(n+1)^{1/4}}\right)\left[-\frac{1}{(n+1)^{1/6}}w_n + \left(1 - \frac{1}{(n+1)^{1/6}}\right)P_{[-1/2,1/2]}w_n\right]
\end{aligned} \tag{2.44}$$

so this time $x_n = w_n$.

Reassuming, we cannot affirm that our method is more convenient or better than the Mainge-Moudafi method, but only that seems to us that it is the first time that it is introduced a two-step iterative approach to the VIP (1.1). In some case, our method approximates the solution more rapidly than Mainge-Moudafi method, in some other case it happens the contrary and in some other cases, both methods give the same sequence.

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