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*Research Article*

# On the Cauchy Functional Inequality in Banach Modules

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We investigate the following functional inequality: $\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\|$ in Banach modules over a $C^*$-algebra, and prove the generalized Hyers-Ulam stability of linear mappings in Banach modules over a $C^*$-algebra.



## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let $X$ and $Y$ be Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. The result of Găvruţa [5] is a special case of a more general theorem, which was obtained by Forti [6].

Th. M. Rassias [7] during the 27th international symposium on functional equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [8], following the same approach as in Th. M. Rassias [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [8], as well as by Th. M. Rassias and Šemrl [9] that one cannot prove a Th. M. Rassias'-type theorem when $p = 1$.

J. M. Rassias [10] followed the innovative approach of Th. M. Rassias' theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [11–21]).

Gilányi [22] showed that if $f$ satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|, \tag{1.1}$$

then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y). \tag{1.2}$$

See also [23]. Fechner [24] and Gilányi [25] proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park et al. [19] investigated the functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\| \tag{1.3}$$

in Banach spaces, and proved the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces.

Throughout this paper, let $A$ be a unital $C^*$-algebra with unitary group $U(A)$ and unit $e$. Assume that $X$ is a Banach $A$-module with norm $\|\cdot\|_X$ and that $Y$ is a Banach $A$-module with norm $\|\cdot\|_Y$.

In this paper, we investigate an $A$-linear mapping associated with the functional inequality (1.3) and prove the generalized Hyers-Ulam stability of $A$-linear mappings in Banach $A$-modules associated with the functional inequality (1.3).

The computations in the proofs of the main theorems are special cases of the general results obtained by Forti [26].

## 2. Functional inequalities in Banach modules over a $C^*$-algebra

**Lemma 2.1.** *Let $f: X \to Y$ be a mapping such that*

$$\|f(x)+f(y)+uf(z)\|_Y \le \|f(x+y+uz)\|_Y \tag{2.1}$$

*for all $x, y, z \in X$ and all $u \in U(A)$. Then $f$ is $A$-linear.*

*Proof.* Letting $x=y=z=0$ and $u=e\in U(A)$ in (2.1), we get

$$\|3f(0)\|_Y \le \|f(0)\|_Y. \tag{2.2}$$

So, $f(0)=0$.

Letting $z=0$ and $y=-x$ in (2.1), we get

$$\|f(x)+f(-x)\|_Y \le \|f(0)\|_Y = 0 \tag{2.3}$$

for all $x \in X$. Hence $f(-x)=-f(x)$ for all $x \in X$.

Letting $z=-x-y$ and $u=e\in U(A)$ in (2.1), we get

$$\|f(x)+f(y)-f(x+y)\|_Y = \|f(x)+f(y)+f(-x-y)\|_Y \le \|f(0)\|_Y = 0 \tag{2.4}$$

for all $x, y \in X$. Thus,

$$f(x+y)=f(x)+f(y) \tag{2.5}$$

for all $x, y \in X$.

Letting $x = -uz$ and $y = 0$ in (2.1), we get

$$\|-f(uz) + uf(z)\|_Y = \|f(-uz) + uf(z)\|_Y \leq \|f(0)\|_Y = 0 \tag{2.6}$$

for all $z \in X$ and all $u \in U(A)$. Thus,

$$f(uz) = uf(z) \tag{2.7}$$

for all $u \in U(A)$ and all $z \in X$.

Now let $a \in A$ $(a \neq 0)$ and $M$ an integer greater than $4|a|$. Then $|a/M| < 1/4 < 1 - 2/3 = 1/3$. By [27, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3(a/M) = u_1 + u_2 + u_3$. So by (2.7)

$$\begin{aligned} f(ax) &= f\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot f\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3} f\left(3\frac{a}{M}x\right) = \frac{M}{3} f(u_1 x + u_2 x + u_3 x) \\ &= \frac{M}{3}\left(f(u_1 x) + f(u_2 x) + f(u_3 x)\right) = \frac{M}{3}(u_1 + u_2 + u_3) f(x) = \frac{M}{3} \cdot 3\frac{a}{M} f(x) = af(x) \end{aligned} \tag{2.8}$$

for all $x \in X$. So, $f : X \to Y$ is $A$-linear, as desired. □

Now, we prove the generalized Hyers-Ulam stability of $A$-linear mappings in Banach $A$-modules.

