Research Article

New Retarded Integral Inequalities with Applications

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Some new nonlinear integral inequalities of Gronwall type for retarded functions are established, which extend the results Lipovan (2003) and Pachpatte (2004). These inequalities can be used as basic tools in the study of certain classes of functional differential equations as well as integral equations. A existence and a uniqueness on the solution of the functional differential equation involving several retarded arguments with the initial condition are also indicated.

1. Introduction

The celebrated Gronwall inequality [1] states that if u and f are nonnegative continuous functions on the interval $[a,b]$ satisfying

$$u(t) \leq c + \int_a^t f(s)u(s)ds, \quad t \in [a,b], \tag{1.1}$$

for some constant $c \geq 0$, then

$$u(t) \leq c \exp\left(\int_a^t f(s)ds\right), \quad t \in [a,b]. \tag{1.2}$$

Since the inequality (1.2) provides an explicit bound to the unknown function u and hence furnishes a handy tool in the study of solutions of differential equations. Because of its

fundamental importance, several generalizations and analogous results of the inequality (1.2) have been established over years. Such generalizations are, in general, referred to as Gronwall type inequalities [2–6]. These inequalities provide necessary tools in the study of the theory of differential equations, integral equations, and inequalities of various types. Many authors [2–21] have established several other very useful Gronwall-like integral inequalities. Among these inequalities, the following one (Theorem A) due to Ou-Yang [15] needs specific mention. It is useful in the study of boundedness of certain second-order differential equations.

Theorem A (Ou-Yang [15]). *If u and f are nonnegative continuous functions on $[0, \infty)$ such that*

$$u^2(t) \leq u_0^2 + 2\int_0^t f(s)u(s)ds \tag{1.3}$$

for all $t \in [0, \infty)$, where $u_0 \geq 0$ is a constant, then

$$u(t) \leq u_0 + \int_0^t f(s)ds, \quad t \in [0, \infty). \tag{1.4}$$

The Ou-Yang inequality prompted researchers to devote considerable time for its generalization and consequent applications [3, 4, 9, 11, 14]. For instance, Lipovan established the following generalization (Theorem B) of Ou-Yang's inequality in the process of establishing a connection between stability and the second law of thermodynamics [14].

Theorem B (Lipovan [14]). *Let u, f, and g be continuous nonnegative functions on R_+ and c be a nonnegative constant. Also, let $w \in C(R_+, R_+)$ be a nondecreasing function with $w(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(R_+, R_+)$ be nondecreasing with $\alpha(t) \leq t$ on R_+. If*

$$u^2(t) \leq c^2 + 2\int_0^{\alpha(t)} \left[f(s)u(s)w(u(s)) + g(s)u(s)\right]ds \tag{1.5}$$

for all $t \in R_+$, then for $0 \leq t \leq t_1$,

$$u(t) \leq \Omega^{-1}\left[\Omega\left(c + \int_0^{\alpha(t)} g(s)ds\right) + \int_0^{\alpha(t)} f(s)ds\right],$$

$$\Omega(r) = \int_1^r \frac{ds}{w(s)}, \quad r > 0, \tag{1.6}$$

Ω^{-1} *is the inverse function of Ω, and $t_1 \in R_+$ is chosen so that $[\Omega(c + \int_0^{\alpha(t)} g(s)ds) + \int_0^{\alpha(t)} f(s)ds] \in Dom(\Omega^{-1})$ for all $t \in R_+$ lying in the interval $0 \leq t \leq t_1$.*

More recently, Pachpatte established further generalization (Theorem C) of Theorem B as follows [20], which is handy in the study of the global existence of solutions to certain integral equations and functional differential equations.

Theorem C (Pachpatte [20]). *Let $u, a_i, b_i \in C(I, R_+)$ and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$ on I for $i = 1, \ldots, n$. Let $p > 1$ and $c \geq 0$ be constants and $w \in C(R_+, R_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$. If for $t \in I$,*

$$u^p(t) \leq c + p\sum_{i=1}^n \int_{\alpha_i(t_0)}^{\alpha_i(t)} u(s)\left[a_i(s)\psi(u(s)) + b_i(s)\right]ds, \tag{1.7}$$

then for $t_0 \leq t \leq t_1$,

$$u(t) \leq \left[G^{-1}\left(G(A(t)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma) d\sigma \right) \right]^{1/(p-1)}, \tag{1.8}$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s^{1/(p-1)})}, \quad r \geq r_0 > 0,$$

$$A(t) = c^{(p-1)/p} + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(\sigma) d\sigma, \tag{1.9}$$

$r_0 > 0$ is arbitrary, G^{-1} is the inverse function of G and $t_1 \in I$ is so chosen that

$$G(A(t)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma) d\sigma \in \text{Dom}(G^{-1}). \tag{1.10}$$

The present paper establishes some nonlinear retarded inequalities which extend the foregoing theorems. In addition, it illustrates the use/application of these inequalities.

