Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 154632, 8 pages
doi:10.1155/2009/154632

Research Article

Approximation of Second-Order Moment Processes from Local Averages

Zhanjie Song,[1,2] Ping Wang,[1] and Weisong Xie[1]

[1] *School of Science, Tianjin University, Tianjin 300072, China*
[2] *Institute of TV and Image Information, Tianjin University, Tianjin 300072, China*

Correspondence should be addressed to Weisong Xie, weis_xie@tju.edu.cn

Received 6 March 2009; Accepted 8 July 2009

Recommended by Jozef Banas

We use local averages to approximate processes that have finite second-order moments and are continuous in quadratic mean. We also provide some insight and generalization of the connection between Bernstein polynomials and Brownian motion, which was investigated by Kowalski in 2006.

1. Introduction

In the literature, very few researchers considered approximating Brownian motion using Bernstein polynomials. Kowalski [1] is the first one who uses this method. In fact, if we restrict Brownian motion on $[0, 1]$, it is a real process with finite second order moment. In this paper, we will approximate all of the complex second order moment processes on $[a, b]$ by Bernstein polynomials and other classical operators by [2]. Therefore the research obtained generalize that of [1].

On the other hand, it is well known that the sampling theorem is one of the most powerful tools in signal analysis. It says that to recover a function in certain function spaces, it suffices to know the values of the function on a sequence of points.

Due to physical reasons, for example, the inertia of the measurement apparatus, the measured sampled values obtained in practice may not be values of $f(t)$ precisely at times t_k $(k \in \mathbb{Z})$, but only local average of $f(t)$ near t_k. Specifically, the measured sampled values are

$$\langle f, u_k \rangle = \int f(t) u_k(t) dt \qquad (1.1)$$

for some collection of averaging functions $u_k(t), k \in \mathbb{Z}$, which satisfy the following properties:

$$\text{supp } u_k \subset \left[t_k - \frac{\delta}{2}, t_k + \frac{\delta}{2}\right], \quad u_k(t) \geq 0, \quad \int u_k(t)dt = 1. \tag{1.2}$$

Gröchenig [3] proved that every band-limited signal can be reconstructed exactly by local averages providing $t_{k+1} - t_k \leq \delta < 1/\sqrt{2}\Omega$, where Ω is the maximal frequency of the signal $f(t)$. Recently, several average sampling theorems have been established, for example, see [4–7].

Since signals are often of random characters, random signals play an important role in signal processing, especially in the study of sampling theorems. For this purpose, one usually uses stochastic processes which are stationary in the wide sense as a model [8, 9]. A wide sense stationary process is only a kind of second order moment processes. In this paper, we study complex second order moment processes on $[a,b]$ by some classical operators.

Given a probability space $(\mathbb{A}, \mathcal{F}, P)$, a stochastic process $\{X(t,\omega) : t \in T, T \subset \mathbb{R}\}$ is said to be a second order moment process on T if $E|X(t,\cdot)|^2 = E(X(t,\cdot)\overline{X(t,\cdot)}) = R_X(t,t) < \infty$, $\forall t \in T$. Now for each $n \in \mathbb{Z}_+$, let $t_{k,n} = k/n$ and $0 \leq \delta_1(n),\ \delta_2(n) \leq C_1/n$, where $k \in \mathbb{Z}$ and C_1 is a constant. Then for each $n \in \mathbb{Z}_+$, let the averaging functions $u_{k,n}(t),\ k \in \mathbb{Z}$, satisfy the following properties:

$$\text{supp } u_{k,n} \subset [t_{k,n} - \delta_1(n), t_{k,n} + \delta_2(n)], \quad u_{k,n}(t) \geq 0, \quad \int u_{k,n}(t)dt = 1, \tag{1.3}$$

$$\int_{t_{k,n}-\delta_1(n)}^{t_{k,n}+\delta_2(n)} t^i u_{k,n}(t)dt = \left(\frac{k}{n}\right)^i + o\left(\frac{1}{n}\right), \quad \text{for } i = 0, 1, 2. \tag{1.4}$$

