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Research Article

Regularity of Parabolic Hemivariational Inequalities with Boundary Conditions

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We prove the regularity for solutions of parabolic hemivariational inequalities of dynamic elasticity in the strong sense and investigate the continuity of the solution mapping from initial data and forcing term to trajectories.

1. Introduction

In this paper, we deal with the existence and a variational of constant formula for solutions of a parabolic hemivariational inequality of the form:

$$\dot{u}(x,t) + \Delta u(x,t) - \operatorname{div} C\left[\varepsilon(u(x,t)\right] + \Xi(x,t) = f(x,t) \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.2}$$

$$C\left[\varepsilon\big(u(x,t)\big)\right]\nu = -(\beta\cdot\nu)u(x,t) \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.3}$$

$$\Xi(x,t) \in \varphi\big(u(x,t)\big) \quad \text{a.e. } (x,t) \in \Omega \times (0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary Γ. Let $x^0 \in \mathbb{R}^N$, $\beta(x) = x - x^0$, $R = \max_{x\in\overline{\Omega}}|x - x^0|$. The boundary Γ is composed of two pieces Γ_0 and Γ_1, which are nonempty sets and defined by

$$\Gamma_0 := \{x \in \Gamma : \beta(x) \cdot \nu \geq \alpha > 0\}, \qquad \Gamma_1 := \{x \in \Gamma : \beta(x) \cdot \nu \leq 0\}, \tag{1.6}$$

where ν is the unit outward normal vector to Γ. Here $\dot{u} = \partial u/\partial t$, $u = (u_1, \dots, u_N)^T$ is the displacement, $\varepsilon(u) = (1/2)\{\nabla u + (\nabla u)^T\} = (1/2)((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))$ is the strain tensor, $\varphi(u) = (\varphi_1(u_1), \dots, \varphi_N(u_N))^T$, φ_i is a multi-valued mapping by filling in jumps of a locally bounded function b_i, $i = 1, \dots, N$. A continuous map C from the space S of $N \times N$ symmetric matrices into itself is defined by

$$C[\varepsilon] = a(\operatorname{tr}\varepsilon)I + b\varepsilon, \quad \text{for } \varepsilon \in S, \tag{1.7}$$

where I is the identity of S, $\operatorname{tr}\varepsilon$ denotes the trace of ε, and $a > 0$, $b > 0$. For example, in the case $N = 2$, $C[\varepsilon] = (E/d(1-\mu^2))[\mu(\operatorname{tr}\varepsilon)I + (1-\mu)\varepsilon]$, where $E > 0$ is Young's modulus, $0 < \mu < 1/2$ is Poisson's ratio and d is the density of the plate.

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality. Namely, we formulated the problem (1.1) as

$$\dot{u} + Au - \operatorname{div} C[\varepsilon(u)] + \Xi = f \quad \text{in } \Omega \times (0, \infty). \tag{1.8}$$

The existence of global weak solutions for a class of hemivariational inequalities has been studied by many authors, for example, parabolic type problems in [1–4], and hyperbolic types in [5–7]. Rauch [8] and Miettinen and Panagiotopoulos [1, 2] proved the existence of weak solutions for elliptic one. The background of these variational problems are physics, especially in solid mechanics, where nonconvex and multi-valued constitutive laws lead to differential inclusions. We refer to [3, 4] to see the applications of differential inclusions. Most of them considered the existence of weak solutions for differential inclusions of various forms by using the Faedo-Galerkin approximation method.

In this paper, we prove the existence and a variational of constant formula for strong solutions of parabolic hemivariational inequalities. The plan of this paper is as follows. In Section 2, the main results besides notations and assumptions are stated. In order to prove the solvability of the linear case with $\Xi(x, t) = 0$ we establish necessary estimates applying the result of Di Blasio et al. [9] to (1.1)–(1.5) considered as an equation in H as well as V^*. The existence and regularity for the nondegenerate nonlinear systems has been developed as seen in [10, Theorem 4.1] or [11, Theorem 2.6], and the references therein. In Section 3, we will obtain the existence for solutions of (1.1)–(1.5) by converting the problem into the contraction mapping principle and the norm estimate of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$. Consequently, if u is a solution asociated with u_0, and f, in view of the monotonicity of A, we show that the mapping

$$H \times L^2(0, T; V^*) \ni (u_0, f) \longmapsto u \in L^2(0, T; V) \cap C([0, T]; H), \tag{1.9}$$

is continuous.

2. Preliminaries and Linear Hemivariational Inequalities

We denote $\xi \cdot \zeta = \sum_{i=1}^{N} \xi_i \zeta_i$ for $\xi = (\xi_1, \ldots, \xi_N)$, $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N$ and $\varepsilon \cdot \tilde{\varepsilon} = \sum_{i,j=1}^{N} \varepsilon_{ij} \tilde{\varepsilon}_{ij}$ for ε, $\tilde{\varepsilon} \in S$. Throughout this paper, we consider

$$V = \{u \in (H^1(\Omega))^N : u = 0 \text{ on } \Gamma_1\}, \qquad H = (L^2(\Omega))^N,$$
$$(u, v) = \int_\Omega u(x) \cdot v(x) dx, \qquad (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x) \cdot v(x) d\Gamma. \tag{2.1}$$

We denote V^* the dual space of V, $(\cdot, \cdot)$ the dual pairing between V and V^*.

The norms on V, H, and V^* will be denoted by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$, respectively. For the sake of simplicity, we may consider

$$\|u\|_* \leq |u| \leq \|u\|, \quad u \in V. \tag{2.2}$$

We denote $\|\cdot\|_{(L^2(\Gamma_0))^N}$ by $\|\cdot\|_{\Gamma_0}$. Let A be the operator associated with a sesquilinear form $a(u, v)$ which is defined Gårding's inequality

$$\operatorname{Re} a(u, u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2, \quad \omega_1 > 0, \ \omega_2 \geq 0, \ \text{for } u \in V, \tag{2.3}$$

that is,

$$(Au, v) = a(u, v), \quad u, v \in V. \tag{2.4}$$

Then A is a symmetric bounded linear operator from V into V^* which satisfies

$$(Au, u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2 \tag{2.5}$$

and its realization in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\} \tag{2.6}$$

is also denoted by A. Here, we note that $D(A)$ is dense in V. Hence, it is also dense in H. We endow the domain $D(A)$ of A with graph norm, that is, for $u \in D(A)$, we define $\|u\|_{D(A)} = |u| + |Au|$. So, for the brevity, we may regard that $|u| \leq \|u\| \leq \|u\|_{D(A)}$ for all $u \in V$. It is known that—A generates an analytic semigroup $S(t)$ $(t \geq 0)$ in both H and V^*.

From the following inequalities

$$\omega_1 \|u\|^2 \leq \operatorname{Re} a(u, u) + \omega_2 |u|^2 \leq C|Au||u| + \omega_2 |u|^2$$
$$\leq (C|Au| + \omega_2 |u|)|u| \leq \max\{C, \omega_2\} \|u\|_{D(A)} |u|, \tag{2.7}$$

it follows that there exists a constant $C_0 > 0$ such that

$$\|u\| \leq C_0 \|u\|_{D(A)}^{1/2} |u|^{1/2}. \tag{2.8}$$

So, we may regard as $V = (D(A), H)_{1/2,2}$ where $(D(A), H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H.