2. Main results

Let R denote the set of real numbers, $R_+ = [0, \infty)$ and $R_1 = [1, \infty)$. Also, let $I = [t_0, T)$ be the given subset of R. Denote by $C^i(M, N)$ the class of all i-times continuously differentiable functions defined on the set M to the set N for $i = 1, 2, \ldots$ and $C^0(M, N) = C(M, N)$.

Theorem 2.1. *Let $u, f_i, g_i \in C(I, R_+)$, $i = 1, \ldots, n$, and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \ldots, n$. Suppose that $c \geq 0$ and $q > 0$ are constants, $\varphi \in C^1(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ on I, and $\psi(u)$ is a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. If*

$$\varphi(u(t)) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) \left[f_i(s) \psi(u(s)) + g_i(s) \right] ds \tag{2.1}$$

for $t \in I$, then

$$u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[\Psi^{-1} \left(\Psi(k(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) ds \right) \right] \right\} \tag{2.2}$$

for $t \in [t_0, t_1)$, where

$$G(r) = \int_{r_0}^{r} \frac{ds}{[\varphi^{-1}(s)]^q}, \quad r \geq r_0 > 0,$$

$$\Psi(r) = \int_{r_0}^{r} \frac{ds}{\psi[\varphi^{-1}(G^{-1}(s))]}, \quad r \geq r_0 > 0, \tag{2.3}$$

$$k(t_0) = G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} g_i(s) ds,$$

G^{-1} and Ψ^{-1} denote the inverse functions of G and Ψ, respectively, for $t \in I. t_1 \in I$ is so chosen that

$$\Psi(k(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)ds \in \text{Dom}(\Psi^{-1}). \tag{2.4}$$

Proof. Assume that $c > 0$. Define a function $z(t)$ by the right-hand side of (2.1). Clearly, $z(t)$ is nondecreasing, $u(t) \leq \varphi^{-1}(z(t))$ for $t \in I$ and $z(t_0) = c$. Differentiating $z(t)$ we get

$$z'(t) = \sum_{i=1}^{n} [u(\alpha_i(t))]^q [f_i(\alpha_i(t))\psi(u(\alpha_i(t))) + g_i(\alpha_i(t))]\alpha_i'(t)$$

$$\leq [\varphi^{-1}(z(t))]^q \sum_{i=1}^{n} [f_i(\alpha_i(t))\psi(\varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t))]\alpha_i'(t). \tag{2.5}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$[\varphi^{-1}(z(t))]^q \geq [\varphi^{-1}(z(t_0))]^q = [\varphi^{-1}(c)]^q > 0. \tag{2.6}$$

That is

$$\frac{z'(t)}{[(\varphi^{-1}(z(t)))]^q} \leq \sum_{i=1}^{n} [f_i(\alpha_i(t))\psi(\varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t))]\alpha_i'(t). \tag{2.7}$$

Setting $t = s$ in the inequality (2.7), integrating it from t_0 to t, using the function G in the left-hand side, and changing variable in the right-hand side, we obtain

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s)\psi(\varphi^{-1}(z(s))) + g_i(s)]ds. \tag{2.8}$$

From the inequality (2.8), we find

$$G(z(t)) \leq p(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\psi(\varphi^{-1}(z(s)))ds, \tag{2.9}$$

where

$$p(t) = G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} g_i(s)ds. \tag{2.10}$$

From the inequality (2.9), we observe that

$$G(z(t)) \leq p(t_1) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\psi(\varphi^{-1}(z(s)))ds, \tag{2.11}$$

for $t \leq t_1$. Now, define a function $k(t)$ by the right-hand side of (2.11). Clearly, $k(t)$ is nondecreasing, $z(t) \leq G^{-1}(k(t))$ for $t \in I$ and $k(t_0) = p(t_1)$. Differentiating $k(t)$, we get

$$k'(t) = \sum_{i=1}^{n} [f_i(\alpha_i(t))\psi(\varphi^{-1}(z(\alpha_i(t))))]\alpha_i'(t) \leq \psi(\varphi^{-1}(G^{-1}(k(t)))) \sum_{i=1}^{n} [f_i(\alpha_i(t))]\alpha_i'(t). \tag{2.12}$$