The local averages of $X(t,\omega)$ near $t_{k,n} = k/n$ are

$$\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle = \int X(t,\omega)u_{k,n}(t)dt. \tag{1.5}$$

The operator M_n is defined as

$$[M_n X](t,\omega) = \sum_{k=0}^{+\infty} \langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle K_{k,n}(t), \tag{1.6}$$

where $K_{k,n}(t) \geq 0$ are kernel functions and satisfy the following equations for all constant C

$$\sum_{k=0}^{+\infty} C K_{k,n}(t) = C + O\left(\frac{1}{n}\right). \tag{1.7}$$

2. Main Results

In this paper, let $T = [a,b]$ and let $\mathbb{C}[a,b]$ denote the space of all continuous real functions on $[a,b]$. $\mathbb{M}[a,b]$ denotes the space of all bounded real functions on $[a,b]$. $\mathbb{H}(\mathbb{A}, [a,b])$ denotes the space of all second order moment processes on $[a,b]$. $\mathbb{H}^C(\mathbb{A},[a,b])$ denotes the space of all second order moment processes in quadratic mean continuous on $[a,b]$. Let us begin with the following proposition.

Proposition 2.1 (Korovkin [10]). *Assume that* $L_n : \mathbb{C}[a,b] \to \mathbb{M}[a,b]$ *are a sequence of linear positive operators. If for* $f(t) = 1, t$, *and* t^2, *one has*

$$\lim_{n\to\infty} \left\| \left[L_n f\right](t) - f(t) \right\|_M = 0, \tag{2.1}$$

where

$$\|f(t)\|_M = \sup_{t\in[a,b]} \{ |f(t)| : f(t) \in M \}, \tag{2.2}$$

then for any $f \in \mathbb{C}[a,b]$, *one has*

$$\lim_{n\to\infty} \left\| \left[L_n f\right](t) - f(t) \right\|_M = 0. \tag{2.3}$$

Notice that for $X(t,\omega) = f(t) \in \mathbb{C}[a,b]$, (1.6) can be changed as

$$\left[M_n f\right](t) = \sum_{k=0}^{+\infty} \langle f(\cdot), u_{k,n}(\cdot) \rangle K_{k,n}(t). \tag{2.4}$$

Then our main result is the following.

Theorem 2.2. *Let* $\{[M_n f](t),\ n \ge 0\}$ *be a sequence of operators defined as* (2.4) *such that for* $f(t) = 1, t$, *and* t^2, *one has*

$$\lim_{n\to\infty} \left\| \left[M_n f\right](t) - f(t) \right\|_M = 0. \tag{2.5}$$

Then for any second order moment processes in quadratic mean continuous $X(t,\omega)$ *on any finite closed interval* $[a,b]$, *one has*

$$\lim_{n\to\infty} E[[M_n X](t,\omega) - X(t,\omega)]^2 = 0, \tag{2.6}$$

where $\{[M_n X](t,\omega),\ n \ge 0\}$ *is a sequence of operators defined as* (1.6).

Proof. Let $X(t,\omega) \in \mathbb{H}^C(\mathbb{A},[a,b])$, and let $R_X(t,s)$ be the correlation functions of $X(t,\omega)$. Then we have $R_X(t,s) \in \mathbb{C}([a,b]\times[a,b])$. For any fixed $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|R(t,t) - R(t,t^*)| < \frac{\varepsilon}{3}, \tag{2.7}$$

whenever $|t-t^*| < \delta$. Then there is $N > 0$ such that $0 \le \delta_1(n),\ \delta_2(n) \le \delta/2$ for all $n \ge N$. Thus when $n \ge N$ and $|t_{k,n} - t| \le \delta/2$, we have