Consider the following initial value problem for the abstract linear parabolic type equation:

$$\begin{gathered}
\dot{u}(t) + Au(t) - \operatorname{div} C\left[\varepsilon(u(t))\right] = f(t), \quad t > 0, \\
u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\
C\left[\varepsilon(u(x,t))\right]\nu = -(\beta \cdot \nu) u(x,t) \quad \text{on } \Gamma_0 \times (0, \infty), \\
u(x, 0) = u_0(x), \quad x \in \Omega.
\end{gathered} \tag{LE}$$

A continuous map C from the space S of $N \times N$ symmetric matrices into itself is defined by

$$C[\varepsilon] = a(\operatorname{tr} \varepsilon) I + b\varepsilon, \quad \text{for } a > 0,\ b > 0,\ \varepsilon \in S. \tag{2.9}$$

It is easily known that

$$(\operatorname{div} C[\varepsilon(w)], v) = -(C[\varepsilon(w)]\nu, v)_{\Gamma_0} + (C[\varepsilon(w)], \varepsilon(v)), \quad v, w \in V, \tag{2.10}$$

$$C\left[\varepsilon(w_1)\right] - C\left[\varepsilon(w_2)\right] = C\left[\varepsilon(w_1 - w_2)\right], \quad w_1,\ w_2 \in V. \tag{2.11}$$

Note that the map C is linear and symmetric and it can be easily verified that the tensor C satisfies the condition

$$\lambda_0 \left|\varepsilon\right|^2 \leq C[\varepsilon] \cdot \varepsilon \leq \lambda_1 \left|\varepsilon\right|^2, \quad \varepsilon \in S \text{ for some } \lambda_0,\ \lambda_1 > 0. \tag{2.12}$$

Let λ be the smallest positive constant such that

$$\|v\|^2 \leq \lambda \|\nabla v\|^2 \quad \forall v \in V. \tag{2.13}$$

Simple calculations and Korn's inequality yield that

$$\lambda_2 |\nabla u|^2 \leq \left|\varepsilon(u)\right|^2 \leq \lambda_3 |\nabla u|^2, \tag{2.14}$$

and hence $|\varepsilon(u)|$ is equivalent to the $(H^1(\Omega))^N$ norm on V. Then by virtue of [9, Theorem 3.3], we have the following result on the linear parabolic type equation (LE).

Proposition 2.1. *Suppose that the assumptions stated above are satisfied. Then the following properties hold.*

(1) *For any* $u_0 \in V = (D(A), H)_{1/2,2}$ *and* $f \in L^2(0,T;H)$ $(T > 0)$, *there exists a unique solution* u *of* (LE) *belonging to*

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V) \tag{2.15}$$

and satisfying

$$\|u\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_1\left(\|u_0\| + \|f\|_{L^2(0,T;H)}\right), \tag{2.16}$$

where C_1 *is a constant depending on* T.

(2) *Let* $u_0 \in H$ *and* $f \in L^2(0,T;V^*)$ *for any* $T > 0$. *Then there exists a unique solution* u *of* (LE) *belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H) \tag{2.17}$$

and satisfying

$$\|u\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1\left(|u_0| + \|f\|_{L^2(0,T;V^*)}\right), \tag{2.18}$$

where C_1 *is a constant depending on* T.

Proof. (1) Let $\widehat{a}(u,v)$ be a bounded sesquilinear form defined in $V \times V$ by

$$\widehat{a}(u,v) = (Au,v) - \left(\operatorname{div} C\left[\varepsilon(u)\right], v\right), \quad u,v \in V. \tag{2.19}$$

Noting that by(2.10)

$$-\left(\operatorname{div} C\left[\varepsilon(u)\right], u\right) = \left(C\left[\varepsilon(u)\right], \varepsilon(u)\right) + ((\beta\cdot\nu)u,u)_{\Gamma_0}, \tag{2.20}$$

and by (2.12), (2.14), and (1.6),

$$\lambda_0\lambda_2\|u\|^2 \leq \left(C\left[\varepsilon(u)\right], \varepsilon(u)\right), \qquad \alpha|u|^2 \leq ((\beta\cdot\nu)u,u)_{\Gamma_0}, \tag{2.21}$$

it follows that there exist $\widehat{\omega}_1 > 0$ and $\widehat{\omega}_2 \geq 0$ such that

$$\operatorname{Re}\widehat{a}(u,u) \geq \widehat{\omega}_1\|u\|^2 - \widehat{\omega}_2|u|^2, \quad \text{for } u \in V. \tag{2.22}$$

Let $\widehat{A}$ be the operator associated with this sesquilinear form:

$$(\widehat{A}u,v) = \widehat{a}(u,v), \quad u,v \in V. \tag{2.23}$$

Then $\widehat{A}$ is also a symmetric continuous linear operator from V into V^* which satisfies

$$(\widehat{A}u, u) \geq \widehat{\omega}_1 \|u\|^2 - \widehat{\omega}_2 |u|^2. \tag{2.24}$$

So we know that—$\widehat{A}$ generates an analytic semigroup $\widehat{S}(t)$ $(t \geq 0)$ in both H and V^*. Hence, by applying [9, Theorem 3.3] to the regularity for the solution of the equation:

$$\begin{gathered}
\dot{u}(t) + \widehat{A}u(t) = f(t), \quad t > 0, \\
u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\
C\left[\varepsilon\left(u(x,t)\right)\right]\nu = -(\beta \cdot \nu)u(x,t) \quad \text{on } \Gamma_0 \times (0, \infty), \\
u(x, 0) = u_0(x), \quad x \in \Omega,
\end{gathered} \tag{2.25}$$

in the space H, we can obtain a unique solution u of (LE) belonging to

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V) \tag{2.26}$$

and satisfying the norm estimate (2.16).

(2) It is easily seen that

$$H = \left\{ x \in V^* : \int_0^T \left\| Ae^{tA}x \right\|_*^2 dt < \infty \right\}, \tag{2.27}$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H \tag{2.28}$$

and follow the argument of (1) term by term to deduce the proof of (2) results. □

3. Existence of Solutions in the Strong Sense

This Section is to investigate the regularity of solutions for the following parabolic hemivariational inequality of dynamic elasticity in the strong sense:

$$\begin{gathered}
\dot{u}(t) + Au(t) - \operatorname{div} C\left[\varepsilon\left(u(t)\right)\right] + \Xi(x,t) = f(t), \quad t \geq 0, \\
u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\
C\left[\varepsilon\left(u(x,t)\right)\right]\nu = -(\beta \cdot \nu)u(x,t) \quad \text{on } \Gamma_0 \times (0, \infty) \\
\Xi(x,t) \in \varphi\left(u(x,t)\right) \quad \text{a.e. } (x,t) \in \Omega \times (0, \infty), \\
u(0) = u_0.
\end{gathered} \tag{HIE}$$

Now, we formulate the following assumptions.

