Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 161405, 22 pages
doi:10.1155/2009/161405

Research Article

A Kind of Estimate of Difference Norms in Anisotropic Weighted Sobolev-Lorentz Spaces

Jiecheng Chen[1] and Hongliang Li[1, 2]

[1] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[2] *Department of Mathematics, Zhejiang Education Institute, Hangzhou 310012, China*

Correspondence should be addressed to Hongliang Li, honglli@126.com

Received 27 April 2009; Accepted 2 July 2009

Recommended by Shusen Ding

We investigate the functions spaces on $\mathbb{R}^n$ for which the generalized partial derivatives $D_k^{r_k} f$ exist and belong to different Lorentz spaces $\Lambda^{p_k,s_k}(w)$, where $p_k > 1$ and w is nonincreasing and satisfies some special conditions. For the functions in these weighted Sobolev-Lorentz spaces, the estimates of the Besov type norms are found. The methods used in the paper are based on some estimates of nonincreasing rearrangements and the application of B_p, $B_{p,\infty}$ weights.

1. Introduction

In this paper we study functions f on $\mathbb{R}^n$ which possess the generalized partial derivatives

$$D_k^{r_k} f \equiv \frac{\partial^{r_k} f}{\partial x_k^{r_k}} \quad (r_k \in \mathbb{N}). \tag{1.1}$$

Our main goal is to obtain some norm estimates for the differences

$$\Delta_k^{r_k}(h) f(x) \equiv \sum_{j=0}^{r_k} (-1)^{r_k - j} \binom{r_k}{j} f(x + jhe_k) \quad (h \in \mathbb{R}) \tag{1.2}$$

(e_k being the unit coordinate vector).

The classic Sobolev embedding theorem asserts that for any function f in Sobolev space $W_p^1(\mathbb{R}^n)$ $(1 \le p < n)$

$$\|f\|_{q^*} \le C \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p, \quad q^* = \frac{np}{n-p}. \tag{1.3}$$

Sobolev proved this inequality in 1938 for $p > 1$. His method, based on integral representations, did not work in the case $p = 1$. Only at the end of fifties Gagliardo and Nirenberg gave simple proofs of inequality (1.3) for all $1 \le p < n$. Inequality (1.3) has been generalized in various directions (see [1–6] for details). It was proved that the left hand side in (1.3) can be replaced by the stronger Lorentz norm, that is, there holds the inequality

$$\|f\|_{q^*,p} \le C \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p, \quad 1 \le p < n. \tag{1.4}$$

For $p > 1$ the result follows by interpolation (see [7, 8]). In the case $p = 1$ some geometric inequalities were applied to prove (1.4) (see [9–13]).

The sharp estimates of the norms of differences for the functions in Sobolev spaces have firstly been proved by Besov et al. [1, Volume 2, page 72]. For the space $W_p^1(\mathbb{R}^n)$ $(1 \le p < n)$ Il'in's result reads as follows: If $n \in \mathbb{N}$, $1 < p < q < \infty$ and $\alpha \equiv 1 - n(1/p - 1/q) > 0$, then

$$\sum_{k=1}^{n} \left(\int_0^\infty \left[h^{-\alpha} \|\Delta_k^1(h) f\|_q \right]^p \frac{dh}{h} \right)^{1/p} \le C \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p. \tag{1.5}$$

Actually, this means that there holds the continuous embedding to the Besov space

$$W_p^1(\mathbb{R}^n) \hookrightarrow B_{p,q}^{\alpha}(\mathbb{R}^n). \tag{1.6}$$

It is easy to see that inequality (1.5) fails to hold for $p = n = 1$, but, it was proved in [14] that (1.5) is true for $p = 1$ and $n \ge 2$.

The generalization of the inequality (1.5) to the spaces $W_p^{r_1,\dots,r_n}$ was given in [12]. That is

$$\sum_{k=1}^{n} \left(\int_0^\infty \left[h^{-\alpha_k} \|\Delta_k^{r_k}(h) f\|_{q,p} \right]^p \frac{dh}{h} \right)^{1/p} \le C \sum_{k=1}^{n} \|D_k^{r_k} f\|_p, \tag{1.7}$$

where $0 < 1/p - 1/q < r/n$, $r = n(\sum_{i=1}^{n} r_i^{-1})^{-1}$, and $\alpha_k = r_k[1 - (r/n)(1/p - 1/q)]$; the inequality is valid if $p > 1$, $n \ge 1$ or $p = 1$, $n \ge 2$. Using (1.7), we get the following continuous embedding:

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,p}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n). \tag{1.8}$$

For $p > 1$ this embedding was proved by Besov et al. [1, Volume 2, page 72]. The main result in [12] is the proof of (1.7) for $p = 1$, $n \ge 2$.

In [15], there was the sharp estimates of the type (1.7) when the derivatives $D_k^{r_k} f$ belong to different Lorentz spaces L^{p_k,s_k}. Before stating the theorem, we give some notations. Let $S_0(\mathbb{R}^n)$ be the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that for each $y > 0$,

$$\lambda_f(y) = \left|\left\{x \in \mathbb{R}^n : \left|f(x)\right| > y\right\}\right| < \infty. \tag{1.9}$$

Let $r_k \in \mathbb{N}$ and $1 \leq p_k,\ s_k < \infty$ for $k = 1, \ldots, n$ $(n \geq 2)$. Denote

$$r = n\left(\sum_{k=1}^{n} \frac{1}{r_k}\right)^{-1}, \qquad p = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{p_k r_k}\right)^{-1}, \qquad s = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{s_k r_k}\right)^{-1}. \tag{1.10}$$

Now we state the main theorem in [15].

Theorem 1.1. *Let $n \geq 2$, $r_k \in \mathbb{N}$, $1 \leq p_k$, $s_k < \infty$, and $s_k = 1$ if $p_k = 1$. Let r, p, and s be the numbers defined by* (1.10). *For every p_j $(1 \leq j \leq n)$ satisfying the condition*

$$\rho_j \equiv \frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} > 0, \tag{1.11}$$

take arbitrary $q_j > p_j$ such that

$$\frac{1}{q_j} > \frac{1}{p} - \frac{r}{n}, \tag{1.12}$$

and denote

$$H_j = 1 - \frac{1}{\rho_j}\left(\frac{1}{p_j} - \frac{1}{q_j}\right), \qquad \alpha_j = H_j r_j, \qquad \frac{1}{\theta_j} = \frac{1 - H_j}{s} + \frac{H_j}{s_j}, \tag{1.13}$$

then for any function $f \in S_0(\mathbb{R}^n)$ which has the weak derivatives $D_k^{r_k} f \in L^{p_k,s_k}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$ there holds the inequality

$$\left(\int_0^\infty \left[h^{-\alpha_j}\left\|\Delta_j^{r_j}(h) f\right\|_{q_j,1}\right]^\theta \frac{dh}{h}\right)^{1/\theta_j} \leq C \sum_{k=1}^{n} \|D_k^{r_k} f\|_{p_k,s_k}, \tag{1.14}$$

where C is a constant that does not depend on f.

