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Research Article

On Logarithmic Convexity for Ky-Fan Inequality

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We give an improvement and a reversion of the well-known Ky-Fan inequality as well as some related results.

1. Introduction and preliminaries

Let $x_1, x_2, \dots, x_n$ and $p_1, p_2, \dots, p_n$ be real numbers such that $x_i \in [0, 1/2]$, $p_i > 0$ with $P_n = \sum_{i=1}^{n} p_i$. Let G_n and A_n be the weighted geometric mean and arithmetic mean, respectively, defined by $G_n = (\prod_{i=1}^{n} x_i^{p_i})^{1/P_n}$, and $A_n = (1/P_n)\sum_{i=1}^{n} p_i x_i = \overline{x}$. In particular, consider the above-mentioned means $G_n' = (\prod_{i=1}^{n}(1-x_i)^{p_i})^{1/P_n}$, and $A_n' = (1/P_n)\sum_{i=1}^{n} p_i(1-x_i)$. Then the well-known Ky-Fan inequality is

$$\frac{G_n}{G_n'} \leq \frac{A_n}{A_n'}. \tag{1.1}$$

It is well known that Ky-Fan inequality can be obtained from the Levinson inequality [1], see also [2, page 71].

Theorem 1.1. *Let f be a real-valued 3-convex function on $[0, 2a]$, then for $0 < x_i < a$, $p_i > 0$,*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right). \tag{1.2}$$

In [3], the second author proved the following result.

Theorem 1.2. *Let f be a real-valued 3-convex function on $[0,2a]$ and $x_i\,(1 \le i \le n)\, n$ points on $[0,2a]$, then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a+x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (a+x_i)\right). \tag{1.3}$$

In this paper, we will give an improvement and reversion of Ky-Fan inequality as well as some related results.

2. Main results

Lemma 2.1. *Define the function*

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)(s-2)}, & s \neq 0,1,2, \\ \dfrac{1}{2}\log x, & s = 0, \\ -x\log x, & s = 1, \\ \dfrac{1}{2}x^2\log x, & s = 2. \end{cases} \tag{2.1}$$

Then $\phi_s'''(x) = x^{s-3}$, that is, $\varphi_s(x)$ is 3-convex for $x > 0$.

Theorem 2.2. *Define the function*

$$\xi_s = \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(\varphi_s(2a - x_i) - \varphi_s(x_i)\right) - \varphi_s(2a - \overline{x}) + \varphi_s(\overline{x}) \tag{2.2}$$

for $x_i,\ p_i$ as in (1.2). Then

(1) *for all $s,t \in I \subseteq R$,*

$$\xi_s\xi_t \ge \xi_r^2 = \xi_{(s+t)/2}^2, \tag{2.3}$$

that is, ξ_s is log *convex in the Jensen sense;*

(2) *ξ_s is continuous on $I \subseteq R$, it is also* log *convex, that is, for $r < s < t$,*

$$\xi_s^{t-r} \le \xi_r^{t-s}\xi_t^{s-r} \tag{2.4}$$

with

$$\xi_0 = \frac{1}{2}\ln\left(\frac{G_n^a A_n}{G_n A_n^a}\right), \tag{2.5}$$

where $G_n^a = \left(\prod_{i=1}^{n}(2a - x_i)^{p_i}\right)^{1/P_n}$, $A_n^a = (1/P_n)\sum_{i=1}^{n} p_i(2a - x_i)$.

Proof. (1) Let us consider the function

$$f(x,u,v,r,s,t) = f(x) = u^2\varphi_s(x) + 2uv\varphi_r(x) + v^2\varphi_t(x), \tag{2.6}$$

where $r = (s+t)/2$, u, v, r, s, t are reals.

$$f'''(x) = \left(ux^{s/2-3/2} + vx^{t/2-3/2}\right)^2 \geq 0 \tag{2.7}$$

for $x > 0$. This implies that f is 3-convex. Therefore, by (1.2), we have $u^2\xi_s + 2uv\xi_r + v^2\xi_t \geq 0$, that is,

$$\xi_s\xi_t \geq \xi_r^2 = \xi_{(s+t)/2}^2. \tag{2.8}$$

This follows that ξ_s is log convex in the Jensen sense.