(Hb) Let $b_i(i = 1, \dots, N) : \mathbb{R} \to \mathbb{R}$ be a locally bounded function verifying

$$\left|b_i(s)\right| \leq \mu_i|s| \quad \text{for } s \in \mathbb{R}, \tag{3.1}$$

where $\mu_i > 0$. We denote

$$\widetilde{\mu} = \max\{\mu_1, \dots, \mu_N\}. \tag{3.2}$$

The multi-valued function $\varphi_i : \mathbb{R} \to 2^{\mathbb{R}}$ is obtained by filling in jumps of a function $b_i : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b_i^\epsilon}, \overline{b_i^\epsilon}, \underline{b_i}, \overline{b_i} : \mathbb{R} \to \mathbb{R}$ as follows.

$$\begin{gathered} \underline{b_i^\epsilon}(s) = \operatorname{ess}\inf_{|\tau - s| \leq \epsilon} b_i(\tau), \qquad \overline{b_i^\epsilon}(s) = \operatorname{ess}\sup_{|\tau - s| \leq \epsilon} b_i(\tau), \\ \underline{b_i}(s) = \lim_{\epsilon \to 0^+} \underline{b_i^\epsilon}(s), \qquad \overline{b_i}(s) = \lim_{\epsilon \to 0^+} \overline{b_i}^\epsilon(s), \\ \varphi_i(s) = \left[\underline{b_i}(s), \overline{b_i}(s)\right]. \end{gathered} \tag{3.3}$$

We denote $b(\xi) := (b_1(\xi_1), \dots, b_N(\xi_N))$, $\varphi(\xi) := (\varphi_1(\xi_1), \dots, \varphi_N(\xi_N))$ for $\widetilde{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$. We will need a regularization of b_i defined by

$$b_i^n(s) = n \int_{-\infty}^{\infty} b_i(s - \tau)\rho(n\tau)d\tau, \tag{3.4}$$

where $\rho \in C_0^\infty((-1, 1))$, $\rho \geq 0$ and $\int_{-1}^{1} \rho(\tau)d\tau = 1$. It is easy to show that b_i^n is continuous for all $n \in \mathbb{N}$ and $\underline{b_i^\epsilon}$, $\overline{b_i^\epsilon}$, $\underline{b_i}$, $\overline{b_i}$, b_i^n satisfy the same condition (Hb) with possibly different constants if b_i satisfies (Hb). It is also known that $b_i^n(s)$ is locally Lipschitz continuous in s, that is for any $r > 0$, there exists a number $L_i(r) > 0$ such that

(Hb-1)

$$\left|b_i^n(s_1) - b^n(s_2)\right| \leq L_i(r)\left|s_1 - s_2\right| \tag{3.5}$$

holds for all $s_1, s_2 \in \mathbb{R}$ with $|s_1| < r$, $|s_2| < r$. We denote

$$L(r) = \max\{L_1(r), \dots, L_N(r)\}. \tag{3.6}$$

The following lemma is from [[12]; Lemma A.5].

Lemma 3.1. *Let $m \in L^1(0,T;\mathbb{R})$ satisfying $m(t) \geq 0$ for all $t \in (0,T)$ and $a \geq 0$ be a constant. Let d be a continuous function on $[0,T] \subset \mathbb{R}$ satisfying the following inequality:*

$$\frac{1}{2}d^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)d(s)ds, \quad t \in [0,T]. \tag{3.7}$$

Then,

$$\left|d(t)\right| \leq a + \int_0^t m(s)ds, \quad t \in [0,T]. \tag{3.8}$$

Proof. Let

$$\beta_\epsilon(t) = \frac{1}{2}(a+\epsilon)^2 + \int_0^t m(s)d(s)ds, \quad \epsilon > 0. \tag{3.9}$$

Then

$$\frac{d\beta_\epsilon(t)}{dt} = m(t)d(t), \quad t \in (0,T), \tag{3.10}$$

and

$$\frac{1}{2}d^2(t) \leq \beta_0(t) \leq \beta_\epsilon(t), \quad t \in (0,T). \tag{3.11}$$

Hence, we have

$$\frac{d\beta_\epsilon(t)}{dt} \leq m(t)\sqrt{2}\sqrt{\beta_\epsilon(t)}. \tag{3.12}$$

Since $t \longrightarrow \beta_\epsilon(t)$ is absolutely continuous and

$$\frac{d}{dt}\sqrt{\beta_\epsilon(t)} = \frac{1}{2\sqrt{\beta_\epsilon(t)}}\frac{d\beta_\epsilon(t)}{dt} \tag{3.13}$$

for all $t \in (0,T)$, it holds

$$\frac{d}{dt}\sqrt{\beta_\epsilon(t)} \leq \frac{1}{\sqrt{2}}m(t), \tag{3.14}$$

that is,

$$\sqrt{\beta_\epsilon(t)} \leq \sqrt{\beta_\epsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)ds, \quad t \in (0,T). \tag{3.15}$$

Therefore, combining this with (3.11), we conclude that

$$
\begin{aligned}
|d(t)| &\le \sqrt{2}\sqrt{\beta_\epsilon(t)} \le \sqrt{2}\sqrt{\beta_\epsilon(0)} + \int_0^t m(s)ds \\
&= a + \epsilon + \int_0^t m(s)ds, \quad t \in [0,T]
\end{aligned}
\tag{3.16}
$$

for arbitrary $\epsilon > 0$. □

From now on, we establish the following results on the local solvability of the following equation,

$$
\begin{gathered}
\dot{u}(t) + Au(t) - \operatorname{div} C\left[\varepsilon(u(t))\right] = -b^n(u(t)) + f(t), \quad t \ge 0,\ n \in \mathbb{N}, \\
u = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\
C\left[\varepsilon(u(x,t))\right]\nu = -(\beta\cdot\nu)u(x,t) \quad \text{on } \Gamma_0 \times (0,\infty), \\
u(0) = u_0.
\end{gathered}
\tag{HIE-1}
$$

Lemma 3.2. *Let u be a solution of* (HIE-1) *and $u \in B_r = \{v \in L^2(0,T;V) : \|v\| \le r\}$. Then, the following inequality holds, for any $0 < t \le T$,*

$$
|u(t)|^2 + \|u\|^2_{L^2(0,t;\Gamma_0)} + \|u\|^2_{L^2(0,t;V)} \le c_1^{-1}\left(\frac{1}{2}|u_0|^2 + \|f\|^2_{L^2(0,t;H)}\right)e^{(\omega_2+L(r)+1)t},
\tag{3.17}
$$

where $c_1 = \min\{1/2, \alpha, \omega_1 + c_0\}$.