Using the monotonicity of ψ, φ^{-1}, G^{-1}, and k, we deduce

$$\frac{k'(t)}{\psi(\varphi^{-1}(G^{-1}(k(t))))} \leq \sum_{i=1}^{n} \left[f_i(\alpha_i(t))\right]\alpha_i'(t). \tag{2.13}$$

Setting $t = s$ in the inequality (2.13), integrating it from t_0 to t, using the function Ψ in the left-hand side, and changing variable in the right-hand side, we obtain

$$\Psi(k(t)) \leq \Psi(k(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)ds. \tag{2.14}$$

From the inequalities (2.11) and (2.14), we conclude that

$$z(t) \leq G^{-1}\left[\Psi^{-1}\left(\Psi(p(t_1)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)ds\right)\right] \tag{2.15}$$

for $t_0 \leq t \leq t_1$. Now a combination of $u(t) \leq \varphi^{-1}(z(t))$ and the last inequality in (2.15) for $t_1 = t$ produces the required inequality.

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\varphi(u) = u^p$ ($p > q > 0$ is a constant), Theorem 2.1 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.2. *Let $u, f_i, g_i \in C(I, R_+)$, $i = 1, \ldots, n$, and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \ldots, n$. Suppose that $c \geq 0$ and $p > q > 0$ are constants, and $\psi(u)$ is a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. If*

$$u^p(t) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)\left[f_i(s)\psi(u(s)) + g_i(s)\right]ds \tag{2.16}$$

for $t \in I$, then

$$u(t) \leq \left[\Psi_0^{-1}\left(\Psi_0(k_1(t_0)) + \frac{p-q}{p}\sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)ds\right)\right]^{1/(p-q)} \tag{2.17}$$

for $t \in [t_0, \bar{t})$, where

$$\Psi_0(r) = \int_{r_0}^{r} \frac{ds}{\psi(s^{1/(p-q)})}, \quad r \geq r_0 > 0,$$

$$k_1(t_0) = c^{(p-q)/p} + \frac{p-q}{p}\sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} g_i(s)ds, \tag{2.18}$$

Ψ_0^{-1} denotes the inverse function of Ψ_0 for $t \in I.\bar{t} \in I$ is so chosen that

$$\Psi_0(k_1(t_0)) + \frac{p-q}{p}\sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)ds \in \mathrm{Dom}(\Psi_0^{-1}). \tag{2.19}$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.1 with suitable modification. We omit the details here. □

Remark 2.3. When $q = 1$, from Corollary 2.2, one derives Theorem C. When $p = 2$, $q = 1$, from Corollary 2.2, one derives Theorem B.

Theorem 2.1 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, one has the following result (Theorem 2.4).

Theorem 2.4. *Let $u \in C(I, R_1)$, $f_i, g_i \in C(I, R_+)$, $i = 1, \ldots, n$, and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \ldots, n$. Suppose that $c \geq 1$ is a constant, $\varphi \in C^1(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ and $\psi_j(u)$, $j = 1, 2$ are nondecreasing continuous functions for $u \in R_+$ with $\psi_j(u) > 0$ for $u > 0$. If*

$$\varphi(u(t)) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) \big[f_i(s)\psi_1(u(s)) + g_i(s)\psi_2(\log(u(s))) \big] ds \tag{2.20}$$

for $t \in I$, then

(i) *as the case $\psi_1(u) \geq \psi_2(\log(u))$,*

$$u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[\Psi_1^{-1} \left(\Psi_1(G(c)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \right) \right] \right\} \tag{2.21}$$

for $t \in [t_0, t_1)$,

(ii) *as the case $\psi_1(u) < \psi_2(\log(u))$,*

$$u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[\Psi_2^{-1} \left(\Psi_2(G(c)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \right) \right] \right\} \tag{2.22}$$

for $t \in [t_0, t_2)$, where

$$\Psi_j(r) = \int_{r_0}^{r} \frac{ds}{\psi_j(\varphi^{-1}(G^{-1}(s)))}, \quad r \geq r_0 > 0, \tag{2.23}$$

G^{-1}, Ψ_j^{-1}, $j = 1, 2$, denote the inverse functions of G, Ψ_j, $j = 1, 2$, respectively, the function $G(t)$ is as defined in Theorem 2.1 for $t \in I$, and $t_j \in I$, $j = 1, 2$ are so chosen that

$$\Psi_j(G(c)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \in \text{Dom}(\Psi_j^{-1}). \tag{2.24}$$