In many cases, the Lorentz space should be substituted by more general space, the weighted Lorentz space. In this paper, we will generalize the above result when the weighted Lorentz spaces $\Lambda^{p_k,s_k}(w)$ take place of L^{p_k,s_k}, where w is a weight on $\mathbb{R}_+$ which satisfies some special conditions.

2. Auxiliary Proposition

Let $\mathcal{M}(X,\mu)$ be the class of all measurable and almost everywhere finite functions on X. For $f \in \mathcal{M}(X,\mu)$, a nonincreasing rearrangement of f is a nonincreasing function f^* on $\mathbb{R}_+ \equiv (0,+\infty)$, that is, equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}, \quad 0 < t < \infty, \tag{2.1}$$

where

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\}, \quad \lambda \ge 0. \tag{2.2}$$

If $X = \mathbb{R}^n$, $\mu(E) = |E|$, then the following relation holds [16, Chapter 2]:

$$\sup_{|E|=t} \int_E |f(x)| dx = \int_0^t f^*(u) du. \tag{2.3}$$

Set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s) ds. \tag{2.4}$$

Assume that $0 < q$, $p < \infty$. A function $f \in \mathcal{M}(X,\mu)$ belongs to the Lorentz space $L^{q,p}(X)$ if

$$\|f\|_{q,p} = \left(\int_0^\infty \left(t^{1/q} f^*(t)\right)^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{2.5}$$

For $0 < p < \infty$, the space $L^{p,\infty}(X)$ is defined as the class of all $f \in \mathcal{M}(X,\mu)$ such that

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty. \tag{2.6}$$

We also let $L^{\infty,\infty}(X) = L^\infty(X)$. Let w be a weight in $\mathbb{R}_+$ (nonnegative locally integrable functions in $\mathbb{R}_+$).

If $(X,\mu) = (\mathbb{R}_+, w(t)dt)$, we replace $L^{q,p}(X)$ with $L^{q,p}(w)$. For $0 < p$, $q < \infty$, or $0 < p \le \infty$ and $q = \infty$, the weighted Lorentz space $\Lambda^{p,q}_{\mathbb{R}^n}(w) = \Lambda^{p,q}(w)$ is defined in [9, Chapter 2] by

$$\Lambda^{p,q}(w) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{\Lambda^{p,q}(w)} = \|f^*\|_{L^{p,q}(w)} < \infty \right\}. \tag{2.7}$$

If $p = q$, denote $\Lambda^p(w) = \Lambda^{p,p}(w)$. It is well known that

$$\Lambda^{p,q}(1) = L^{p,q}(\mathbb{R}^n), \tag{2.8}$$

and if $0 < p,\ q < \infty$, then

$$\Lambda^{p,q}(w) = \Lambda^{q}(\tilde{w}), \tag{2.9}$$

where

$$\tilde{w}(t) = W^{q/p-1}(t)w(t), \qquad W(t) = \int_0^t w(s)ds. \tag{2.10}$$

In following part of this paper, we will always denote $W(t) = \int_0^t w(s)ds$.

The weighted Lorentz spaces have close connection with weights of B_p, $B_{p,\infty}$ for $0 < p < \infty$ (see [9, Chapter 1]). Let A be the Hardy operator as follows:

$$Af(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t > 0. \tag{2.11}$$

The space L^p_{dec} is the cone of all nonnegative nonincreasing functions in L^p. We denote $w \in B_p$ if

$$A : L^p_{\mathrm{dec}}(w) \longrightarrow L^p(w) \tag{2.12}$$

is bounded and denote $w \in B_{p,\infty}$ if

$$A : L^p_{\mathrm{dec}}(w) \longrightarrow L^{p,\infty}(w) \tag{2.13}$$

is bounded.

Lemma 2.1 (Generalized Hardy's inequalities). *Let φ be nonnegative, measurable on $(0,\infty)$ and suppose $-\infty < \lambda < 1$, $1 \le q \le \infty$, and w is a weight in $\mathbb{R}_+$, $W(\infty) = \infty$, then one has*

$$\begin{aligned}
\left\{\int_0^\infty \left(W(t)^{\lambda}\frac{1}{W(t)}\int_0^t \varphi(s)w(s)ds\right)^q \frac{w(t)}{W(t)}dt\right\}^{1/q} &\le \frac{1}{1-\lambda}\left\{\int_0^\infty \left(W(t)^{\lambda}\varphi(t)\right)^q \frac{w(t)}{W(t)}dt\right\}^{1/q}, \\
\left\{\int_0^\infty \left(W(t)^{1-\lambda}\int_t^\infty \varphi(s)\frac{w(s)}{W(s)}ds\right)^q \frac{w(t)}{W(t)}dt\right\}^{1/q} &\le \frac{1}{1-\lambda}\left\{\int_0^\infty \left(W(t)^{1-\lambda}\varphi(t)\right)^q \frac{w(t)}{W(t)}dt\right\}^{1/q}
\end{aligned} \tag{2.14}$$

(with the obvious modification if $q = \infty$).

Proof. It is easy to obtain this result applying Hardy's inequality [16]. □

Lemma 2.2. *Let $\varphi \in \Lambda^{p,s}(w)$ $(1 \le p, s < \infty)$ be a nonnegative nonincreasing function on $\mathbb{R}_+$, w be a nonincreasing weight on $\mathbb{R}_+$ and there exists $A > 0$, such that*

$$W(\xi t) \ge \xi^A W(t), \quad \forall \xi > 1,\ \forall t > 0, \tag{2.15}$$

Then for $\delta > 0$ there exists a continuously differentiable ϕ on $\mathbb{R}_+$ such that

(i) $\psi(t) \leq C\phi(t)$, $t \in \mathbb{R}_+$,

(ii) $\phi(t)W(t)^{1/p-\delta}$ *decreases and* $\phi(t)W(t)^{1/p+\delta}$ *increases on* $\mathbb{R}_+$,

(iii) $\|\phi\|_{\Lambda^{p,s}(w)} \leq C\|\psi\|_{\Lambda^{p,s}(w)}$,

where C is a constant depends only on p, δ, and A.

Proof. Without loss of generality, we may suppose that $\delta < 1/p$. Set

$$\phi_1(t) = W(t)^{\delta-1/p}\int_{t/2}^{\infty}\psi(u)W(u)^{1/p-\delta}\frac{w(u)}{W(u)}du. \tag{2.16}$$

Then $\phi_1(t)W(t)^{1/p-\delta}$decreases and

$$\begin{aligned}\phi_1(t) &\geq W(t)^{\delta-1/p}\int_{t/2}^{t}\psi(u)W(u)^{1/p-\delta}\frac{w(u)}{W(u)}du\\ &\geq W(t)^{\delta-1/p}\psi(t)\frac{W(t)^{1/p-\delta}-W(t/2)^{1/p-\delta}}{1/p-\delta}.\end{aligned} \tag{2.17}$$