Proof. We remark that from (2.11), (2.12), it follows that there is a constant $c_0 > 0$ such that

$$
c_0\|u_1(t) - u_2(t)\|^2 \le \left(C\left[\varepsilon(u_1(t))\right] - C\left[\varepsilon(u_2(t))\right], \varepsilon(u_1(t)) - \varepsilon(u_2(t))\right).
\tag{3.18}
$$

Consider the following equation:

$$
\dot{u}(t) + Au(t) - \operatorname{div} C\left[\varepsilon(u(t))\right] = -b^n(u(t)) + f(t), \quad t > 0,\ n \in \mathbb{N}.
\tag{3.19}
$$

Multipying on both sides of $u(t)$, we get

$$
\begin{aligned}
&(\dot{u}(t), u(t)) + (Au(t), u(t)) + (C\left[\varepsilon(u(t))\right], \varepsilon(u(t))) + ((\beta\cdot\nu)u(t), u(t)) \\
&\quad + (b^n(u(t)), u(t)) = (f(t), u(t)),
\end{aligned}
\tag{3.20}
$$

and integrating this over $(0,t)$, by (1.6), (2.5), (3.18) and (Hb-1), we have

$$\begin{aligned} &\frac{1}{2}|u(t)|^2+\alpha\int_0^t\|u(\tau)\|_{\Gamma_0}^2d\tau+(\omega_1+c_0)\int_0^t\|u(\tau)\|^2d\tau \\ &\quad\le\frac{1}{2}|u_0|^2+(\omega_2+L(r))\int_0^t|u(\tau)|^2d\tau+\int_0^t\left\{|f(\tau)|^2+|u(\tau)|^2\right\}d\tau, \end{aligned} \tag{3.21}$$

that is,

$$\begin{aligned} &c_1\left(|u(t)|^2+\|u\|_{L^2(0,t;\Gamma_0)}^2+\|u\|_{L^2(0,t;V)}^2\right) \\ &\quad\le\frac{1}{2}|u_0|^2+\|f\|_{L^2(0,t;H)}^2+(\omega_2+L(r)+1)\int_0^t|u(\tau)|^2d\tau. \end{aligned} \tag{3.22}$$

Applying Gronwall lemma, the proof of the lemma is complete. □

Theorem 3.3. *Assume that $u_0\in H$, $f\in L^2(0,T;V^*)$ and (Hb). Then, there exists a time $T_0>0$ such that* (HIE-1) *admits a unique solution*

$$u\in L^2(0,T_0;V)\cap W^{1,2}(0,T_0;V^*)\cap C([0,T_0];H),\quad 0<T_0\le T. \tag{3.23}$$

Proof. Assume that (2.5) holds for $\omega_2\neq 0$. Let the constant r satisfy the following inequality:

$$c_1^{-1}\left(\frac{1}{2}|u_0|^2+\|f\|_{L^2(0,T;H)}^2\right)e^{(\omega_2+L(r)+1)T}<r. \tag{3.24}$$

Let us fix $T\ge T_0>0$ such that

$$\frac{\max\{\tilde{\mu},L(r)\}}{4\omega_2(\omega_1+c_0)}\left(e^{2\omega_2T_0}-1\right)<1, \tag{3.25}$$

where $\tilde{\mu}$ is given by (Hb).

Invoking Proposition 2.1, for a given $w\in B_r=\{v\in L^2(0,T_0;V):\|v\|\le r\}$, the problem

$$\begin{aligned} &\dot{u}(t)+Au(t)-\operatorname{div}C[\varepsilon(u(t))]=-b^n(w(t))+f(t),\quad t\ge 0,n\in\mathbb{N}, \\ &u=0\quad\text{on }\Gamma_1\times(0,\infty), \\ &C[\varepsilon(u(x,t))]\nu=-(\beta\cdot\nu)u(x,t)\quad\text{on }\Gamma_0\times(0,\infty), \\ &u(0)=u_0. \end{aligned} \tag{HIE-2}$$

has a unique solution $u\in L^2(0,T;V)\cap C([0,T];H)$. To prove the existence and uniqueness of solutions of semilinear type (HIE-1), by virtue of Lemma 3.2, we are going to show that the mapping defined by $w\mapsto u$ maps is strictly contractive from B_r into itself if the condition (3.25) is satisfied. □

Lemma 3.4. *Let u_1, u_2 be the solutions of* (HIE-2) *with w replaced by $w_1, w_2 \in B_r$ where B_r is the ball of radius r centered at zero of $L^2(0, T_0; V)$, respectively. Then the following inequality holds:*

$$\left|u_1(t) - u_2(t)\right| \le \int_0^t e^{\omega_2(t-s)} G(s) ds, \tag{3.26}$$

where

$$G(t) = L(r)\left\|w_1(t) - w_2(t)\right\|. \tag{3.27}$$

Proof. Let u_1, u_2 be the solutions of (HIE-2) with w replaced by $w_1, w_2 \in L^2(0, T_0; V)$, respectively. Then, we have that

$$\begin{aligned}&\frac{d}{dt}\left(u_1(t) - u_2(t)\right) + A\left(u_1(t) - u_2(t)\right) - \left(\operatorname{div} C\left[\varepsilon\left(u_1(t)\right)\right] - \operatorname{div} C\left[\varepsilon\left(u_2(t)\right)\right]\right) \\ &\quad = -\left\{b^n\left(w_1(t)\right) - b^n\left(w_2(t)\right)\right\}, \quad t > 0,\ n \in \mathbb{N}.\end{aligned} \tag{3.28}$$

Multiplying on both sides of $u_1(t) - u_2(t)$ and by (2.8), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|u_1(t) - u_2(t)\right|^2 + a\left(u_1(t) - u_2(t), u_1(t) - u_2(t)\right) \\ &\qquad + \left(C\left[\varepsilon\left(u_1(t)\right)\right] - C\left[\varepsilon\left(u_2(t)\right)\right], \varepsilon\left(u_1(t)\right) - \varepsilon\left(u_2(t)\right)\right) \\ &\qquad + \left(\left(\beta(x) \cdot \nu\right)\left(u_1(t) - u_2(t)\right), u_1(t) - u_2(t)\right)_{\Gamma_0} \\ &\quad = -\left(b^n\left(w_1(t)\right) - b^n\left(w_2(t)\right), u_1(t) - u_2(t)\right),\end{aligned} \tag{3.29}$$

and so, by (3.18), (2.5), (Hb), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|u_1(t) - u_2(t)\right|^2 + \left(\omega_1 + c_0\right)\left\|u_1(t) - u_2(t)\right\|^2 \\ &\quad \le \omega_2\left|u_1(t) - u_2(t)\right|^2 + L(r)\left\|w_1(t) - w_2(t)\right\|\left|u_1(t) - u_2(t)\right|.\end{aligned} \tag{3.30}$$

Putting

$$G(t) = L(r)\left\|w_1(t) - w_2(t)\right\|, \qquad H(t) = G(t)\left|u_1(t) - u_2(t)\right| \tag{3.31}$$

and integrating (3.30) over $(0, t)$, this yields

$$\frac{1}{2}\left|u_1(t) - u(t)\right|^2 + \left(\omega_1 + c_0\right)\int_0^t \left\|u_1(s) - u_2(s)\right\|^2 ds \le \omega_2 \int_0^t \left|u_1(s) - u_2(s)\right|^2 ds + \int_0^t H(s) ds. \tag{3.32}$$

From (3.32) it follows that

$$\begin{aligned} &\frac{d}{dt}\left\{e^{-2\omega_2 t}\int_0^t \left|u_1(s)-u_2(s)\right|^2 ds\right\} \\ &\quad = 2e^{-2\omega_2 t}\left\{\frac{1}{2}\left|u_1(t)-u_2(t)\right|^2 - \omega_2\int_0^t \left|u_1(s)-u_2(s)\right|^2 ds\right\} \\ &\quad \le 2e^{-2\omega_2 t}\int_0^t H(s)ds. \end{aligned} \tag{3.33}$$