**Theorem 2.2.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + uf(z)\|_Y \leq \|f(x + y + uz)\|_Y + \theta\left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \tag{2.9}$$

*for all $x, y, z \in X$ and all $u \in U(A)$. Then, there exists a unique $A$-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2^r + 2}{2^r - 2}\theta\|x\|_X^r \tag{2.10}$$

*for all $x \in X$.*

*Proof.* Since $f$ is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$.

Letting $u = e \in U(A)$, $y = x$ and $z = -2x$ in (2.9), we get

$$\|2f(x) - f(2x)\|_Y = \|2f(x) + f(-2x)\|_Y \leq (2 + 2^r)\theta\|x\|_X^r \tag{2.11}$$

for all $x \in X$. So,

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \leq \frac{2 + 2^r}{2^r}\theta\|x\|_X^r \tag{2.12}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_Y \\ &\le \frac{2+2^r}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|_X^r\end{aligned} \tag{2.13}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.14}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.10).

It follows from (2.9) that

$$\begin{aligned}\|L(x) + L(y) + uL(z)\|_Y &= \lim_{n\to\infty} 2^n\left\|f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + uf\left(\frac{z}{2^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} 2^n\left\|f\left(\frac{x+y+uz}{2^n}\right)\right\|_Y + \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \\ &= \|L(x+y+uz)\|_Y\end{aligned} \tag{2.15}$$

for all $x, y, z \in X$ and all $u \in U(A)$. So,

$$\|L(x) + L(y) + uL(z)\|_Y \le \|L(x+y+uz)\|_Y \tag{2.16}$$

for all $x, y, z \in X$ and all $u \in U(A)$. By Lemma 2.1, the mapping $L : X \to Y$ is $A$-linear.

Now, let $T : X \to Y$ be another $A$-linear mapping satisfying (2.10). Then, we have

$$\begin{aligned}\|L(x) - T(x)\|_Y &= 2^n\left\|L\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y \\ &\le 2^n\left(\left\|L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y\right) \\ &\le \frac{2(2^r+2)2^n}{(2^r-2)2^{nr}}\theta\|x\|_X^r,\end{aligned} \tag{2.17}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $L$. Thus, the mapping $L : X \to Y$ is a unique $A$-linear mapping satisfying (2.10). □

**Theorem 2.3.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.9). *Then, there exists a unique $A$-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2+2^r}{2-2^r}\theta\|x\|_X^r \tag{2.18}$$

*for all $x \in X$.*

*Proof.* It follows from (2.11) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{2+2^r}{2}\theta\|x\|_X^r \tag{2.19}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|_Y &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_Y \\ &\leq \frac{2+2^r}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|_X^r\end{aligned} \tag{2.20}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{(1/2^n)f(2^n x)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{(1/2^n)f(2^n x)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right) \tag{2.21}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.18).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Theorem 2.4.** *Let $r > 1/3$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + uf(z)\|_Y \leq \|f(x+y+uz)\|_Y + \theta\cdot\|x\|_X^r\cdot\|y\|_X^r\cdot\|z\|_X^r \tag{2.22}$$

*for all $x, y, z \in X$ and all $u \in U(A)$. Then, there exists a unique $A$-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2^r\theta}{8^r - 2}\|x\|_X^{3r} \tag{2.23}$$

*for all $x \in X$.*

*Proof.* Since $f$ is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$.