(2) Note that ξ_s is continuous at all points $s = 0,\ s = 1$, and $s = 2$ since

$$\begin{aligned}
\xi_0 &= \lim_{s\to 0}\xi_s = \frac{1}{2}\ln\left(\frac{G_n^a A_n}{G_n A_n^a}\right),\\
\xi_1 &= \lim_{s\to 1}\xi_s = \frac{1}{P_n}\sum_{i=1}^n p_i\left(x_i\ln x_i - (2a-x_i)\ln(2a-x_i)\right) + (2a-\overline{x})\ln(2a-\overline{x}) - \overline{x}\ln\overline{x},\\
\xi_2 &= \lim_{s\to 2}\xi_s = \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i\left((2a-x_i)^2\ln(2a-x_i) - x_i^2\ln x_i\right) - (2a-\overline{x})^2\ln(2a-\overline{x}) + \overline{x}^2\ln\overline{x}\right].
\end{aligned} \tag{2.9}$$

Since ξ_s is a continuous and convex in Jensen sense, it is log convex. That is,

$$(t-r)\ln\xi_s \leq (t-s)\ln\xi_r + (s-r)\ln\xi_t, \tag{2.10}$$

which completes the proof. □

Corollary 2.3. *For x_i, p_i as in* (1.2),

$$1 < \exp\left(2\xi_3^4\,\xi_4^{-3}\right) \leq \frac{G_n^a A_n}{G_n A_n^a} \leq \exp\left(2\xi_{-1}^{3/4}\,\xi_3^{1/4}\right). \tag{2.11}$$

Proof. Setting $s = 0,\ r = -1$, and $t = 3$ in Theorem 1.2, we get $\xi_0^4 \leq \xi_{-1}^3\,\xi_3$ or

$$\xi_0 \leq \xi_{-1}^{3/4}\,\xi_3^{1/4}. \tag{2.12}$$

Again setting $s = 3,\ r = 0$, and $t = 4$ in Theorem 1.2, we get $\xi_3^4 \leq \xi_0\,\xi_4^3$ or

$$\xi_0 \geq \xi_3^4\,\xi_4^{-3}. \tag{2.13}$$

Combining both inequalities (2.12), (2.13), we get

$$\xi_3^4\,\xi_4^{-3} \leq \xi_0 \leq \xi_{-1}^{3/4}\,\xi_3^{1/4}. \tag{2.14}$$

Also we have ξ_s positive for $s > 2$; therefore, we have

$$0 < \xi_3^4 \xi_4^{-3} \le \xi_0 \le \xi_{-1}^{3/4} \xi_3^{1/4}. \tag{2.15}$$

Applying exponentional function, we get

$$1 < \exp\left(2\xi_3^4 \xi_4^{-3}\right) \le \frac{G_n^a A_n}{G_n A_n^a} \le \exp\left(2\xi_{-1}^{3/4} \xi_3^{1/4}\right). \tag{2.16}$$

□

Remark 2.4. In Corollary 2.3, putting $2a = 1$ we get an improvement of Ky-Fan inequality.

Theorem 2.5. *Define the function*

$$\rho_s = \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(\varphi_s(a+x_i) - \varphi_s(x_i)\right) - \varphi_s(a+\overline{x}) + \varphi_s(\overline{x}), \tag{2.17}$$

for x_i, p_i, a as for Theorem 1.1. Then

(1) *for all $s, t \in I \subseteq R$,*

$$\rho_s\rho_t \ge \rho_r^2 = \rho_{(s+t)/2}^2, \tag{2.18}$$

that is, ρ_s is log *convex in the Jensen sense;*

(2) *ρ_s is continuous on $I \subseteq R$, it is also* log *convex. That is for $r < s < t$,*

$$\rho_s^{t-r} \le \rho_r^{t-s} \rho_t^{s-r} \tag{2.19}$$

with

$$\rho_0 = \frac{1}{2}\ln\left(\frac{\widetilde{G}_n A_n}{G_n \widetilde{A}_n}\right), \tag{2.20}$$

where $\widetilde{G}_n = (\prod_{i=1}^n (a+x_i)^{p_i})^{1/P_n}$, $\widetilde{A}_n = (1/P_n)\sum_{i=1}^n p_i(a+x_i)$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Remark 2.6. Let us note that similar results for difference of power means were recently obtained by Simic in [4].

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