Using the conditions which w satisfy, it gives

$$\phi_1(t) \geq C\psi(t). \tag{2.18}$$

Furthermore, noticing w is nonincreasing and applying Lemma 2.1, we get that

$$\begin{aligned}\|\phi_1\|_{\Lambda^{p,s}(w)} &= \left\{2\int_0^{\infty}\left[W(2h)^{\delta}\int_h^{\infty}W(u)^{1/p-\delta}\psi(u)\frac{w(u)}{W(u)}du\right]^s\frac{w(2h)}{W(2h)}dh\right\}^{1/s}\\ &\leq 2^{1/s+\delta}\left\{\int_0^{\infty}\left[W(h)^{\delta}\int_h^{\infty}W(u)^{1/p-\delta}\psi(u)\frac{w(u)}{W(u)}du\right]^s\frac{w(h)}{W(h)}dh\right\}^{1/s}\\ &\leq C\left(\int_0^{\infty}\left(W(h)^{1/p}\psi(h)\right)^s\frac{w(h)}{W(h)}dh\right)^{1/s}\\ &= C\|\psi\|_{\Lambda^{p,s}(w)}.\end{aligned} \tag{2.19}$$

now set

$$\phi(t) = \left(\delta+\frac{1}{p}\right)W(t)^{-1/p-\delta}\int_0^{t}\phi_1(u)W(u)^{\delta+1/p}\frac{w(u)}{W(u)}du. \tag{2.20}$$

Then $\phi(t)W(t)^{1/p+\delta}$ increases on $\mathbb{R}_+$, and

$$\phi(t) \geq \phi_1(t) \geq C\psi(t). \tag{2.21}$$

Furthermore,

$$\phi(t)W(t)^{1/p-\delta} = \left(\delta + \frac{1}{p}\right)W(t)^{-2\delta}\int_0^t \phi_1(u)W(u)^{\delta+1/p}\frac{w(u)}{W(u)}du$$
$$= W(t)^{-2\delta}\int_0^t \phi_1(u)dW(u)^{\delta+1/p} \tag{2.22}$$
$$= W(t)^{-2\delta}\int_0^{W(t)^{2\delta}} \phi_1(h(v))v^{(1/p-\delta)/(2\delta)}dv,$$

where $v = W(u)^{2\delta}$, $h(v) = u$, that is, $h(v) = W^{-1}(v^{1/(2\delta)})$. Since $\phi_1(t)W(t)^{1/p-\delta}$ is decreasing function on $\mathbb{R}_+$, thus $\phi_1(h(v))v^{(1/p-\delta)/(2\delta)}$ is decreasing and $\phi(t)W(t)^{1/p-\delta}$ is also decreasing on $\mathbb{R}_+$.

Finally, using Lemma 2.1 and (2.19), we get (iii). The Lemma 2.2 is proved. □

Let $r_k \in \mathbb{N}$ and $1 < p_k < \infty$ for $k = 1, \dots, n$ $(n \geq 2)$. Denote

$$r = n\left(\sum_{j=1}^{n}\frac{1}{r_j}\right)^{-1}, \qquad p = \frac{n}{r}\left(\sum_{j=1}^{n}\frac{1}{p_j r_j}\right)^{-1},$$
$$\gamma_k = 1 - \frac{1}{r_k}\left(\frac{r}{n} + \frac{1}{p_k} - \frac{1}{p}\right). \tag{2.23}$$

Then $\gamma_k > 0$ and

$$\sum_{k=1}^{n}\gamma_k = n - 1. \tag{2.24}$$

To prove our main results we use the estimates of the rearrangement of a given function in term of its derivatives $D_k^{r_k} f$ $(k = 1, \dots, n)$.

We will use the notations (2.23).

Lemma 2.3. *Let $r_k \in \mathbb{N}$, $1 < p_k < \infty$, $1 \leq s_k < \infty$ for $k = 1, \dots, n$ $(n \geq 2)$ and w is continuous weight on $\mathbb{R}_+$. Set*

$$s = \frac{n}{r}\left(\sum_{j=1}^{n}\frac{1}{s_j r_j}\right)^{-1}. \tag{2.25}$$

Let

$$0 < \delta < \frac{1}{4}\min_{\gamma_j<1}(1 - \gamma_j), \tag{2.26}$$

and suppose that $\phi_k \in \Lambda^{p_k,s_k}(w)$ $(k = 1,\dots,n)$ *are positive continuously differentiable functions with* $\phi_k'(t) < 0$ *on* $\mathbb{R}_+$ *such that* $\phi_k(t)W(t)^{1/p_k-\delta}$ *decreases and* $\phi_k(t)W(t)^{1/p_k+\delta}$ *increases on* $\mathbb{R}_+$. *Set for* $u,t > 0$,

$$\eta_k(u,t) = \left(\frac{W(t)}{u}\right)^{r_k}\phi_k(t), \tag{2.27}$$

$$\sigma(t) = \sup\left\{\min_{1\le k\le n}\eta_k(u_k,t) : \prod_{k=1}^{n}u_k = W(t)^{n-1},\ u_k > 0\right\}. \tag{2.28}$$

Then

(i) *there holds the inequality*

$$\left(\int_0^\infty W(t)^{s(1/p-r/n)-1}\sigma(t)^s w(t)dt\right)^{1/s} \le C'\prod_{k=1}^{n}\|\phi_k\|_{\Lambda^{p_k,s_k}(w)}^{r/(nr_k)}; \tag{2.29}$$

(ii) *there exist continuously differentiable functions* $u_k(t)$ *on* $\mathbb{R}_+$ *such that*

$$\prod_{k=1}^{n}u_k(t) = W(t)^{n-1},$$
$$\sigma(t) = \eta_k(u_k(t),t) \quad (t\in\mathbb{R}_+,\ k = 1,\dots,n); \tag{2.30}$$

(iii) *for any* k *such that*

$$\frac{1}{p_k} > \frac{1}{p} - \frac{r}{n} \tag{2.31}$$

the function $u_k(t)W(t)^{\delta-1}$ *decreases on* $\mathbb{R}_+$.

Proof. The proof is similar to [15, Lemma 2.2]. All the argument holds true when we substitute the weight $w(t)$ in this lemma for $w(t) = 1$. □

The Lebesgue measure of a measurable set $A \subset \mathbb{R}^k$ will be denoted by $\text{mes}_k A$.

For any F_σ − set $E \subset \mathbb{R}^n$ denote by E^j the orthogonal projection of E onto the coordinate hyperplane $x_j = 0$. By the Loomis-Whitney inequality [17, Chapter 4]

$$(\text{mes}_n E)^{n-1} \le \prod_{j=1}^{n}\text{mes}_{n-1}E^j. \tag{2.32}$$

Let $f \in S_0(\mathbb{R}^n)$, $t > 0$, and let E_t be a set of type F_σ and measure t such that $|f(x)| \geq f^*(t)$ for all $x \in E_t$. Denote by $\lambda_j(t)$ the $(n-1)$-dimensional measure of the projection E_t^j $(j = 1, \ldots, n)$. By (2.32), we have that

$$\prod_{j=1}^{n} \lambda_j(t) \geq t^{n-1}. \tag{2.33}$$

Lemma 2.4. *Let $n \geq 2$, $r_k \in \mathbb{N}$ $(k = 1, \ldots, n)$, w be nonincreasing, and $w(t) \to a$ when $t \to \infty$ where $a > 0$. Function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in L_{\text{loc}}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$. Then for all $0 < t < \tau < \infty$ and $k = 1, \ldots, n$ one has*

$$f^*(t) \leq K\left[f^*(\tau) + \left(\frac{\tau}{t}\right)^{r_k} \left(\frac{W(t)}{\lambda'_k(t)}\right)^{r_k} (D_k^{r_k} f)^{**}(\tau)\right], \tag{2.34}$$

where $\prod_{k=1}^{n} \lambda'_k(t) \geq W(t)^{n-1}$ and K is a constant depending on $r_1, \ldots, r_n$ and a.