Proof. Let $c > 0$. Define a function $z(t)$ by the right-hand side of (2.20). Clearly, $z(t)$ is nondecreasing, $u(t) \leq \varphi^{-1}(z(t))$ for $t \in I$ and $z(t_0) = c$. Differentiating $z(t)$, we get

$$z'(t) = \sum_{i=1}^{n} [u(\alpha_i(t))]^q [f_i(\alpha_i(t))\psi_1(u(\alpha_i(t))) + g_i(\alpha_i(t))\psi_2(\log(u(\alpha_i(t))))] \alpha_i'(t)$$

$$\leq [\varphi^{-1}(z(t))]^q \sum_{i=1}^{n} [f_i(\alpha_i(t))\psi_1(\varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t))\psi_2(\log(\varphi^{-1}(z(\alpha_i(t)))))] \alpha_i'(t). \tag{2.25}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\left[\varphi^{-1}(z(t))\right]^q \geq \left[\varphi^{-1}(z(t_0))\right]^q = \left[\varphi^{-1}(c)\right]^q > 0. \tag{2.26}$$

That is

$$\frac{z'(t)}{\left[(\varphi^{-1}(z(t)))\right]^q} \leq \sum_{i=1}^{n} \left[f_i(\alpha_i(t))\psi_1(\varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t))\psi_2(\log(\varphi^{-1}(z(\alpha_i(t)))))\right]\alpha_i'(t). \tag{2.27}$$

Setting $t = s$ in the inequality (2.27), integrating it from t_0 to t, using the function G in the left-hand side, and changing variable in the right-hand side, we obtain

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s)\psi_1(\varphi^{-1}(z(s))) + g_i(s)\psi_2(\log(\varphi^{-1}(z(s))))\right]ds. \tag{2.28}$$

When $\psi_1(u) \geq \psi_2(\log(u))$, from the inequality (2.28), we find

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s) + g_i(s)\right]\psi_1(\varphi^{-1}(z(s)))ds. \tag{2.29}$$

Now, define a function $k(t)$ by the right-hand side of (2.29). Clearly, $k(t)$ is nondecreasing, $z(t) \leq G^{-1}(k(t))$ for $t \in I$ and $k(t_0) = G(c)$. Differentiating $k(t)$, we get

$$\begin{aligned} k'(t) &= \sum_{i=1}^{n} \left[f_i(\alpha_i(t)) + g_i(\alpha_i(t))\right]\psi_1(\varphi^{-1}(z(\alpha_i(t))))\alpha_i'(t) \\ &\leq \psi_1(\varphi^{-1}(G^{-1}(k(t))))\sum_{i=1}^{n} \left[f_i(\alpha_i(t)) + g_i(\alpha_i(t))\right]\alpha_i'(t). \end{aligned} \tag{2.30}$$

Using the monotonicity of ψ_1, φ^{-1}, G^{-1}, and k, we deduce

$$\frac{k'(t)}{\psi_1(\varphi^{-1}(G^{-1}(k(t))))} \leq \sum_{i=1}^{n} \left[f_i(\alpha_i(t))\right]\alpha_i'(t). \tag{2.31}$$

Setting $t = s$ in the inequality (2.31), integrating it from t_0 to t, using the function Ψ_1 in the left-hand side, and changing variable in the right-hand side, we obtain

$$\Psi_1(k(t)) \leq \Psi_1(k(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s) + g_i(s)\right]ds. \tag{2.32}$$

From the inequality (2.32), we conclude that

$$z(t) \leq G^{-1}\left[\Psi_1^{-1}\left(\Psi_1(G(c)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s) + g_i(s)\right]ds\right)\right] \tag{2.33}$$

for $t \in I$. Now a combination of $u(t) \leq \varphi^{-1}(z(t))$ and the last inequality produces the required inequality in (2.21).

When $\psi_1(u) < \psi_1(\log(u))$, from the inequality (2.28), we find

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] \psi_1(\varphi^{-1}(z(s))) ds. \tag{2.34}$$

Now, by a suitable application of the process from (2.29) to (2.32) in the inequality (2.34), we conclude that

$$z(t) \leq G^{-1}\left[\Psi_2^{-1}\left(\Psi_2(G(c)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \right) \right] \tag{2.35}$$

for $t \in I$. Now a combination of $u(t) \leq \varphi^{-1}(z(t))$ and the last inequality produces the required inequality in (2.22).