Letting $u = e \in U(A)$, $y = x$ and $z = -2x$ in (2.22), we get

$$\|2f(x) - f(2x)\|_Y = \|2f(x) + f(-2x)\|_Y \leq 2^r\theta\|x\|_X^{3r} \tag{2.24}$$

for all $x \in X$. So,

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \leq \frac{2^r}{8^r}\theta\|x\|_X^{3r} \tag{2.25}$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_Y &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_Y \\ &\le \frac{2^r}{8^r} \sum_{j=l}^{m-1} \frac{2^j}{8^{rj}} \theta \|x\|_X^{3r} \end{aligned} \tag{2.26}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.26) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.27}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.26), we get (2.23).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Theorem 2.5.** *Let $r < 1/3$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.22). *Then, there exists a unique $A$-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{2^r \theta}{2 - 8^r} \|x\|_X^{3r} \tag{2.28}$$

*for all $x \in X$.*

*Proof.* It follows from (2.24) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_Y \le \frac{2^r}{2} \theta \|x\|_X^{3r} \tag{2.29}$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_Y &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|_Y \\ &\le \frac{2^r}{2} \sum_{j=l}^{m-1} \frac{8^{rj}}{2^j} \theta \|x\|_X^{3r} \end{aligned} \tag{2.30}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.30) that the sequence $\{(1/2^n) f(2^n x)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{(1/2^n) f(2^n x)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \tag{2.31}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.30), we get (2.28).

The rest of the proof is similar to the proof of Theorem 2.2. □

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## References

[1] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[6] G. L. Forti, "An existence and stability theorem for a class of functional equations," *Stochastica*, vol. 4, no. 1, pp. 23–30, 1980.

[7] Th. M. Rassias, "Problem 16; 2, Report of the 27th International Symposium on Functional Equations," *Aequationes Mathematicae*, vol. 39, no. 2-3, pp. 292–293, 1990.

[8] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[9] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[10] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[11] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser Boston, Boston, Mass, USA, 1998.

[13] K.-W. Jun and Y.-H. Lee, "A generalization of the Hyers-Ulam-Rassias stability of the pexiderized quadratic equations," *Journal of Mathematical Analysis and Applications*, vol. 297, no. 1, pp. 70–86, 2004.

[14] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[15] C. Park, "On the stability of the linear mapping in Banach modules," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 711–720, 2002.

[16] C. Park, "Lie $*$-homomorphisms between Lie $C^*$-algebras and Lie $*$-derivations on Lie $C^*$-algebras," *Journal of Mathematical Analysis and Applications*, vol. 293, no. 2, pp. 419–434, 2004.

[17] C. Park, "Homomorphisms between Poisson $JC^*$-algebra," *Bulletin of the Brazilian Mathematical Society*, vol. 36, no. 1, pp. 79–97, 2005.

[18] C. Park, "Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras," *Fixed Point Theory and Applications*, vol. 2007, Article ID 50175, 15 pages, 2007.

[19] C. Park, Y. S. Cho, and M.-H. Han, "Functional inequalities associated with Jordan-von Neumann-type additive functional equations," *Journal of Inequalities and Applications*, vol. 2007, Article ID 41820, 13 pages, 2007.

[20] C. Park and J. Cui, "Generalized stability of $C^*$ -ternary quadratic mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 23282, 6 pages, 2007.

[21] C. Park and A. Najati, "Homomorphisms and derivations in $C^*$ -algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.

[22] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.

[23] J. Rätz, "On inequalities associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 66, no. 1-2, pp. 191–200, 2003.

[24] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.

[25] A. Gilányi, "On a problem by K. Nikodem," *Mathematical Inequalities and Applications*, vol. 5, no. 4, pp. 707–710, 2002.

[26] G.-L. Forti, "Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 127–133, 2004.

[27] R. V. Kadison and G. K. Pedersen, "Means and convex combinations of unitary operators," *Mathematica Scandinavica*, vol. 57, no. 2, pp. 249–266, 1985.