Proof. Let $\lambda'_k(t) = (1/\sqrt[n]{a})(W(t)/t)\lambda_k(t)$, then

$$\prod_{k=1}^{n} \lambda'_k(t) = \frac{1}{a}\left(\frac{W(t)}{t}\right)^n \prod_{k=1}^{n} \lambda_k(t). \tag{2.35}$$

Due to the conditions of w and (2.33), we can get

$$\prod_{k=1}^{n} \lambda'_k(t) \geq W(t)^{n-1}. \tag{2.36}$$

In [2, 12, 15], we have

$$f^*(t) \leq K\left[f^*(\tau) + \left(\frac{\tau}{\lambda_k(t)}\right)^{r_k} (D_k^{r_k} f)^{**}(\tau)\right]. \tag{2.37}$$

So we immediately get (2.34). □

Lemma 2.5. *If $w \in B_{1,\infty}$, $1 < p_0 < \infty$ and $1 \leq s_0 < \infty$, then $v \equiv W(t)^{s_0/p_0 - 1} w(t) \in B_{s_0}$.*

Proof. Let $w \in B_{1,\infty}$. Since $B_{1,\infty} \subset B_{p_0}$, so by [9, Chapter 1] we get

$$\int_0^r \frac{1}{W(t)^{1/p_0}} dt \leq C \frac{r}{W(r)^{1/p_0}}, \quad \forall r > 0. \tag{2.38}$$

Then

$$\int_0^r \frac{1}{V(t)^{1/s_0}} dt \leq C \frac{r}{V(r)^{1/s_0}}, \quad \forall r > 0, \tag{2.39}$$

where

$$V(t) = \int_0^t v(t)dt. \tag{2.40}$$

So $v \in B_{s_0}$. □

Lemma 2.6. *Let $n \geq 2$, $r_k \in \mathbb{N}$, $1 < p_k < \infty$, $1 \leq s_k < \infty$ for $k = 1, \dots, n$. Assume that weight w on $\mathbb{R}_+$ satisfies the following conditions:*

(i) *it is nonincreasing, continuous, and $\lim_{t\to\infty} w(t) = a$, $a > 0$,*

(ii) *exists $A > 0$, such that*

$$W(\xi t) \geq \xi^A W(t), \quad \forall \xi > 1, \forall t > 0. \tag{2.41}$$

Set

$$r = n\left(\sum_{k=1}^{n} \frac{1}{r_k}\right)^{-1}, \qquad p = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{p_k r_k}\right)^{-1},$$
$$s = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{s_k r_k}\right)^{-1}. \tag{2.42}$$

Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in \Lambda^{p_k, s_k}(w)$ ($k = 1, \dots, n$). Then for any $\xi > 1$

$$f^*(t) \leq K\left[f^*(\xi t) + \xi^{\overline{r}} \sigma(t)\right], \tag{2.43}$$

where $\overline{r} = \max r_k$, the constants K depends only on $r_1, \dots, r_n$, w, and

$$\left(\int_0^\infty W(t)^{s(1/p - r/n)-1} w(t) \sigma(t)^s dt\right)^{1/s} \leq C \prod_{k=1}^{n} \|D_k^{r_k} f\|_{\Lambda^{p_k, s_k}(w)}^{r/(n r_k)}. \tag{2.44}$$

Proof. For every fixed $k = 1, \dots, n$ we take

$$\psi_k(t) = \left(D_k^{r_k} f\right)^{**}(t). \tag{2.45}$$

Thanks to Lemma 2.5, and $w \in B_{1,\infty}$ (for w is nonincreasing), we know

$$v = W(t)^{s_k/p_k - 1} w(t) \in B_{s_k}. \tag{2.46}$$

Thus

$$\|\psi_k\|_{\Lambda^{p_k, s_k}(w)} = \left\|\left(D_k^{r_k} f\right)^{**}\right\|_{L^{s_k}(v)} \leq C \left\|\left(D_k^{r_k} f\right)^{*}\right\|_{L^{s_k}(v)} = C \left\|D_k^{r_k} f\right\|_{\Lambda^{p_k, s_k}(w)}. \tag{2.47}$$

Next we apply Lemma 2.2 with δ defined as in Lemma 2.3. In this way we obtain the functions which we denote by $\phi_k(t)$ $(k = 1,\dots,n)$. Further, with these functions $\phi_k(t)$ we define the function $\sigma(t)$ by (2.28). By Lemma 2.3, we have the inequality (2.44). Using Lemma 2.4 with $\tau = \xi t$, we obtain

$$f(t) \leq K\left[f^*(\xi t) + \xi^{\overline{r}}\left(\frac{W(t)}{\lambda_k'(t)}\right)^{r_k}\phi_k\right], \tag{2.48}$$

where $\prod_{k=1}^n \lambda_k'(t) \geq W(t)^{n-1}$. Taking into account (2.28), we get (2.43). □

Corollary 2.7. *Let* $0 < \theta \leq 1$, $n \geq 2$, $r_k \in \mathbb{N}$, $1 < p_k < \infty$, $1 \leq s_k < \infty$ *for* $k = 1,\dots,n$, *and* r, p, s *be the numbers defined by* (2.42). *Assume weight* w *on* $\mathbb{R}_+$ *satisfies the following conditions:*

(i) *it is nonincreasing, continuous, and* $\lim_{t\to\infty} w(t) = a$, $a > 0$,

(ii) *there exist two constants* η, β *with* $\beta < 1$ *such that*

$$W\left(\frac{t}{\xi}\right)^{\theta/\eta-1} w\left(\frac{t}{\xi}\right) \leq C\xi^{\beta} W(t)^{\theta/\eta-1} w(t), \quad \forall t > 0, \forall \xi > 1, \tag{2.49}$$

and there holds

$$\widetilde{q} \equiv \sup\{\eta;\ \exists \beta < 1,\ (2.49)\ \textit{holds}\} > 1. \tag{2.50}$$

Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ *has weak derivatives* $D_k^{r_k} f \in \Lambda^{p_k,s_k}(w)$ $(k = 1,\dots,n)$ *and* $f \in \Lambda^1(w) + \Lambda^{p_0}(w)$ *for some* p_0 *with* $1 \leq p_0 < \widetilde{q}$ *such that*

$$\frac{1}{p_0} > \frac{1}{p} - \frac{r}{n}. \tag{2.51}$$

Let $p_0 < q < \widetilde{q}$ *and*

$$\frac{1}{q} > \frac{1}{p} - \frac{r}{n}. \tag{2.52}$$

Then $f \in \Lambda^{q,\theta}(w)$ *and*

$$\|f\|_{\Lambda^{q,\theta}(w)} \leq C\left[\|f\|_{\Lambda^1(w)+\Lambda^{p_0}(w)} + \prod_{k=1}^{n} \|D_k^{r_k} f\|_{\Lambda^{p_k,s_k}(w)}^{r/(nr_k)}\right]. \tag{2.53}$$