If $c = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\varphi(u) = u^p$ ($p > q > 0$ is a constant), Theorem 2.4 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.5. *Let $u \in C(I, R_1)$, $f_i, g_i \in C(I, R_+)$, $i = 1, \ldots, n$, and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \ldots, n$. Suppose that $c \geq 0$ and $p > q > 0$ are constants, and $\psi_j(u)$, $j = 1, 2$ are nondecreasing continuous functions for $u \in R_+$ with $\psi_j(u) > 0$ for $u > 0$. If*

$$u^p(t) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) [f_i(s)\psi_1(u(s)) + g_i(s)\psi_2(\log u(s))] ds \tag{2.36}$$

for $t \in I$, then

(i) *as the case $\psi_1(u) \geq \psi_2(\log(u))$,*

$$u(t) \leq \left[G_1^{-1}\left(G_1(c^{(p-q)/p}) + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \right) \right]^{1/(p-q)} \tag{2.37}$$

for $t \in [t_0, t_1)$,

(ii) *as the case $\psi_1(u) < \psi_2(\log(u))$,*

$$u(t) \leq \left[G_2^{-1}\left(G_2(c^{(p-q)/p}) + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \right) \right]^{1/(p-q)} \tag{2.38}$$

for $t \in [t_0, t_2)$, where G_j^{-1}, $j = 1, 2$, denote the inverse functions of G_j, $j = 1, 2$,

$$G_j(r) = \int_{r_0}^{r} \frac{ds}{\psi_j(s^{1/(p-q)})}, \quad r \geq r_0 > 0, \tag{2.39}$$

for $t \in I$, and $t_j \in I$, $j = 1, 2$, are chosen so that

$$G_j(c^{(p-q)/p}) + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) + g_i(s)] ds \in \mathrm{Dom}(G_j^{-1}). \tag{2.40}$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.4 with suitable modification. We omit the details here. □

Theorem 2.1 can easily be applied to generate another useful nonlinear integral inequalities in more general situations. For example, we have the following result (Theorem 2.6).

Theorem 2.6. *Let $u, f_i, g_i \in C(I, R_+)$, $i = 1, \ldots, n$, and let $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \ldots, n$. Suppose that $c \geq 0$ and $q > 0$ are constants, $\varphi \in C^1(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ on I, and $L, M \in C(R_+^2, R_+)$ satisfy*

$$0 \leq L(t, v) - L(t, w) \leq M(t, w)(v - w) \tag{2.41}$$

for $t, v, w \in R_+$ with $v \geq w \geq 0$. If

$$\varphi(u(t)) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) \big[f_i(s) L(s, u(s)) + g_i(s) u(s) \big] ds \tag{2.42}$$

for $t \in I$, then

$$u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[\Omega^{-1} \left(\Omega(k_2(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \big[f_i(s) M(s) + g_i(s) \big] ds \right) \right] \right\} \tag{2.43}$$

for $t \in [t_0, t_1)$, where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(G^{-1}(s))}, \quad r \geq r_0 > 0,$$

$$k_2(t_0) = G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) L(s) ds, \tag{2.44}$$

G^{-1} and Ω^{-1} denote the inverse function of G and Ω, respectively, the function G is as defined in Theorem 2.1 for $t \in I$ and $t_1 \in I$ is so chosen that

$$\Omega(k_2(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) ds \in \mathrm{Dom}(\Omega^{-1}). \tag{2.45}$$

Proof. Let $c > 0$. Define a function $z(t)$ by the right-hand side of (2.42). Clearly, $z(t)$ is nondecreasing, $u(t) \leq \varphi^{-1}(z(t))$ for $t \in I$ and $z(t_0) = c$. Differentiating $z(t)$, we get

$$z'(t) = \sum_{i=1}^{n} \big[u(\alpha_i(t)) \big]^q \big[f_i(\alpha_i(t)) L(\alpha_i(t), u(\alpha_i(t))) + g_i(\alpha_i(t)) u(\alpha_i(t)) \big] \alpha_i'(t)$$

$$\leq \big[\varphi^{-1}(z(t)) \big]^q \sum_{i=1}^{n} \big[f_i(\alpha_i(t)) L(\alpha_i(t), \varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t)) \varphi^{-1}(z(\alpha_i(t))) \big] \alpha_i'(t). \tag{2.46}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\frac{z'(t)}{\big[(\varphi^{-1}(z(t))) \big]^q} \leq \sum_{i=1}^{n} \big[f_i(\alpha_i(t)) L(\alpha_i(t), \varphi^{-1}(z(\alpha_i(t)))) + g_i(\alpha_i(t)) \varphi^{-1}(z(\alpha_i(t))) \big] \alpha_i'(t). \tag{2.47}$$

Setting $t = s$ in the inequality (2.47), integrating it from t_0 to t, using the function G in the left-hand side, and changing variable in the right-hand side, we obtain

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s)L(s, \varphi^{-1}(z(s))) + g_i(s)\varphi^{-1}(z(s)) \right] ds. \tag{2.48}$$