$$\begin{aligned}
&E|\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle - X(t,\omega)|^2 \\
&\quad = E|\langle [X(\cdot,\omega) - X(t,\omega)], u_{k,n}(\cdot)\rangle|^2 \\
&\quad = \iint_{t_{k,n}-\delta_1(n)}^{t_{k,n}+\delta_2(n)} [R_X(x,y) - R_X(x,t) - R_X(t,y) + R(t,t)] u_{k,n}(x) u_{k,n}(y)\, dx dy \\
&\quad = \iint_{t_{k,n}-\delta_1(n)}^{t_{k,n}+\delta_2(n)} [[R_X(x,y) - R_X(x,t)] + [R_X(x,t) - R_X(x,t)] \\
&\qquad\qquad + [R(t,t) - R_X(t,y)]] u_{k,n}(x) u_{k,n}(y)\, dx\, dy \\
&\quad \le \varepsilon.
\end{aligned} \tag{2.8}$$

At the same time, since $X(t,\omega) \in \mathbb{H}^C(\mathbb{A},[a,b])$, $E|X(t,\omega)|^2 = R_X(t,t) \le M < \infty$. Then using (2.8), that for any given $\varepsilon > 0$ and any $t_{k,n},\ t \in [a,b]$, we have

$$E|\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle - X(t,\omega)|^2 \le \varepsilon + \frac{16M}{\delta^2}(t - t_{k,n})^2. \tag{2.9}$$

From (1.7), (2.5), and (2.9), we have

$$\begin{aligned}
&E[[M_n X](t,\omega) - X(t,\omega)]^2 \\
&\quad = E\left[[M_n X](t,\omega) - X(t,\omega)\sum_{k=0}^{+\infty} K_{k,n}(t) + X(t,\omega)\sum_{k=0}^{+\infty} K_{k,n}(t) - X(t,\omega)\right]^2 \\
&\quad \le 2E\left[[M_n X](t,\omega) - X(t,\omega)\sum_{k=0}^{+\infty} K_{k,n}(t)\right]^2 + 2E\left[X(t,\omega)\sum_{k=0}^{+\infty} K_{k,n}(t) - X(t,\omega)\right]^2 \\
&\quad = 2E\left[[M_n X](t,\omega) - X(t,\omega)\sum_{k=0}^{+\infty} K_{k,n}(t)\right]^2 + 2\left|\sum_{k=0}^{+\infty} 1 \cdot K_{k,n}(t) - 1\right|^2 R_X(t,t) \\
&\quad = 2E\left|\sum_{k=0}^{+\infty} (\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle - X(t,\omega)) K_{k,n}(t)\right|^2 + O\left(\frac{1}{n}\right) \\
&\quad \le 2\left|\sum_{k=0}^{+\infty} E|\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle - X(t,\omega)|^2 K_{k,n}(t)\right| \left|\sum_{k=0}^{+\infty} K_{k,n}(t)\right| + O\left(\frac{1}{n}\right) \\
&\quad = 2\left|\sum_{k=0}^{+\infty} E|\langle X(\cdot,\omega), u_{k,n}(\cdot)\rangle - X(t,\omega)|^2 K_{k,n}(t)\right| + O\left(\frac{1}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\left|\sum_{k=0}^{+\infty}\left(\varepsilon + \frac{16M}{\delta^2}(t-t_{k,n})^2\right)K_{k,n}(t)\right| + O\left(\frac{1}{n}\right)\\
&= \frac{32M}{\delta^2}\left|\sum_{k=0}^{+\infty}\left(t^2 - 2t_{k,n}t + t_{k,n}^2\right)K_{k,n}(t)\right| + 2\varepsilon + O\left(\frac{1}{n}\right)\\
&\leq \frac{32M}{\delta^2}\left|\left(t^2[M_n1](t) - 2t[M_nx](t) + \left[M_nx^2\right](t)\right)\right|^2 + 2\varepsilon + O\left(\frac{1}{n}\right)\\
&\longrightarrow 0 \quad (\text{when } n \longrightarrow \infty).
\end{aligned}
\tag{2.10}
$$

This completes the proof. □

3. Applications

As the application of Theorem 2.2, we give a new kind of operators.