Proof. Let $f = g + h$, with $g \in \Lambda^1(w)$ and $h \in \Lambda^{p_0}(w)$. Applying Hölder's inequality and noticing $W(\infty) = \infty$ and w is nonincreasing, we obtain

$$\begin{aligned} J_1 &\equiv \int_1^\infty f^{*\theta}(t) W(t)^{\theta/q-1} w(t)\,dt \\ &\le \int_1^\infty g^{*\theta}\left(\frac{t}{2}\right) W(t)^{\theta/q-1} w(t)\,dt + \int_1^\infty h^{*\theta}\left(\frac{t}{2}\right) W(t)^{\theta/q-1} w(t)\,dt \\ &\le C\left[\left(\int_{1/2}^\infty g^*(t) w(t)\,dt\right)^\theta + \left(\int_{1/2}^\infty h^{*p_0}(t) w(t)\,dt\right)^{\theta/p_0}\right]. \end{aligned} \tag{2.54}$$

So

$$J_1 \le C' \|f\|_{\Lambda^1(w)+\Lambda^{p_0}(w)}. \tag{2.55}$$

Let $0 < \delta < 1$. Using (2.43) with $\xi > 1$, which satisfies $C_1 K^\theta \xi^{\beta-1} \le 1/2$ (C_1, β are two constants in (2.49) for $\eta = q$), combining (2.49), (2.52), and Hölder's inequality, we get

$$\begin{aligned} J_\delta &\equiv \int_\delta^\infty f^{*\theta}(t) W(t)^{\theta/q-1} w(t)\,dt \\ &\le J_1 + K^\theta \int_\delta^1 f^{*\theta}(\xi t) W(t)^{\theta/q-1} w(t)\,dt + K\xi^{\overline{r}} \int_\delta^1 \sigma(t)^\theta W(t)^{\theta/q-1} w(t)\,dt \\ &\le J_1 + K^\theta \frac{C_1}{\xi^{1-\beta}} \int_\delta^\infty f^{*\theta}(t) W(t)^{\theta/q-1} w(t)\,dt + C\int_\delta^1 \sigma(t)^\theta W(t)^{\theta/q-1} w(t)\,dt \\ &\le J_1 + \frac{1}{2} J_\delta + C'\left(\int_0^1 \sigma(t)^s W(t)^{(1/p-r/n)s} \frac{w(t)}{W(t)}\right)^{\theta/s}. \end{aligned} \tag{2.56}$$

By (2.55), $J_\delta < \infty$. Furthermore, from (2.49), we can get

$$W(\xi t) \ge \xi^{(1-\beta)q/\theta} W(t), \quad \forall t > 0,\ \forall \xi > 1. \tag{2.57}$$

Inequality (2.53) now follows from (2.44) and (2.55). □

Remark 2.8. If $w = a$ ($a > 0$) in Corollary 2.7, then it is easy to get $\tilde{q} = \infty$.

Remark 2.9. Let $r_k \in \mathbb{N}$, $1 < p_k < \infty$, $1 \le s_k < \infty$ for $k = 1, \dots, n$ ($n \ge 2$). Let r, p, and s be the numbers defined by (2.42). Assume that $p < n/r$, $q^* = np/(n - rp)$ and w satisfies the conditions of Corollary 2.7 with $\tilde{q} > q^*$. Then for any function $f \in C^\infty(\mathbb{R}^n)$ with compact support we have

$$\|f\|_{\Lambda^{q^*,s}(w)} \le C \prod_{k=1}^n \|D_k^{r_k} f\|_{\Lambda^{p_k,s_k}(w)}^{r/(nr_k)}. \tag{2.58}$$

This statement can be easily got from Lemma 2.6. Inequality (2.58) gives a generalization of Remark 2.6 of [15] when $p_k > 1$, $k = 1, \ldots, n$ because $w = 1$ satisfies the preceding conditions.

Remark 2.10. Beyond constant weights, there are many weights satisfying conditions of Corollary 2.7. For example,

(i) $w = t^{-\alpha} + a$, where $0 < \alpha < \theta$, $0 < a < \infty$,

(ii)

$$w = \begin{cases} t^{-\alpha}, & \text{if } 0 < t < 1, \\ 1, & \text{if } t \geq 1, \end{cases} \tag{2.59}$$

where $0 \leq \alpha < 1$.

For weight w in (i) or (ii), it is easy to see the weighted Lorentz space $\Lambda^{p,q}(w)$ for $0 < p$, $q < \infty$ does not coincide with any Lorentz space $L^{r,s}$.

3. The Main Theorem

Theorem 3.1. *Let $n \geq 2$, $r_k \in \mathbb{N}$, $1 < p_k < \infty$, $1 \leq s_k < \infty$ for $k = 1, \ldots, n$. Let r, p, and s be the numbers defined by* (2.42)*. Suppose weight w on $\mathbb{R}_+$ satisfies the following conditions:*

(i) *it is nonincreasing, continuous, and $\lim_{t\to\infty} w(t) = a$, $a > 0$,*

(ii) *there exist two constants η, β with $\beta < 1$ such that*

$$W\left(\frac{t}{\xi}\right)^{1/\eta-1} w\left(\frac{t}{\xi}\right) \leq C\xi^{\beta} W(t)^{1/\eta-1} w(t), \quad \forall t > 0, \forall \xi > 1, \tag{3.1}$$

and there holds

$$\widetilde{q} \equiv \sup\{\eta;\ \exists \beta < 1,\ (3.1)\ \textit{holds}\} > \max\{p_i;\ i = 1, \ldots, n\}. \tag{3.2}$$

For every p_j $(1 \leq j \leq n)$ satisfying the condition

$$\rho_j \equiv \frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} > 0, \tag{3.3}$$

take arbitrary q_j such that $p_j < q_j < \widetilde{q}$ and

$$\frac{1}{q_j} > \frac{1}{p} - \frac{r}{n} \tag{3.4}$$

and denote

$$H_j = 1 - \frac{1}{\rho_j}\left(\frac{1}{p_j} - \frac{1}{q_j}\right), \qquad \alpha_j = H_j r_j, \qquad \frac{1}{\theta_j} = \frac{1-H_j}{s} + \frac{H_j}{s_j}. \tag{3.5}$$

Then for any function $f \in S_0(\mathbb{R}^n)$ with the weak derivatives $D_k^{r_k} f \in \Lambda^{p_k,s_k}(w)$ $(k = 1,\dots,n)$ there holds the inequality

$$\left(\int_0^\infty \left[h^{-\alpha_j}\left\|\Delta_j^{r_j}(h)f\right\|_{\Lambda^{q_j,1}(w)}\right]^{\theta_j}\frac{dh}{h}\right)^{1/\theta_j} \leq C\sum_{k=1}^n \left\|D_k^{r_k} f\right\|_{\Lambda^{p_k,s_k}(w)}, \tag{3.6}$$

where C is a constant that does not depend on f.