From the inequalities (2.41), (2.48), we find

$$G(z(t)) \leq G(c) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t_1)} f_i(s)L(s)\, ds + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s)M(s) + g_i(s) \right] \varphi^{-1}(z(s))\, ds, \tag{2.49}$$

for $t \leq t_1$. Now, define a function $k_2(t)$ by the right-hand side of (2.49). Clearly, $k_2(t)$ is nondecreasing, $z(t) \leq G^{-1}(k_2(t))$ for $t \in I$. Differentiating $k_2(t)$, we get

$$
\begin{aligned}
k_2'(t) &= \sum_{i=1}^{n} \left[f_i(\alpha_i(t))M(\alpha_i(t)) + g_i(\alpha_i(t)) \right] \varphi^{-1}(z(s))\alpha_i'(t) \\
&\leq \varphi^{-1}(G^{-1}(k_2(t))) \sum_{i=1}^{n} \left[f_i(\alpha_i(t))M(\alpha_i(t)) + g_i(\alpha_i(t)) \right] \alpha_i'(t).
\end{aligned}
\tag{2.50}
$$

Using the monotonicity of φ^{-1}, G^{-1}, and k_2, we deduce

$$\frac{k_2'(t)}{\psi(\varphi^{-1}(G^{-1}(k_2(t))))} \leq \sum_{i=1}^{n} \left[f_i(\alpha_i(t))M(\alpha_i(t)) + g_i(\alpha_i(t)) \right] \alpha_i'(t). \tag{2.51}$$

Setting $t = s$ in the inequality (2.51), integrating it from t_0 to t, using the function Ω in the left-hand side, and changing variable in the right-hand side, we obtain

$$\Omega(k_2(t)) \leq \Omega(k_2(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s)M(s) + g_i(s) \right] ds. \tag{2.52}$$

From the inequalities (2.49) and (2.52), we conclude that

$$z(t) \leq G^{-1} \left[\Omega^{-1} \left(\Omega(k_2(t_0)) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \left[f_i(s)M(s) + g_i(s) \right] ds \right) \right] \tag{2.53}$$

for $t_0 \leq t \leq t_1$. Now a combination of $u(t) \leq \varphi^{-1}(z(t))$ and the last inequality produces the required inequality in (2.43) for $t_1 = t$.

If $c = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\varphi(u) = u^p$ ($p > q > 0$ is a constant), Theorem 2.6 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.7. *Let u, f_i, g_i, and α_i be as defined in Theorem 2.6. Suppose that $c \geq 0$ and $p > q > 0$ are constants, and $L, M \in C(R_+^2, R_+)$ satisfy*

$$0 \leq L(t, v) - L(t, w) \leq M(t, w)(v - w) \tag{2.54}$$

for $t, v, w \in R_+$ with $v \geq w \geq 0$. If

$$u^p(t) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) [f_i(s) L(s, u(s)) + g_i(s) u(s)] ds \tag{2.55}$$

for $t \in I$, then

$$u(t) \leq \left[\Omega_1^{-1} \left(\Omega_1(k_3(t_0)) + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) M(s) + g_i(s)] ds \right) \right]^{1/(p-q)} \tag{2.56}$$

for $t \in [t_0, t_1)$, where

$$\Omega_1(r) = \int_{r_0}^{r} \frac{ds}{s^{1/(p-q)}}, \quad r \geq r_0 > 0,$$

$$k_3(t_0) = c^{(p-q)/p} + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) L(s) ds, \tag{2.57}$$

Ω_1^{-1} denotes the inverse function of Ω_1 for $t \in I$ and $t_1 \in I$ is so chosen that

$$\Omega_1(k_3(t_0)) + \frac{p-q}{p} \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [f_i(s) M(s) + g_i(s)] ds \in \mathrm{Dom}(\Omega_1^{-1}). \tag{2.58}$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.6 with suitable modification. We omit the details here. □

3. Applications

We will show that our results are useful in proving the global existence of solutions to certain differential equations with time delay. Consider the functional differential equation involving several retarded arguments with the initial condition:

$$\varphi'(x(t)) x'(t) = F(t, x(t - h_1(t)), \ldots, x(t - h_n(t))), \quad t \in I,$$

$$x(t_0) = x_0, \tag{3.1}$$

where x_0 is a constant, $F \in C(I \times R^n, R)$, $h_i \in C(I, R_+)$, $i = 1, \ldots, n$ be nonincreasing such that $t - h_i(t) \geq 0$, $t - h_i(t) \in C^1(I, I)$, $h_i'(t) < 1$, and $\varphi \in C^1(R, R)$ is an increasing function with $\varphi(|x|) \leq |\varphi(x)|$. The following theorem deals with a bound on the solution of the problem (3.1).