For a signal function defined as

$$
\operatorname{sgn}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{3.1}
$$

let $\{\alpha_n,\ n \in \mathbb{R}^+\}$ be a monotonic sequence that satisfies

$$
\lim_{n \to +\infty} \alpha_n = +\infty, \tag{3.2}
$$

and let

$$
h_n(c,t) = \begin{cases} e^{-nt}, & c = 0, \\ (1+ct)^{-\operatorname{sgn}(c)\cdot\alpha_n}, & c \neq 0. \end{cases} \tag{3.3}
$$

Obviously, function $h(c,t)$ is continuous in $\mathbb{R}$, now we let

$$
b_{k,n}(c,t) = (-1)^k \frac{t^k}{k!} h_n^{(k)}(c,t). \tag{3.4}
$$

Using Gamma-function, $b_{n,k}(c,t)$ can be noted by

$$
b_{k,n}(c,t) = \begin{cases} e^{-nt}\dfrac{(nt)^k}{k!}, & c = 0, \\ \dfrac{\Gamma(\operatorname{sgn}(c)\cdot\alpha_n + k)}{\Gamma(\operatorname{sgn}(c)\cdot\alpha_n)k!}(ct)^k(1+ct)^{-\operatorname{sgn}(c)\cdot\alpha_n - k}, & c \neq 0, \end{cases} \tag{3.5}
$$

where

$$\frac{\Gamma(\operatorname{sgn}(c)\cdot\alpha_n+k)}{\Gamma(\operatorname{sgn}(c)\cdot\alpha_n)}=(\operatorname{sgn}(c)\cdot\alpha_n+k-1)(\operatorname{sgn}(c)\cdot\alpha_n+k-2)\cdots(\operatorname{sgn}(c)\cdot\alpha_n). \tag{3.6}$$

If $c<0$, $t\in[0,-1/c]$ we need $\{\alpha_n, n\in\mathbb{Z}^+\}$; if $c\geq 0$, $t\in[0,+\infty)$, then $\{\alpha_n, n\in\mathbb{R}^+\}$ is enough. Let $c=-1,0,1$ and $\alpha_n=n$ then we have the kernel function of Bernstein polynomials, Szász-Mirakian operators, and Baskakov operators [11].

Now we define Gamma-Radom operators by local averages

$$[M_n^*X](t,\omega,c)=\sum_{k=0}^{+\infty}\langle X(\cdot,\omega),u_{k,n}(\cdot)\rangle b_{k,n}(c,t), \tag{3.7}$$

where $u_{k,n}(t),\ k\in\mathbb{Z}$ satisfy (1.3).

Similarly, let

$$t_{k,n}=\begin{cases}\dfrac{k}{(-c\alpha_n)}, & c<0,\ k=0,1,2,\ldots-c\alpha_n,\\ \dfrac{k}{n}, & c=0,\ k=0,1,2,\ldots,\\ \dfrac{k}{(c\alpha_n)}, & c>0,\ k=0,1,2,\ldots.\end{cases} \tag{3.8}$$

The Nyquist rate is $1/(|c|\alpha_n)$ or $1/n$.

For $c=-1,1$, let $u_{k,n}=\delta(\cdot-k/|c|\alpha_n)$, for $c=0$, let $u_{k,n}=\delta(\cdot-k/n)$, for example, using Dirac-function, then for deterministic signals we have the Bernstein polynomials, Szász-Mirakian operators and Baskakov operators [11]. Let $u_{k,n}$ be a uniform ditributed function on $[k/(n+1),(k+1)/(n+1)]$ or $[k/n,(k+1)/n]$. We can get the BernsteinKantorovich operators, Szász- Kantorovich operators, and Baskakov-Kantorovich operators [11]. For random signals, the following results can be setup.

Corollary 3.1. *For a second order moment processes $X(t),\ t\in[0,D]$ in quadratic mean continuous on $[0,D]$, one has*

$$\lim_{n\to\infty}E[[M_n^*X](t,\omega,c)-X(t)]^2=0, \tag{3.9}$$

where $D=-1/c$ for $c<0$, $D>0$ for $c\geq 0$, and $[M_n^*X](t,\omega,c)$ is defined by (3.7).