Proof. First we can get $0 < H_j < 1$ by our conditions. denote

$$g_k(x) = \left|D_k^{r_k} f(x)\right|. \tag{3.7}$$

Further, assume that $j = 1$ and set for $h > 0$

$$f_h(x) = \left|\Delta_1^{r_1}(h)f(x)\right|. \tag{3.8}$$

For almost all $x \in \mathbb{R}^n$ we have [1, Volume 1, page 101]

$$f_h(x) \leq \int_0^h \cdots \int_0^h g_1(x + (u_1 + \cdots + u_{r_1})e_1)du_1 \cdots du_{r_1}. \tag{3.9}$$

Thus,

$$f_h^*(t) \leq h^{r_1} g_1^{**}(t). \tag{3.10}$$

Indeed, for any subset $A \subset \mathbb{R}^n$ with $|A| = t$

$$\int_A f_h(x)dx \leq h^{r_1} \sup_{B\subset\mathbb{R}^n,\,|B|=t}\int_B g_1(y)dy = h^{r_1} t g_1^{**}(t), \tag{3.11}$$

(3.10) then follows.

For $p_k > 1$, w is nonincreasing ($w \in B_{1,\infty}$), we get $W(t)^{s_k/p_k-1}w(t) \in B_{s_k}$ by Lemma 2.5. Thus from (3.10)

$$\begin{aligned}\|f_h\|_{\Lambda^{p_1,s_1}(w)} &= \left(\int_0^\infty f_h^{*s_1}(t)W(t)^{s_1/p_1-1}w(t)dt\right)^{1/s_1}\\ &\le h^{r_1}\left(\int_0^\infty g_1^{**s_1}(t)W(t)^{s_1/p_1-1}w(t)dt\right)^{1/s}\\ &\le Ch^{r_1}\|g_1\|_{\Lambda^{p_1,s_1}(w)}.\end{aligned} \tag{3.12}$$

It follows $f_h \in \Lambda^{p_1,s_1}(w)$. Furthermore

$$\begin{aligned}\|D_1^{r_1}f_h\|_{\Lambda^{p_1,s_1}(w)} &\le C\left(\int_0^\infty\left(\left(D_1^{r_1}f\right)^*\left(\frac{t}{2^{r_1}}\right)\right)^{s_1}W(t)^{s_1/p_1-1}w(t)dt\right)^{1/s_1}\\ &= C\left(\int_0^\infty\left(\left(D_1^{r_1}f\right)^*(t)\right)^{s_1}W(2^{r_1}t)^{s_1/p_1-1}w(2^{r_1}t)dt\right)^{1/s_1}.\end{aligned} \tag{3.13}$$

Then due to Hardy lemma [16, page 56]

$$\begin{aligned}\|D_1^{r_1}f_h\|_{\Lambda^{p_1,s_1}(w)} &\le C\left(\int_0^\infty\left(\left(D_1^{r_1}f\right)^*(t)\right)^{s_1}W(t)^{s_1/p_1-1}w(t)dt\right)^{1/s_1}\\ &= C\|D_1^{r_1}f\|_{\Lambda^{p_1,s_1}(w)}.\end{aligned} \tag{3.14}$$

It follows $D_1^{r_1}f_h \in \Lambda^{p_1,s_1}(w)$. Analogically we get $D_k^{r_k}f_h \in \Lambda^{p_k,s_k}(w)$. Thus by Corollary 2.7 we have $f_h \in \Lambda^{q_1,1}(w)$.

Denote for $h > 0$

$$J(h) \equiv \|f_h\|_{\Lambda^{q_1,1}(w)} = \int_0^\infty (f_h)^*(t)W(t)^{1/q_1-1}w(t)dt. \tag{3.15}$$

Set $\xi_0 = (4KC_1)^{1/(-\beta+1)}$ (C_1, β are two constants in (3.1) for $\eta = q_1$), and

$$Q(h) = \left\{t > 0 : f_h^*(t) \ge 2Kf_h^*(\xi_0 t)\right\}, \tag{3.16}$$

where K is the constant in Lemma 2.5. Then by (3.1)

$$\begin{aligned}\int_{\mathbb{R}_+\setminus Q(h)} f_h^*(t)W(t)^{1/q_1-1}w(t)dt &\le 2K\int_{\mathbb{R}_+\setminus Q(h)} f_h^*(\xi_0 t)W(t)^{1/q_1-1}w(t)dt\\ &\le 2K\int_0^\infty f_h^*(\xi_0 t)W(t)^{1/q_1-1}w(t)dt\\ &\le \frac{2KC_1}{\xi_0^{1-\beta}}\int_0^\infty f^*(t)W(t)^{1/q_1-1}w(t)dt.\end{aligned} \tag{3.17}$$

Therefore ,

$$J(h) \leq 2\int_{Q(h)} f_h^*(t) W(t)^{1/q_1-1} w(t)\,dt \equiv 2J'(h). \tag{3.18}$$

Let

$$0 < \delta < \frac{1}{4}\min_{\gamma_i<1}(1-\gamma_i). \tag{3.19}$$

Now for every $k = 1,\ldots,n$ by applying Lemma 2.2 with $\psi(t) = g_k^{**}(t)$. We obtain $\phi_k(t)$ $(k = 1,\ldots,n)$ on $\mathbb{R}_+$ such that

$$\phi_k(t) W(t)^{1/p_k-\delta} w(t) \downarrow, \qquad \phi_k(t) W(t)^{1/p_k+\delta} w(t) \uparrow, \tag{3.20}$$

$$g_k^{**}(t) \leq C\phi_k(t), \tag{3.21}$$

$$\|\phi_k\|_{\Lambda^{p_k,s_k}(w)} \leq C\|g_k^{**}\|_{\Lambda^{p_k,s_k}(w)}. \tag{3.22}$$

For $W(t)^{s_k/p_k-1} w(t) \in B_{s_k}$, it follows that

$$\|g_k^{**}\|_{\Lambda^{p_k,s_k}(w)} \leq C\|g_k\|_{\Lambda^{p_k,s_k}(w)}. \tag{3.23}$$

Thus

$$\|\phi_k\|_{\Lambda^{p_k,s_k}(w)} \leq C\|D_k^{r_k} f\|_{\Lambda^{p_k,s_k}(w)}. \tag{3.24}$$

We will estimate $f_h^*(t)$ for fixed $h > 0$ and $t \in Q(h)$. By Lemma 2.4, (3.21), we have that for each $t \in Q(h)$

$$f_h^*(t) \leq C\left(\frac{W(t)}{\lambda_k'(t,h)}\right)^{r_k} \phi_k(t), \tag{3.25}$$

where $\prod_{k=1}^{n}\lambda'_k(t,h) \geq W(t)^{n-1}$. Applying Lemma 2.3, we obtain that there exist a nonnegative function $\sigma(t)$ and positive continuously differentiable functions $u_k(t)$ $(k = 1,\dots,n)$ on $\mathbb{R}_+$ satisfying the following conditions:

$$f_h^*(t) \leq C\sigma(t), \quad t \in Q(h), \tag{3.26}$$

$$\left(\int_0^\infty W(t)^{s(1/p-r/n)-1} w(t)\sigma(t)^s dt\right)^{1/s} \leq C\prod_{k=1}^{n} \|D_k^{r_k} f\|_{\Lambda^{p_k,s_k}(w)}^{r/(nr_k)}, \tag{3.27}$$

$$\sigma(t) = \left(\frac{W(t)}{u_k(t)}\right)^{r_k} \phi_k(t), \tag{3.28}$$

$$\prod_{k=1}^{n} u_k(t) = W(t)^{n-1}, \tag{3.29}$$

$$u_1(t)W(t)^{\delta-1} \text{ decreases.} \tag{3.30}$$

Denote

$$\beta(t) = \frac{W(t)}{u_1(t)}. \tag{3.31}$$

We will prove that for any $h > 0$ and any $t \in Q(h)$

$$f_h^*(t) \leq Ch^{r_1}\chi(t), \tag{3.32}$$

where

$$\chi(t) \equiv \sigma(t)\beta(t)^{-r_1} = \phi_1(t) \quad \text{(see (3.28))}. \tag{3.33}$$