Theorem 3.1. *Assume that $F : I \times R^n \to R$ is a continuous function for which there exist continuous nonnegative functions $f_i(t)$, $g_i(t)$, $i = 1, \ldots, n$ for $t \in I$ such that*

$$|F(t, u_1, \ldots, u_n)| \leq \sum_{i=1}^{n} |u_i|^q [f_i(t) \psi(|u_i|) + g_i(t)], \tag{3.2}$$

where $q > 0$ is a constant and ψ is as in Theorem 2.1. Let

$$Q_i = \max_{t \in I} \frac{1}{1 - h_i'(t)}, \quad i = 1, \ldots, n. \tag{3.3}$$

If $x(t)$ is any solution of the problem (3.1), then

$$|x(t)| \le \varphi^{-1}\left\{ G^{-1}\left[\Psi^{-1}\left(\Psi(\overline{k}(t_0)) + \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} \overline{f_i}(\sigma)d\sigma \right)\right]\right\} \tag{3.4}$$

for $t \in I$, where G, Ψ are as in Theorem 2.1 and

$$\overline{k}(t_0) = G(|\varphi(x_0)|) + \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} \overline{g_i}(\sigma)d\sigma, \tag{3.5}$$

$\overline{f_i}(\sigma) = Q_i f_i(\sigma + h_i(s))$, $\overline{g_i}(\sigma) = Q_i g_i(\sigma + h_i(s))$ *for $s, \sigma \in I$.*

Proof. It is easy to see that the solution $x(t)$ of the problem (3.1) satisfies the equivalent integral equation:

$$\varphi(x(t)) = \varphi(x(t_0)) + \int_{t_0}^{t} F(s, x(s - h_1(s)), \ldots, x(s - h_n(s)))ds. \tag{3.6}$$

From (3.2) and (3.6), we have

$$\begin{aligned}|\varphi(x(t))| &\le |\varphi(x(t_0))| + \int_{t_0}^{t} |F(s, x(s - h_1(s)), \ldots, x(s - h_n(s)))|ds \\ &\le |\varphi(x_0)| + \int_{t_0}^{t} \sum_{i=1}^{n} |x(s - h_i(s))|^q [f_i(t)\psi(|x(s - h_i(s))|) + g_i(t)]ds \end{aligned} \tag{3.7}$$

for $t, s \in I$. By making the change of variables on the right side of the inequality of (3.7) and rewriting, we have

$$\varphi(|x(t)|) \le |\varphi(x_0)| + \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} |x(\sigma)|^q [\overline{f_i}(\sigma)\psi(|x(\sigma)|) + \overline{g_i}(\sigma)]d\sigma, \tag{3.8}$$

where $\overline{f_i}(\sigma) = Q_i f_i(\sigma + h_i(s))$, $\overline{g_i}(\sigma) = Q_i g_i(\sigma + h_i(s))$ for $s, \sigma \in I$. Now an immediate application of the inequality established in Theorem 2.1 to the inequality (3.8) yields the result. $\square$

Remark 3.2. Consider the functional differential equation involving several retarded arguments with the initial condition:

$$px^{p-1}(t)x'(t) = F(t, x(t - h_1(t)), \ldots, x(t - h_n(t))), \quad t \in I,$$
$$x(t_0) = x_0,$$

(3.9)

where $p > 0$ and x_0 are constants, $F \in C(I \times R^n, R)$, $h_i \in C(I, R_+)$, $i = 1, \ldots, n$ be nonincreasing such that $t - h_i(t) \geq 0$, $t - h_i(t) \in C^1(I, I)$, $h_i'(t) < 1$.

Assume that $F : I \times R^n \to R$ is a continuous function for which there exist continuous nonnegative functions $f_i(t)$, $g_i(t)$, $i = 1, \ldots, n$ for $t \in I$ such that

$$\left| F(t, u_1, \ldots, u_n) \right| \leq \sum_{i=1}^{n} |u_i|^q \left[f_i(t)\psi(|u_i|) + g_i(t) \right],$$

(3.10)

where $q > 0 \, (p > q)$ is a constant and ψ is as in Theorem 2.1. If $x(t)$ is any solution of the problem (3.9), then it satisfies the equivalent integral equation:

$$x^p(t) = x^p(t_0) + \int_{t_0}^{t} F(s, x(s - h_1(s)), \ldots, x(s - h_n(s))) ds.$$