Integrating (3.33) over $(0,t)$ we have

$$\begin{aligned} e^{-2\omega_2 t}\int_0^t \left|u_1(s)-u_2(s)\right|^2 ds &\le 2\int_0^t e^{-2\omega_2\tau}\int_0^\tau H(s)ds\,d\tau \\ &= 2\int_0^t\int_s^t e^{-2\omega_2\tau}d\tau H(s)ds = 2\int_0^t \frac{e^{-2\omega_2 s}-e^{-2\omega_2 t}}{2\omega_2}H(s)ds \\ &= \frac{1}{\omega_2}\int_0^t \left(e^{-2\omega_2 s}-e^{-2\omega_2 t}\right)H(s)ds, \end{aligned} \tag{3.34}$$

thus, we get

$$\omega_2\int_0^t \left|u_1(s)-u_2(s)\right|^2 ds \le \int_0^t \left(e^{2\omega_2(t-s)}-1\right)H(s)ds. \tag{3.35}$$

From (3.32) and (3.35) it follows that

$$\begin{aligned} &\frac{1}{2}\left|u_1(t)-u_2(t)\right|^2 + (\omega_1+c_0)\int_0^t \left\|u_1(s)-u_2(s)\right\|^2 ds \\ &\qquad \le \int_0^t e^{2\omega_2(t-s)}H(s)ds \\ &\qquad = \int_0^t e^{2\omega_2(t-s)}G(s)\left|u_1(s)-u_2(s)\right|ds, \end{aligned} \tag{3.36}$$

which implies

$$\begin{aligned} &\frac{1}{2}\left(e^{-2\omega_2 t}\left|u_1(t)-u_2(t)\right|\right)^2 + (\omega_1+c_0)e^{-2\omega_2 t}\int_0^t \left\|u_1(s)-u_2(s)\right\|^2 ds \\ &\quad \le \int_0^t e^{-\omega_2 s}G(s)e^{-\omega_2 s}\left|u_1(s)-u_2(s)\right|ds. \end{aligned} \tag{3.37}$$

By using Lemma 3.1, we obtain that

$$e^{-\omega_2 t}\left|u_1(t)-u_2(t)\right| \le \int_0^t e^{-\omega_2 s}G(s)ds. \tag{3.38}$$

The proof of lemma is complete.

From (3.26) and (3.36) it follows that

$$\begin{aligned}
&\frac{1}{2}\left|u_1(t)-u_2(t)\right|^2+\left(\omega_1+c_0\right)\int_0^t \left\|u_1(s)-u_2(s)\right\|^2 ds \\
&\quad\le \int_0^t e^{2\omega_2(t-s)}G(s)\int_0^s e^{\omega_2(s-\tau)}G(\tau)d\tau\, ds \\
&\quad= e^{2\omega_2 t}\int_0^t e^{-\omega_2 s}G(s)\int_0^s e^{-\omega_2\tau}G(\tau)d\tau\, ds \\
&\quad= e^{2\omega_2 t}\int_0^t \frac{1}{2}\frac{d}{ds}\left\{\int_0^s e^{-\omega_2\tau}G(\tau)d\tau\right\}^2 ds \\
&\quad= \frac{1}{2}e^{2\omega_2 t}\left\{\int_0^t e^{-\omega_2\tau}G(\tau)d\tau\right\}^2 \\
&\quad\le \frac{1}{2}e^{2\omega_2 t}\int_0^t e^{-2\omega_2\tau}d\tau\int_0^t G(\tau)^2 d\tau \\
&\quad= \frac{1}{2}e^{2\omega_2 t}\frac{1-e^{-2\omega_2 t}}{2\omega_2}\int_0^t G(\tau)^2 d\tau \\
&\quad= \frac{L(r)^2}{4\omega_2}\left(e^{2\omega_2 t}-1\right)\int_0^t \left\|w_1(s)-w_2(s)\right\|^2 ds.
\end{aligned} \tag{3.39}$$

Starting from the initial value $u_0(t)=u_0$, consider a sequence $\{u_n(\cdot)\}$ satisfying

$$\begin{gathered}
\dot{u}_{n+1}(t)+Au_{n+1}(t)-\operatorname{div}C\left[\varepsilon\left(u_{n+1}(t)\right)\right]=-b^n\left(u_n(t)\right)+f(t), \quad t\ge 0 \\
u_{n+1}=0 \quad \text{on } \Gamma_1\times(0,\infty) \\
C\left[\varepsilon\left(u_{n+1}(x,t)\right)\right]\nu=-(\beta\cdot\nu)\dot{u}_{n+1}(x,t), \quad \text{on } \Gamma_0\times(0,\infty) \\
u_{n+1}(0)=u_0.
\end{gathered} \tag{3.40}$$

Then from (3.39) it follows that

$$\begin{aligned}
&\frac{1}{2}\left|u_{n+1}(t)-u_n(t)\right|^2+\left(\omega_1+c_0\right)\int_0^t \left\|u_{n+1}(s)-u_n(s)\right\|^2 ds \\
&\quad\le \frac{L(r)^2}{4\omega_2}\left(e^{2\omega_2 t}-1\right)\int_0^t \left\|u_n(s)-u_{n-1}(s)\right\|^2 ds.
\end{aligned} \tag{3.41}$$

So by virtue of the condition (3.25) the contraction principle gives that there exists $u(\cdot) \in L^2(0,T_0;V)$ such that

$$u_n(\cdot) \longrightarrow u(\cdot) \quad \text{in } L^2(0,T_0;V), \tag{3.42}$$

and hence, from (3.41) there exists $u(\cdot) \in C([0,T_0];H)$ such that

$$u_n(\cdot) \longrightarrow u(\cdot) \quad \text{in } C([0,T_0];H). \tag{3.43}$$

□

Now, we give a norm estimation of the solution (HIE) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.5. *Let the assumption (Hb) be satisfied. Assume that $u_0 \in H$ and $f \in L^2(0,T;V^*)$ for any $T > 0$. Then, the solution u of* (HIE) *exists and is unique in*

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H). \tag{3.44}$$

Furthermore, there exists a constant C_2 depending on T such that

$$\|u\|_{L^2\cap W^{1,2}} \le C_2\left(1 + |u_0| + \|f\|_{L^2(0,T;V^*)}\right). \tag{3.45}$$

Proof. Let $w \in B_r$ be the solution of

$$\begin{gathered} \dot{w}(t) + Aw(t) - \operatorname{div} C\left[\varepsilon(w(t))\right] = f(t), \quad t \ge 0, \\ w = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ C\left[\varepsilon(w(x,t))\right]\nu = -(\beta\cdot\nu)w(x,t) \quad \text{on } \Gamma_0 \times (0,\infty), \\ w(0) = u_0. \end{gathered} \tag{3.46}$$

Then, since

$$\frac{d}{dt}(u(t) - w(t)) + A(u(t) - w(t)) - \operatorname{div} C\left[\varepsilon(u(t))\right] + \operatorname{div} C\left[\varepsilon(w(t))\right] = -b^n(u(t)), \tag{3.47}$$

by multiplying by $u(t) - w(t)$, from (Hb), (3.18) and the monotonicity of A, we obtain

$$\frac{1}{2}\frac{d}{dt}\left|u(t) - w(t)\right|^2 + (\omega_1 + c_0)\left\|u(t) - w(t)\right\|^2 \le \omega_2\left|u(t) - w(t)\right|^2 + \tilde{\mu}\left\|u(t)\right\|\left|u(t) - w(t)\right|. \tag{3.48}$$