By (3.24)

$$\|\chi\|_{\Lambda^{p_1,s_1}(w)} \leq C\|D_1^{r_1} f\|_{\Lambda^{p_1,s_1}(w)}. \tag{3.34}$$

For $h \geq \beta(t)$ $(t \in Q(h))$ the inequality (3.32) follows directly from (3.26) and (3.33). If $0 < h < \beta(t)$, $t \in Q(h)$, then (3.32) is the immediate consequence of (3.10), (3.21), and (3.33).

Now, taking into account (3.26) and (3.32), we obtain that for $h > 0$ and any $t \in Q(h)$

$$f_h^*(t) \leq C\Phi(t,h), \tag{3.35}$$

where

$$\Phi(t,h) = \min\left(\sigma(t), h^{r_1}\chi(t)\right), \tag{3.36}$$

and $\chi(t)$ is defined by (3.33).

Further, we have (see (3.18))

$$J'(h) \leq C\int_0^\infty W(t)^{1/q_1-1}w(t)\Phi(t,h)dt,$$
$$J \equiv \int_0^\infty h^{-\alpha_1\theta_1-1}J(h)^{\theta_1}dh \leq C\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_0^\infty W(t)^{1/q_1-1}w(t)\Phi(t,h)dt\right)^{\theta_1}. \tag{3.37}$$

By (3.30), the function $\beta(t)W(t)^{-\delta}$ increases on $\mathbb{R}_+$. It follows easily that β^{-1} exists on $\mathbb{R}_+$ and satisfies $\beta^{-1}(0) = 0$, $\beta^{-1}(\infty) = \infty$, and

$$\frac{W(\beta^{-1}(2z))}{W(\beta^{-1}(z))} \leq 2^{1/\delta}. \tag{3.38}$$

Furthermore, we have

$$\begin{aligned} J &\leq C\left[\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_0^{\beta^{-1}(h)} W(t)^{1/q_1-1}w(t)\Phi(t,h)dt\right)^{\theta_1}\right] \\ &\quad + C\left[\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_{\beta^{-1}(h)}^\infty W(t)^{1/q_1-1}w(t)\Phi(t,h)dt\right)^{\theta_1}\right] \\ &\equiv C(J_1+J_2). \end{aligned} \tag{3.39}$$

Using Minkowsi's inequality, we obtain

$$\begin{aligned} J_1^{1/\theta_1} &= \left(\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\sum_{k=0}^\infty\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} W(t)^{1/q_1-1}w(t)\sigma(t)dt\right)^{\theta_1}\right)^{1/\theta_1} \\ &\leq \sum_{k=0}^\infty\left[\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} W(t)^{1/q_1-1}w(t)\sigma(t)dt\right)^{\theta_1}\right]^{1/\theta_1} \\ &\leq \sum_{k=0}^\infty 2^{-k\alpha_1}\left[\int_0^\infty z^{-\alpha_1\theta_1-1}dz\left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} W(t)^{1/q_1-1}w(t)\sigma(t)dt\right)^{\theta_1}\right]^{1/\theta_1}. \end{aligned} \tag{3.40}$$

Further, using Hölder's inequality and (3.38), we get when $\theta_1 > 1$ (the case $\theta_1 = 1$ is obvious)

$$\begin{aligned}
&\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} W(t)^{1/q_1-1} w(t)\sigma(t)\,dt \\
&\quad \le \left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{\theta_1}\,dt\right)^{1/\theta_1} \left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} \frac{w(t)}{W(t)}\,dt\right)^{1/\theta_1'} \\
&\quad \le C\left(\int_0^{\beta^{-1}(z)} W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{\theta_1}\,dt\right)^{1/\theta_1}.
\end{aligned} \tag{3.41}$$

Thus, by Fubini's theorem and (3.33)

$$\begin{aligned}
J_1 &\le C\int_0^\infty z^{-\alpha_1\theta_1-1}\,dz \int_0^{\beta^{-1}(z)} W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{\theta_1}\,dt \\
&= C'\int_0^\infty W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{\theta_1}\beta^{-\alpha_1\theta_1}\,dt \\
&= C'\int_0^\infty W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{(1-H_1)\theta_1}\chi(t)^{H_1\theta_1}\,dt.
\end{aligned} \tag{3.42}$$

The same argument gives that

$$\begin{aligned}
J_2 &\le C\int_0^\infty z^{(-\alpha_1+r_1)\theta_1-1}\,dz \int_{\beta^{-1}(z)}^\infty W(t)^{\theta_1/q_1-1} w(t)\chi(t)^{\theta_1}\,dt \\
&\le C'\int_0^\infty W(t)^{\theta_1/q_1-1} w(t)\beta(t)^{(r_1-\alpha_1)\theta_1}\chi(t)^{\theta_1}\,dt.
\end{aligned} \tag{3.43}$$

By (3.33) the last integral is the same as one on the right side of (3.42). So, we have that

$$J \le C\int_0^\infty W(t)^{\theta_1/q_1-1} w(t)\sigma(t)^{(1-H_1)\theta_1}\chi(t)^{H_1\theta_1}\,dt. \tag{3.44}$$

Now we apply Hölder's inequality with the exponents $u = s_1/H_1\theta_1$ and $u' = s_1/(s_1 - H_1\theta_1)$. Observe that

$$(1-H_1)\theta_1 u' = s, \qquad \left(\frac{\theta_1}{q_1} - \frac{s_1}{p_1 u}\right)u' = s\left(\frac{1}{p} - \frac{r}{n}\right). \tag{3.45}$$

Therefore, we get, applying (3.27) and (3.34)

$$\begin{aligned} J^{1/\theta_1} &\le C\left(\int_0^\infty W(t)^{s(1/p-r/n)-1} w(t)\sigma(t)^s dt\right)^{(1-H_1)/s} \|D_1^{r_1} f\|_{\Lambda^{p_1,s_1}(w)}^{H_1} \\ &\le C\left(\prod_{k=1}^n \|D_k^{r_k} f\|_{\Lambda^{p_k,s_k}(w)}^{r/(nr_k)}\right)^{1-H_1} \|D_1^{r_1} f\|_{\Lambda^{p_1,s_1}(w)}^{H_1}. \end{aligned} \tag{3.46}$$

