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Research Article

New Limit Formulas for the Convolution of a Function with a Measure and Their Applications

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Asymptotic behavior of a convolution of a function with a measure is investigated. Our results give conditions which ensure that the exact rate of the convolution function can be determined using a positive weight function related to the given function and measure. Many earlier related results are included and generalized. Our new limit formulas are applicable to subexponential functions, to tail equivalent distributions, and to polynomial-type convolutions, among others.

1. Introduction

This paper investigates the existence of the limit of the ratio of a convolution and a positive valued weight function. The limit is given by an explicit formula in terms of the elements in the convolution and of the weight function. Our results are formulated for the convolution of a function with a measure and also for the convolution of two functions.

Our work was inspired by two different applications. One of them is the asymptotic stability theory of differential and integral equations, where an important question is to determine the exact convergence rate to the steady state. The second one is related to the asymptotic representation of the distribution of the sum of independent random variables. In the above and several similar problems, the weighted limits of convolutions play important role with different types of weights.

Let μ be a given measure on the Borel sets of $[0,\infty)$ and let $f : [0,\infty)\to\mathbb{R}$ be a measurable function. The convolution $f*d\mu$ is defined by

$$(f*d\mu)(t) := \int_0^t f(t-s)d\mu(s) \tag{1.1}$$

for all $t\in[0,\infty)$ for which the integral exists.

The convolution of two locally Lebesgue integrable functions $f, g : [0,\infty) \to \mathbb{R}$ is defined by

$$(f*g)(t) := \int_0^t f(t-s)g(s)ds \tag{1.2}$$

for $t \in [0,\infty)$ for which the integral exists.

The motivation of our work came from the following three known results.

The first well-known result has been used frequently in the asymptotic theory of the solutions of differential and integral equations (see, e.g., [1]).

Theorem 1.1. *Let $f, g : [0,\infty) \to \mathbb{R}$ be locally integrable and assume that*

$$f(\infty) := \lim_{t\to\infty} f(t) \in \mathbb{R}, \qquad \int_0^\infty |g(t)|dt < \infty. \tag{1.3}$$

Then

$$\lim_{t\to\infty}(f*g)(t) = f(\infty)\int_0^\infty g(t)dt. \tag{1.4}$$

The next well-known simple result plays a central role, for instance, in the asymptotic theory of fractional differential and integral equations (see, e.g., [2–5]).

Theorem 1.2. *Let $\alpha, \beta > 0$ be given. Then*

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+\beta-1}}\int_0^t (t-s)^{\alpha-1}s^{\beta-1}\,ds = B(\alpha,\beta), \tag{1.5}$$

where $B(\alpha,\beta)$ is the well-known Beta function.

The third known result is formulated for continuous subexponential weight functions. A continuous function $\gamma : [0,\infty) \to (0,\infty)$ is subexponential if

$$\lim_{t\to\infty}\frac{(\gamma*\gamma)(t)}{\gamma(t)} = 2\int_0^\infty \gamma(t)dt < \infty, \tag{1.6}$$

$$\lim_{t\to\infty}\frac{\gamma(t-s)}{\gamma(t)} = 1, \quad \text{for any fixed } s > 0. \tag{1.7}$$

The terminology is suggested by the fact that (1.7) implies that for every $\alpha > 0$ $\lim_{t\to\infty}\gamma(t)\exp(\alpha t) = \infty$.

The next result has been proved in [6] and it plays a central role to get exact rates of subexponential decay of solutions of Volterra integral and integro-differential equations (see, e.g., [6–8]).

Theorem 1.3. *Let the weight function γ be continuous and subexponential. If $f, g : [0,\infty) \to \mathbb{R}$ are continuous functions such that*

$$L_\gamma(f) := \lim_{t\to\infty} \frac{f(t)}{\gamma(t)}, \qquad L_\gamma(g) := \lim_{t\to\infty} \frac{g(t)}{\gamma(t)} \tag{1.8}$$

are finite, then

$$L_\gamma(f*g) := \lim_{t\to\infty} \frac{(f*g)(t)}{\gamma(t)} = L_\gamma(f)\int_0^\infty g + L_\gamma(g)\int_0^\infty f. \tag{1.9}$$

Based on the above three known results, we conclude the next observations.

(i) All of the above theorems give different limit formulas for the ratio $f*g/\gamma$ at $+\infty$. In fact $\gamma(t) = 1,\ t \geq 0$, in Theorem 1.1 and $f(t) = t^{\alpha-1},\ g(t) = t^{\beta-1},\ \gamma(t) = t^{\alpha+\beta-1},\ t \geq 0$, in Theorem 1.2.

(ii) The weight functions in Theorems 1.1 and 1.2 satisfy condition (1.7), but they do not satisfy condition (1.6).

(iii) The condition for g in (1.8) is not necessarily true in Theorem 1.1. Instead of that $\lim_{t\to\infty}(1/\gamma(t))\int_{t-1}^t g = 0$ holds, where $\gamma(t) = 1,\ t \geq 0$.

(iv) $L_\gamma(f) = L_\gamma(g) = 0$ in Theorem 1.2 and at the same time $L_\gamma(f*g) = B(\alpha,\beta)$ is not zero.

Our first goal is to prove results which unify the above-mentioned theorems. Second, we want to extend the limit formulas for the convolution of a function with a measure. This makes possible the applications of our theorems to not only density but also distribution functions.