By integrating on (3.48) over $(0,t)$ we have

$$\begin{aligned}&\frac{1}{2}|u(t)-w(t)|^2+(\omega_1+c_0)\int_0^t\|u(s)-w(s)\|^2ds\\&\quad\le\omega_2\int_0^t|u(s)-w(s)|^2ds+\tilde{\mu}\int_0^t\|u(s)\||u(s)-w(s)|ds.\end{aligned}\tag{3.49}$$

By the procedure similar to (3.39) we have

$$\frac{1}{2}|u(t)-w(t)|+(\omega_1+c_0)\int_0^t\|u(s)-w(s)\|^2ds\le\frac{\tilde{\mu}^2}{4\omega_2}\left(e^{2\omega_2t}-1\right)\int_0^t\|u(s)\|^2ds.\tag{3.50}$$

Put

$$M=\frac{\tilde{\mu}^2}{4\omega_2(\omega_1+c_0)}\left(e^{2\omega_2t}-1\right).\tag{3.51}$$

Then it holds

$$\|u-w\|_{L^2(0,T_0;V)}\le M^{1/2}\|u\|_{L^2(0,T_0;V)}\tag{3.52}$$

and hence, from (2.16) in Proposition 2.1, we have that

$$\begin{aligned}\|u\|_{L^2(0,T_0;V)}&\le\frac{1}{1-M^{1/2}}\|w\|_{L^2(0,T_0;V)}\\&\le\frac{C_0}{1-M^{1/2}}\left(1+|u_0|+\|f\|_{L^2(0,T_0;V^*)}\right)\\&\le C_2\left(1+|u_0|+\|f\|_{L^2(0,T_0;V^*)}\right)\end{aligned}\tag{3.53}$$

for some positive constant C_2. Noting that by (Hb)

$$\left\|b^n(u)\right\|_{L^2(0,T;H)}\le\text{const}\cdot\|u\|_{L^2(0,T;V)}\tag{3.54}$$

and by Proposition 2.1

$$\|u\|_{W^{1,2}(0,T;V^*)}\le C_1\left\{1+|u_0|+\left\|b^n(u)+f\right\|_{L^2(0,T;V^*)}\right\},\tag{3.55}$$

it is easy to obtain the norm estimate of u in $W^{1,2}(0,T_0;V^*)$ satisfying (3.45).

Now from Theorem 3.3 it follows that

$$\left|u(T_0)\right| \le \|u\|_{C\left([0,T_0],H\right)} \le C_2\left(1 + \left|u_0\right| + \|f\|_{L^2(0,T_0;V^*)}\right). \tag{3.56}$$

So, we can solve the equation in $[T_0, 2T_0]$ and obtain an analogous estimate to (3.53). Since the condition (3.25) is independent of initial values, the solution of (HIE-1) can be extended the internal $[0, nT_0]$ for a natural number n, that is, for the initial $u(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (3.53) holds for the solution in $[0, (n+1)T_0]$. Furthermore, the estimate (3.45) is easily obtained from (3.53) and (3.56).

We show that (u, Ξ) is a solution of the problem (HIE). Lemma 3.4 and (Hb) give that

$$\left|b^n(u(t))\right| \le \tilde{\mu}\left|u(t)\right| \le c, \tag{3.57}$$

and for $u_0 \in H$, there exists a unique solution u of (HIE) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C\left([0,T];H\right) \tag{3.58}$$

and satisfying (3.44).

From (3.44)and (3.57), we can extract a subsequence from $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \longrightarrow u \text{ weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \tag{3.59}$$

$$b^n(u_n) \longrightarrow \Xi \text{ weakly in } L^2(0,T;H). \tag{3.60}$$

Here, we remark that if V is compactly embedded in H and $u \in L^2(0,T;V))$, the following embedding

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H) \tag{3.61}$$

is compact in view [13, Theorem 2]. Hence, the mapping

$$u_n \longrightarrow u \text{ strongly in } L^2(0,T;H). \tag{3.62}$$

By a solution of (HIE-1), we understand a mild solution that has a form

$$u_n(t) = S(t)u_0 + \int_0^t S(t-s)\left\{\operatorname{div} C\left[\varepsilon\left(u_n(t)\right)\right] - b^n\left(u_n(s)\right) + f(s)\right\}ds, \quad t \ge 0, \tag{3.63}$$

so letting $n \to \infty$ and using the convergence results above, we obtain

$$\dot{u}(t) + Au(t) - \operatorname{div} C\left[\varepsilon\left(u(t)\right)\right] + \Xi(t) = f(t), \quad 0 \le t \le T. \tag{3.64}$$

Now, we show that $\Xi(x,t) \in \varphi(u(x,t))$ a.e. in $Q := \Omega \times (0,T_0)$. Indeed, from (3.62) we have $u_{ni} \to u_i$ strongly in $L^2(0,T;L^2(\Omega))$ and hence $u_{ni}(x,t) \to u_i(x,t)$ a.e. in Q for each $i = 1,2,\dots,N$. Let $i \in \{1,2,\dots,N\}$ and $\eta > 0$. Using the theorem of Lusin and Egoroff, we can choose a subset $\omega \subset Q$ such that $|\omega| < \eta$, $u_i \in L^2(Q \setminus \omega)$ and $u_{ni} \to u_i$ uniformly on $Q \setminus \omega$. Thus, for each $\epsilon > 0$, there is an $M > 2/\epsilon$ such that

$$\left|u_{ni}(x,t) - u_i(x,t)\right| < \frac{\epsilon}{2} \quad \text{for } n > M \text{ and } (x,t) \in Q \setminus \omega. \tag{3.65}$$

Then, if $|u_{ni}(x,t) - s| < 1/n$, we have $|u_i(x,t) - s| < \epsilon$ for all $n > M$ and $(x,t) \in Q \setminus \omega$.

Therefore we have

$$\underline{b_i^\epsilon}\left(u_i(x,t)\right) \le b_i^n\left(u_{ni}(x,t)\right) \le \overline{b_i^\epsilon}\left(u_i(x,t)\right), \quad \forall n > M,\ (x,t) \in Q \setminus \omega. \tag{3.66}$$

Let $\phi \in L^2(0,T;L^2(\Omega))$, $\phi \ge 0$. Then

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b_i}^\epsilon(u_i(x,t))\phi(x,t)dx\,dt &\le \int_{Q\setminus\omega} b_i^n\left(u_{ni}(x,t)\right)\phi(x,t)dx\,dt \\ &\le \int_{Q\setminus\omega} \overline{b_i}^\epsilon(u_i(x,t))\phi(x,t)dx\,dt. \end{aligned} \tag{3.67}$$

Letting $n \to \infty$ in this inequality and using (3.60), we obtain

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b_i}^\epsilon(u_i(x,t))\phi(x,t)dx\,dt &\le \int_{Q\setminus\omega} \Xi_i(x,t)\phi(x,t)dx\,dt \\ &\le \int_{Q\setminus\omega} \overline{b_i}^\epsilon(u_i(x,t))\phi(x,t)dx\,dt, \end{aligned} \tag{3.68}$$

where $\Xi = (\Xi_1,\dots,\Xi_N)$. Letting $\epsilon \to 0^+$ in this inequality, we deduce that

$$\Xi_i(x,t) \in \varphi_i\left(u_i(x,t)\right) \quad \text{a.e. in } Q \setminus \omega, \tag{3.69}$$

and letting $\eta \to 0^+$ we get

$$\Xi_i(x,t) \in \varphi_i\left(u_i(x,t)\right) \quad \text{a.e. in } Q. \tag{3.70}$$