Since

$$\sum_{k=1}^n \frac{r}{nr_k} = 1, \tag{3.47}$$

we get the inequality (3.6). The theorem is proved. □

Let $X = X(\mathbb{R}^n)$ be a rearrangement invariant space (r.i. space), Y be an r.i. space over $\mathbb{R}_+$ and $s > 0$. Set $r = [s] + 1$ ($[s]$ = integral part of s). The Besov space $B^s_{X,Y;j}(\mathbb{R}^n)$ is defined as follows (see [18, 19]):

$$B^s_{X,Y;j}(\mathbb{R}^n) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{B^s_{X,Y;j}} = \left\| \frac{t^{-s/n}\omega_{X,j}(f,t^{1/n})_r}{\Phi_Y(t)} \right\|_Y < \infty \right\}, \tag{3.48}$$

where

$$\begin{gathered} \omega_{X,j}(f,t)_r = \sup_{|h|\le t} \left\| \Delta^r_{h,j} f \right\|_X \quad (t > 0), \qquad \Delta^{k+1}_{h,j} f(x) = \Delta^1_{h,j}\left(\Delta^k_{h,j}\right) f(x), \\ \Delta^1_{h,j} f(x) = f\left(x + he_j\right) - f(x), \end{gathered} \tag{3.49}$$

and $\Phi_Y(t)$ denotes the fundamental function of $Y : \Phi_Y(t) = \|\chi_E\|_Y$, with E being any measurable subset of $\mathbb{R}_+$ with $|E| = t$.

Then we have the following.

Corollary 3.2. *Let $n \ge 2$, $r \in \mathbb{N}$, $p > 1$, $1 \le s_k < \infty$ for $k = 1, \dots, n$, and*

$$s = n\left(\sum_{k=1}^n \frac{1}{s_k}\right)^{-1}. \tag{3.50}$$

Let the weight w be the same as that in Theorem 3.1. Take arbitrary q such that

$$p < q < \tilde{q}, \qquad \frac{1}{q} > \frac{1}{p} - \frac{1}{n}, \tag{3.51}$$

and denote

$$H = 1 - \frac{n}{r}\left(\frac{1}{p} - \frac{1}{q}\right), \qquad \alpha = Hr, \qquad \frac{1}{\theta_j} = \frac{1-H}{s} + \frac{H}{s_j}. \tag{3.52}$$

Then for any function $f \in S_0(\mathbb{R}^n)$ *which has the weak derivatives* $D_k^r f \in \Lambda^{p,s_k}(w)$ $(k = 1, \ldots, n)$ *there hold*

$$f \in B^{\alpha}_{\Lambda^{q,1}(w), L^{\theta_j};j}(\mathbb{R}^n),$$
$$\|f\|_{B^{\alpha}_{\Lambda^{q,1}(w), L^{\theta_j};j}} \le C\sum_{k=1}^{n}\|D_k^r f\|_{\Lambda^{p,s_k}(w)}, \tag{3.53}$$

where C is a constant that does not depend on f.

Proof. We can easily obtain the similar result to Lemma 2.4 in [20] by substituting $\Lambda^{q,1}(w)$ for $L^{p,s}(\mathbb{R}^n)$ there. Now the corollary is obvious using the Hardy's inequality and Theorem 3.1. □

Remark 3.3. If there exists j $(1 \le j \le n)$ with $p_j = s_j = 1$, whether Theorem 3.1 remains true is still a question now.

Acknowledgments

This work is supported by NSFC (no. 10571156, 10871173), Natural Science Foundation of Zhejiang Province (no. Y606117), Foundation of Zhejiang Province Education Department (no. Y200803879) and 2008 Excellent Youth Foundation of College of Zhejiang Province (no. 01132047).

References

[1] O. V. Besov, V. P. Il'in, and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Theorems. Vol. I*, V. H. Winston & Sons, Washington, DC, USA, 1978.

[2] V. I. Kolyada, "Rearrangements of functions and embedding of anisotropic spaces of Sobolev type," *East Journal on Approximations*, vol. 4, no. 2, pp. 111–199, 1998.

[3] L. D. Kudryavtsev and S. M. Nikol'skiĭ, "Spaces of differentiable functions of several variables and imbedding theorems," in *Analysis*, vol. 26 of *Encyclopaedia Math. Sci.*, pp. 1–140, Springer, Berlin, Germany, 1991.

[4] S. M. Nikol'skií, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer, New York, NY, USA, 1975.

[5] H. Triebel, *Theory of Function Spaces*, vol. 78 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1983.

[6] H. Triebel, *Theory of Function Spaces. II*, vol. 84 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1992.

[7] J. Peetre, "Espaces d'interpolation et théorème de Soboleff," *Université de Grenoble. Annales de l'Institut Fourier*, vol. 16, no. 1, pp. 279–317, 1966.

[8] R. S. Strichartz, "Multipliers on fractional Sobolev spaces," *Journal of Mathematics and Mechanics*, vol. 16, pp. 1031–1060, 1967.

[9] M. J. Carro, J. A. Raposo, and J. Soria, "Recent developments in the theory of Lorentz spaces and weighted inequalities," *Memoirs of the American Mathematical Society*, vol. 187, no. 877, 2007.

[10] W. G. Faris, “Weak Lebesgue spaces and quantum mechanical binding,” *Duke Mathematical Journal*, vol. 43, no. 2, pp. 365–373, 1976.
[11] V. I. Kolyada, “Rearrangements of functions, and embedding theorems,” *Uspekhi Matematicheskikh Nauk*, vol. 44, no. 5, pp. 61–95, 1989, [English translation: Russian Math. Surveys, vol. 44, no. 5, pp. 73–118, 1989].
[12] V. I. Kolyada, “On the embedding of Sobolev spaces,” *Matematicheskie Zametki*, vol. 54, no. 3, pp. 48–71, 1993, [English translation: Math. Notes, vol. 54, no. 3, pp. 908–922, 1993].
[13] S. Poornima, “An embedding theorem for the Sobolev space $W^{1,1}$,” *Bulletin des Sciences Mathématiques*, vol. 107, no. 3, pp. 253–259, 1983.
[14] V. I. Kolyada, “On the relations between moduli of continuity in various metrics,” *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, vol. 181, pp. 117–136, 1988, [English translation: Proc. Steklov Inst. Math., vol. 4, pp. 127–148, 1989].
[15] V. I. Kolyada and F. J. Pérez, “Estimates of difference norms for functions in anisotropic Sobolev spaces,” *Mathematische Nachrichten*, vol. 267, pp. 46–64, 2004.
[16] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1988.
[17] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, Germany, 1957.
[18] J. Martín and M. Milman, “Symmetrization inequalities and Sobolev embeddings,” *Proceedings of the American Mathematical Society*, vol. 134, no. 8, pp. 2335–2347, 2006.
[19] J. Martín and M. Milman, “Higher-order symmetrization inequalities and applications,” *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 91–113, 2007.
[20] V. I. Kolyada, “Inequalities of Gagliardo-Nirenberg type and estimates for the moduli of continuity,” *Uspekhi Matematicheskikh Nauk*, vol. 60, no. 6, pp. 139–156, 2005, [English translation: Russian Math. Surveys, vol. 60, no. 6, pp. 1147–1164].