Proof. A simple computation shows that for $c=0,\ t\in[0,+\infty)$, we have

$$\begin{aligned}\sum_{k=0}^{+\infty}1\cdot b_{k,n}(0,t)&=1,\\ \sum_{k=0}^{+\infty}\frac{k}{n}\cdot b_{k,n}(0,t)&=t,\\ \sum_{k=0}^{+\infty}\left(\frac{k}{n}\right)^2\cdot b_{k,n}(0,t)&=t^2+\frac{t}{n},\end{aligned} \tag{3.10}$$

and for $c \neq 0,\ t \in [0, +\infty)$, we have

$$\sum_{k=0}^{+\infty} 1 \cdot b_{k,n}(c,t) = 1,$$
$$\sum_{k=0}^{+\infty} \frac{k}{|c| \cdot \alpha_n} \cdot b_{k,n}(c,t) = t, \tag{3.11}$$
$$\sum_{k=0}^{+\infty} \left(\frac{k}{|c| \cdot \alpha_n} \right)^2 \cdot b_{k,n}(c,t) = t^2 + \frac{t(1+ct)}{(|c|n)}.$$

For $0 \leq \delta_1(n),\ \delta_2(n) \leq C_1/n \to 0$ (when $n \to \infty$), $t \in [0, D]$, we have

$$[M_n^* 1](t,c) = 1 + O\left(\frac{1}{n}\right),$$
$$[M_n^* x](t,c) = t + O\left(\frac{1}{n}\right), \tag{3.12}$$
$$\left[M_n^* x^2\right](t,c) = t^2 + O\left(\frac{1}{n}\right).$$

Using Theorem 2.2, we have (3.9).

Obviously, let $c = -1$, $\alpha_n = n$, $u_{k,n} = \delta(\cdot - k/n)$ in Corollary 3.1, we get the first result of Kowalski [1]. □

Acknowledgments

The authors would like to express their sincere gratitude to Professors Liqun Wang, Lixing Han, Wenchang Sun, and Xingwei Zhou for useful suggestions which helped them to improve the paper. This work was partially supported by the National Natural Science Foundation of China (Grant no. 60872161) and the Natural Science Foundation of Tianjin (Grant no. 08JCYBJC09600).

References

[1] E. Kowalski, "Bernstein polynomials and Brownian motion," *American Mathematical Monthly*, vol. 113, no. 10, pp. 865–886, 2006.

[2] T. K. Pogány, "Some Korovkin-type theorems for stochastic processes," *Theory of Probability and Mathematical Statistics*, no. 61, pp. 145–151, 1999.

[3] K. Gröchenig, "Reconstruction algorithms in irregular sampling," *Mathematics of Computation*, vol. 59, no. 199, pp. 181–194, 1992.

[4] A. Aldroubi, "Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces," *Applied and Computational Harmonic Analysis*, vol. 13, no. 2, pp. 151–161, 2002.

[5] F. Marvasti, *Nonuniform Sampling: Theory and Practice*, Information Technology: Transmission, Processing and Storage, Kluwer Academic/Plenum Publishers, New York, NY, USA, 2001.

[6] Z. Song, S. Yang, and X. Zhou, "Approximation of signals from local averages," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1414–1420, 2006.

[7] W. Sun and X. Zhou, "Reconstruction of band-limited signals from local averages," *IEEE Transactions on Information Theory*, vol. 48, no. 11, pp. 2955–2963, 2002.

[8] K. Seip, "A note on sampling of bandlimited stochastic processes," *IEEE Transactions on Information Theory*, vol. 36, no. 5, p. 1186, 1990.

[9] Z. Song, W. Sun, X. Zhou, and Z. Hou, "An average sampling theorem for bandlimited stochastic processes," *IEEE Transactions on Information Theory*, vol. 53, no. 12, pp. 4798–4800, 2007.

[10] P. P. Korovkin, *Linear Operators and Approximation Theory*, Gosizdat Fizmatlit, Moscow, Russia, 1959.

[11] Z. Ditzian and V. Totik, *Moduli of Smoothness*, vol. 9 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1987.