This implies that $\Xi(x,t) \in \varphi(u(x,t))$ a.e. in Q. This completes the proof of theorem. □

Remark 3.6. In terms of Proposition 2.1, we remark that if $u_0 \in V = (D(A),H)_{1/2,2}$ and $f \in L^2(0,T;H)$ for any $T > 0$ then the solution u of (HIE) exists and is unique in

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C\left([0,T];V\right). \tag{3.71}$$

Futhermore, there exists a constant C_2 depending on T such that

$$\|u\|_{L^2\cap W^{1,2}} \leq C_2\left(1+\|u_0\|+\|f\|_{L^2(0,T;H)}\right). \tag{3.72}$$

Theorem 3.7. *Let the assumption (Hb) be satisfied*

(1) *if* $(u_0, f) \in V \times L^2(0,T;H)$, *then the solution* u *of* (HIE) *belongs to* $u \in L^2(0,T;D(A)) \cap C([0,T];V)$ *and the mapping*

$$H \times L^2(0,T;H) \ni (u_0, f) \longmapsto u \in L^2(0,T;D(A)) \cap C([0,T];V) \tag{3.73}$$

is continuous.

(2) *let* $(u_0, f) \in H \times L^2(0,T;V^*)$. *Then the solution* u *of* (HIE) *belongs to* $u \in L^2(0,T;V) \cap C([0,T];H)$ *and the mapping*

$$H \times L^2(0,T;V^*) \ni (u_0, f) \longmapsto u \in L^2(0,T;V) \cap C([0,T];H) \tag{3.74}$$

is continuous.

Proof. (1) It is easy to show that if $u_0 \in V$ and $f \in L^2(0,T;H)$, then u belongs to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. let $(u_{0i}, f_i) \in V \times L^2(0,T;H)$ and $u_i \in B_r$ be the solution of (HIE) with (u_{0n}, f_i) in place of (u_0, f) for $i = 1, 2$. Then in view of Proposition 2.1, we have

$$\begin{aligned} &\|u_1 - u_2\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\ &\quad\leq C_1\left\{\|u_{01}-u_{02}\| + \|b^n(u_1)-b^n(u_2)\|_{L^2(0,T;H)} + \|f_1-f_2\|_{L^2(0,T;H)}\right\} \\ &\quad\leq C_1\left\{\|u_{01}-u_{02}\| + L(r)\|u_1-u_2\|_{L^2(0,T:V)} + \|f_1-f_2\|_{L^2(0,T;H)}\right\}. \end{aligned} \tag{3.75}$$

Since

$$u_1(t) - u_2(t) = u_{01} - u_{02} + \int_0^t (\dot{u}_1(s) - \dot{u}_2(s))\,ds, \tag{3.76}$$

we get

$$\|u_1 - u_2\|_{L^2(0,T;H)} \leq \sqrt{T}|u_{01} - u_{02}| + \frac{T}{\sqrt{2}}\|u_1 - u_2\|_{W^{1,2}(0,T;H)}. \tag{3.77}$$

Hence, arguing as in (2.8), we get

$$\begin{aligned}\|u_1-u_2\|_{L^2(0,T;V)} &\le C_0\|u_1-u_2\|^{1/2}_{L^2(0,T;D(A))}\|u_1-u_2\|^{1/2}_{L^2(0,T;H)}\\ &\le C_0\|u_1-u_2\|^{1/2}_{L^2(0,T;D(A))}\left\{T^{1/4}|u_{01}-u_{02}|^{1/2}+\left(\frac{T}{\sqrt{2}}\right)^{1/2}\|u_1-u_2\|^{1/2}_{W^{1,2}(0,T;H)}\right\}\\ &\le C_0T^{1/4}|u_{01}-u_{02}|^{1/2}\|u_1-u_2\|^{1/2}_{L^2(0,T;D(A_0))}\\ &\quad + C_0\left(\frac{T}{\sqrt{2}}\right)^{1/2}\|u_1-u_2\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}\\ &\le 2^{-7/4}C_0|u_{01}-u_{02}|+2C_0\left(\frac{T}{\sqrt{2}}\right)^{1/2}\|u_1-u_2\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.\end{aligned} \tag{3.78}$$

Combining (3.75) with (3.78), we obtain

$$\begin{aligned}&\|u_1-u_2\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}\\ &\quad\le C_1\{\|u_{01}-u_{02}\|+\|f_1-f_2\|_{L^2(0,T;H)}\}+2^{-7/4}C_0C_1\mu|u_{01}-u_{02}|\\ &\qquad+2C_0C_1\left(\frac{T}{\sqrt{2}}\right)^{1/2}L(r)\|u_1-u_2\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.\end{aligned} \tag{3.79}$$

Suppose that $(u_{0n},f_n)\to(u_0,f)$ in $V\times L^2(0,T;H)$ and let u_n and u be the solutions (HIE) with (u_{0n},f_n) and (u_0,f), respectively. Let $0<T_1\le T$ be such that

$$2C_0C_1(T_1/\sqrt{2})^{1/2}L(r)<1. \tag{3.80}$$

Then by virtue of (3.79) with T replaced by T_1 we see that

$$u_n\longrightarrow u\quad \text{in } L^2(0,T_1;D(A_0))\cap W^{1,2}(0,T_1;H). \tag{3.81}$$

This implies that $u_n(T_1)\mapsto u(T_1)$ in V. Hence the same argument shows that $u_n\to u$ in

$$L^2(T_1,\min\{2T_1,T\};\ D(A_0))\cap W^{1,2}(T_1,\min\{2T_1,T\};H). \tag{3.82}$$

Repeating this process we conclude that $u_n\to u$ in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$.

(2) If $(u_0, f) \in H \times L^2(0,T;H)$ then u belongs to $L^2(0,T;V) \cap C([0,T];H)$ from Theorem 3.5. Let $(u_{0i}, f_i) \in H \times L^2(0,T;H)$ and $u_i \in B_r$ be the solution of (HIE) with (u_{0i}, f_i) in place of (u_0, f) for $i = 1, 2$. Multiplying (HIE) by $u_1(t) - u_2(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|u_1(t) - u_2(t)\right|^2 + (\omega_1 + c_0)\left\|u_1(t) - u_2(t)\right\|^2 \\ &\quad \leq \omega_2\left|u_1(t) - u_2(t)\right|^2 + \left|b^n(u_1(t)) - b^n(u_2(t))\right|\left|u_1(t) - u_2(t)\right| \\ &\qquad + \left|f_1(t) - f_2(t)\right|\left|u_1(t) - u_2(t)\right|. \end{aligned} \tag{3.83}$$

Put

$$G(t) = L(r)\left\|u_1(t) - u_2(t)\right\| + \left|f_1(t) - f_2(t)\right|. \tag{3.84}$$