(3.11)

From (3.10) and (3.11), one has

$$|x(t)|^p \leq |x_0|^p + \int_{t_0}^{t} \left| F(s, x(s - h_1(s)), \ldots, x(s - h_n(s))) \right| ds$$
$$\leq |x_0|^p + \int_{t_0}^{t} \sum_{i=1}^{n} |x(s - h_i(s))|^q \left[f_i(t)\psi(|x(s - h_i(s))|) + g_i(t) \right] ds$$

(3.12)

for $t, s \in I$. By making the change of variables on the right side of (3.12) and rewriting, one has

$$|x(t)|^p \leq |x_0|^p + \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} |x(\sigma)|^q \left[\overline{f_i}(\sigma)\psi(|x(\sigma)|) + \overline{g_i}(\sigma) \right] d\sigma,$$

(3.13)

where $\overline{f_i}(\sigma) = Q_i f_i(\sigma + h_i(s))$, $\overline{g_i}(\sigma) = Q_i g_i(\sigma + h_i(s))$ for $s, \sigma \in I$. Now an immediate application of the inequality established in Corollary 2.2 to the inequality (3.13) yields

$$|x(t)| \leq \left[\Psi_0^{-1}\left(\Psi_0(\overline{k_1}(t_0)) + \frac{p - q}{p} \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} \overline{f_i}(\sigma) d\sigma \right) \right]^{1/(p-q)}$$

(3.14)

for $t \in I$, where Ψ_0 is as in Corollary 2.2,

$$\overline{k_1}(t_0) = x_0^{p-q} + \frac{p - q}{p} \sum_{i=1}^{n} \int_{t_0 - h_i(t_0)}^{t - h_i(t)} \overline{g_i}(\sigma) d\sigma,$$

(3.15)

$\overline{f_i}(\sigma) = Q_i f_i(\sigma + h_i(s))$, and $\overline{g_i}(\sigma) = Q_i g_i(\sigma + h_i(s))$ for $s, \sigma \in I$.

The following theorem provides a uniqueness on the solution of the problem (3.9).

Theorem 3.3. *Assume that $F : I \times R^3 \to R$ is a continuous function for which there exist continuous nonnegative functions $f_i(t)$, $i = 1, \ldots, n$ for $t \in I$ such that*

$$\left| F(t, u_1, \ldots, u_n) - F(t, v_1, \ldots, v_n) \right| \leq \sum_{i=1}^{n} f_i(t) \left| u_i^p - v_i^p \right|, \tag{3.16}$$

where $p > 1$ is a constant, then the problem (3.9) has at most one solution on I.

Proof. Let $x(t)$ and $\overline{x}(t)$ be two distinct solutions of the problem (3.9), one has

$$x^p(t) - \overline{x}^p(t) = \int_{t_0}^{t} \left[F(s, x(s - h_1(s)), \ldots, x(s - h_n(s))) - F(s, \overline{x}(s - h_1(s)), \ldots, \overline{x}(s - h_n(s))) \right] ds. \tag{3.17}$$

From (3.16) and (3.17), one finds

$$\left| x^p(t) - \overline{x}^p(t) \right| \leq \int_{t_0}^{t} \left(\sum_{i=1}^{n} f_i(s) \left| x^p(s - h_i(s)) - \overline{x}^p(s - h_i(s)) \right| \right) ds \tag{3.18}$$

for $t, s \in I$. By making the change of variables on the right side of (3.18) and rewriting, one has

$$\left(\left| x^p(t) - \overline{x}^p(t) \right|^{1/p} \right)^p \leq \sum_{i=1}^{n} \int_{\beta_i(t_0)}^{\beta_i(t)} \left[\left| x^p(\sigma) - \overline{x}^p(\sigma) \right|^{1/p} \right]^{p-1} \overline{f_i}(\sigma) \left[\left| x^p(\sigma) - \overline{x}^p(\sigma) \right|^{1/p} \right] d\sigma, \tag{3.19}$$

where $\beta_i(t) = t - h_i(t)$, $\overline{f_i}(\sigma) = Q_i f_i(\sigma + h_i(s))$ for $s, \sigma \in I$. Now when $\psi(u) = u$, $q = p - 1$, a suitable application of the inequality in Corollary 2.2 to the function $\left| x^p(t) - \overline{x}^p(t) \right|^{1/p}$ and the inequality (3.19), one concludes that

$$\left| x^p(t) - \overline{x}^p(t) \right|^{1/p} \leq 0 \tag{3.20}$$

for all $t \in I$. Hence $x(t) = \overline{x}(t)$. $\qquad\square$

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