Then, by the similar argument in (3.32), we get

$$\begin{aligned} &\frac{1}{2}\left|u_1(t) - u_2(t)\right|^2 + (\omega_1 + c_0)\int_0^t \left\|u_1(s) - u_2(s)\right\|^2 ds \\ &\quad \leq \frac{1}{2}\left|u_{01} - u_{02}\right|^2 + \omega_2\int_0^t \left|u_1(s) - u_2(s)\right|^2 ds + \int_0^t G(s)\left|u_1(s) - u_2(s)\right| ds \end{aligned} \tag{3.85}$$

and we have that

$$\frac{d}{dt}\left\{e^{-2\omega_2 t}\int_0^t \left|u_1(s) - u_2(s)\right|^2 ds\right\} \leq 2e^{-2\omega_2 t}\left\{\frac{1}{2}\left|u_{01} - u_{02}\right|^2 + \int_0^t G(s)\left|u_1(s) - u_2(s)\right| ds\right\}, \tag{3.86}$$

thus, arguing as in (3.35) we have

$$\begin{aligned} &\omega_2\int_0^t \left|u_1(s) - u_2(s)\right|^2 ds \\ &\quad \leq \frac{1}{2}\left(e^{2\omega_2 t} - 1\right)\left|u_{01} - u_{02}\right|^2 + \int_0^t \left(e^{2\omega_2(t-s)} - 1\right)G(s)\left|u_1(s) - u_2(s)\right| ds. \end{aligned} \tag{3.87}$$

Combining this inequality with (3.85) it holds that

$$\begin{aligned} &\frac{1}{2}\left|u_1(t) - x_2(t)\right|^2 + (\omega_1 + c_0)\int_0^t \left\|u_1(s) - u_2(s)\right\|^2 ds \\ &\quad \leq \frac{1}{2}e^{2\omega_2 t}\left|u_{01} - u_{02}\right|^2 + \int_0^t e^{2\omega_2(t-s)}G(s)\left|u_1(s) - u_2(s)\right| ds. \end{aligned} \tag{3.88}$$

By Lemma 3.1 the following inequality

$$\frac{1}{2}\left(e^{-2\omega_2 t}\left|u_1(t)-u_2(t)\right|\right)^2+(\omega_1+c_0)e^{-2\omega_2 t}\int_0^t\left\|u_1(s)-u_2(s)\right\|^2 ds \le \frac{1}{2}\left|u_{01}-u_{02}\right|^2+\int_0^t e^{-\omega_2 s}G(s)e^{-\omega_2 s}\left|u_1(s)-u_2(s)\right|ds \tag{3.89}$$

implies that

$$e^{-\omega_2 t}\left|u_1(t)-u_2(t)\right| \le \left|u_{01}-u_{02}\right|+\int_0^t e^{-\omega_2 s}G(s)ds. \tag{3.90}$$

Hence, from (3.88) and (3.90) it follows that

$$\begin{aligned}\frac{1}{2}\left|u_1(t)-u_2(t)\right|^2+(\omega_1+c_0)\int_0^t\left\|u_1(s)-u_2(s)\right\|^2 ds &\le \frac{1}{2}e^{2\omega_2 t}\left|u_{01}-u_{02}\right|^2\\ &+\int_0^t e^{2\omega_2(t-s)}G(s)e^{\omega_2 s}\left(\left|u_{01}-u_{02}\right|+\int_0^s e^{\omega_2(s-\tau)}G(\tau)ds\right)\\ &\le \frac{1}{2}e^{2\omega_2 t}\left|u_{01}-u_{02}\right|^2+\left|u_{01}-u_{02}\right|e^{2\omega_2 t}\int_0^t G(s)ds+\frac{L(r)^2}{4\omega_2}\left(e^{2\omega_2 t}-1\right)\int_0^t G(s)^2 ds.\end{aligned} \tag{3.91}$$

The last term of (3.91) is estimated as

$$\frac{L(r)^2\left(e^{2\omega_2 t}-1\right)}{4\omega_2}\int_0^t 2\left(L^2\left\|x_1(s)-x_2(s)\right\|^2+\left|k_1(s)-k_2(s)\right|^2\right)ds. \tag{3.92}$$

Let $T_2 < T$ be such that

$$\omega_1+c_0-\frac{L(r)^2}{2\omega_2}\left(e^{2\omega_2 T_2}-1\right)>0. \tag{3.93}$$

Hence, from (3.91) and (3.92) it follows that there exists a constant $C > 0$ such that

$$\left|u_1(T_2)-u_2(T_2)\right|^2+\int_0^{T_2}\left\|u_1(s)-u_2(s)\right\|^2 ds \le C\left(\left|u_{01}-u_{02}\right|^2+\int_0^{T_2}\left|f_1(s)-f_2(s)\right|^2 ds\right). \tag{3.94}$$

Suppose $(u_{0n}, u_n) \to (u_0, f)$ in $H \times L^2(0, T_2; V^*)$, and let u_n and u be the solutions (HIE) with (u_{0n}, f_n) and (u_0, f), respectively. Then, by virtue of (3.94), we see that $u_n \to u$

in $L^2(0,T_2,V) \cap C([0,T_2];H)$. This implies that $u_n(T_2) \rightarrow u(T_2)$ in H. Therefore the same argument shows that $u_n \rightarrow u$ in

$$L^2(T_2, \min\{2T_2, T\}; V) \cap C\big(\big[T_2, \min\{2T_2, T\}\big]; H\big). \tag{3.95}$$

Repeating this process, we conclude that $u_n \rightarrow u$ in $L^2(0,T;V) \cap C([0,T];H)$. □

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References

[1] M. Miettinen, "A parabolic hemivariational inequality," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 4, pp. 725–734, 1996.

[2] M. Miettinen and P. D. Panagiotopoulos, "On parabolic hemivariational inequalities and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 7, pp. 885–915, 1999.

[3] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functions*, Birkhäuser, Boston, Mass, USA, 1985.

[4] P. D. Panagiotopoulos, "Modelling of nonconvex nonsmooth energy problems. Dynamic hemivariational inequalities with impact effects," *Journal of Computational and Applied Mathematics*, vol. 63, no. 1–3, pp. 123–138, 1995.

[5] J. Y. Park, H. M. Kim, and S. H. Park, "On weak solutions for hyperbolic differential inclusion with discontinuous nonlinearities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 1-2, pp. 103–113, 2003.

[6] J. Y. Park and S. H. Park, "On solutions for a hyperbolic system with differential inclusion and memory source term on the boundary," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 57, no. 3, pp. 459–472, 2004.

[7] S. Migórski and A. Ochal, "Vanishing viscosity for hemivariational inequalities modeling dynamic problems in elasticity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 8, pp. 1840–1852, 2007.

[8] J. Rauch, "Discontinuous semilinear differential equations and multiple valued maps," *Proceedings of the American Mathematical Society*, vol. 64, no. 2, pp. 277–282, 1977.

[9] G. Di Blasio, K. Kunisch, and E. Sinestrari, "L^2-regularity for parabolic partial integro-differential equations with delay in the highest-order derivatives," *Journal of Mathematical Analysis and Applications*, vol. 102, no. 1, pp. 38–57, 1984.

[10] N. U. Ahmed, *Optimization and Identification of Systems Governed by Evolution Equations on Banach Space*, vol. 184 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, UK, 1988.

[11] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Nordhoff, Leiden, The Netherlands, 1976.

[12] H. Brézis, *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans un Espace de Hilbert*, North Holland, Amsterdam, The Netherlands, 1973.

[13] J.-P. Aubin, "Un théorème de compacité," *Comptes Rendus de l'Académie des Sciences*, vol. 256, pp. 5042–5044, 1963.