In fact we prove limit formulas which contain three terms, and the weight function does not satisfy condition (1.6). The major idea in the proofs of the main results is borrowed from the theory of subexponential functions. Namely, for large enough t, in fact $t \geq 2T > 0$, the convolution $(f*d\mu)(t)$ can be split into three terms:

$$(f*d\mu)(t) = \int_{[0,T)} f(t-s)d\mu(s) + \int_{[T,t-T)} f(t-s)d\mu(s) + \int_{t-T}^{t} f(t-s)d\mu(s). \tag{1.10}$$

Under suitable assumptions and some time-tricky and technical treatments of the above three terms, we get the limit formula

$$L_\gamma(f*d\mu) = L_\gamma(f)\mu([0,\infty)) + l_\gamma(f,\mu) + L_\gamma(\mu,1)\int_0^\infty f, \tag{1.11}$$

where the following limits are finite:

$$L_\gamma(f) := \lim_{t\to\infty} \frac{f(t)}{\gamma(t)},$$
$$l_\gamma(f,\mu) := \lim_{T\to\infty}\left(\lim_{t\to\infty}\int_{[T,t-T)} f(t-s)\,d\mu(s)\right), \tag{1.12}$$
$$L_\gamma(\mu,1) := \lim_{t\to\infty}\frac{\mu([t-1,t))}{\gamma(t)}.$$

In the limit formula (1.11), the terms $L_\gamma(f)\mu([0,\infty))$ and $L_\gamma(\mu,1)\int_0^\infty f$ are interpreted as zero whenever $L_\gamma(f) = 0$ and $L_\gamma(\mu,1) = 0$, respectively. So the values of $\mu([0,\infty))$ and $\int_0^\infty f$ need not be finite in the applications.

The limit formula (1.11) can be reformulated for the convolution of two functions f and g. Formally, it can be done if the measure μ is such that $\mu(B) := \int_B g$ for every Borel set $B \subset \mathbb{R}$.

In that case $l_\gamma(f,\mu) = l_\gamma(f,g)$ and $L_\gamma(\mu,1) = L_\gamma(g,1)$, where

$$l_\gamma(f,g) := \lim_{T\to\infty}\left(\lim_{t\to\infty}\int_T^{t-T} f(t-s)g(s)\,ds\right),$$
$$L_\gamma(g,1) := \lim_{t\to\infty}\frac{1}{\gamma(t)}\int_{t-1}^{t} g(s)\,ds. \tag{1.13}$$

These indicate that our remarks (i)–(iv) are taking into account and the known Theorems 1.1, 1.2, and 1.3 are unified in our results.

The organization of the paper is as follows. Section 2 contains notations and definitions. Section 3 lists and discusses the main results both for the convolution of a function with a measure and for the convolution of two functions. In Section 4 we present the corollaries of our main results for subexponential and long-tailed distributions. In Section 5 we show that our results can be easily reformulated to an extended set of weight functions. Section 6 gives some corollaries of our main results for the case when the weight function is of polynomial type. These results have possible applications in the asymptotic theory of fractional differential and integral equations. The proofs of the main results are given in Section 8 based on some preliminary statements stated and proved in Section 7.

2. The basic notations and definitions

First we introduce some notations . The set of real numbers is denoted by $\mathbb{R}$, and $\mathbb{R}_+$ denotes the set of nonnegative numbers.

In our investigations we will make use of different sets of measures and functions given in the next definitions.

Definition 2.1. Let $\mathcal{B}$ be the σ-algebra of the Borel sets of $\mathbb{R}_+$. $\mathcal{M}$ denotes the set of measures μ defined on $\mathcal{B}$ such that the μ-measure of any compact subset of $\mathbb{R}_+$ is a nonnegative number.

Note that the classical Lebesgue measure defined on $\mathcal{B}$, denoted by λ, is an element of $\mathcal{M}$.

Let $a, b \in \mathbb{R}$. In this paper we will write $\int_a^b f d\mu$ (or $\int_a^b f(s)d\mu(s)$) for the μ-integral of f on the closed interval $[a,b]$. The μ-integral of f on the interval $[a,b)$ is written as $\int_{[a,b)} f d\mu$ (or $\int_{[a,b)} f(s)d\mu(s)$). When $\mu = \lambda$, instead of $\int_a^b f d\lambda$ we also write $\int_a^b f$ (or $\int_a^b f(s)ds$).

Definition 2.2. $\mathcal{L}$ denotes the set of functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are Lebesgue integrable on any compact subset of $\mathbb{R}_+$. As usual,

$$\mathcal{L}^1 = \left\{ f \in \mathcal{L} : \int_0^\infty |f| < \infty \right\}. \tag{2.1}$$

Definition 2.3. F_b is the set of the Borel measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are bounded on any compact subset of $\mathbb{R}_+$.

Definition 2.4. A measure μ from $\mathcal{M}$ belongs to the set $\mathcal{M}_c$ if it is absolutely continuous with respect to λ. In this case $\mu = g\lambda$ means that g is a nonnegative function from $\mathcal{L}$ such that $\mu(B) = \int_B g d\lambda$ for every $B \in \mathcal{B}$ (g is the Radon-Nikodym derivative of μ with respect to λ).

It is not difficult to show that for any $f \in F_b$ and $\mu \in \mathcal{M}$ the convolution

$$(f * d\mu)(t) := \int_0^t f(t-s)d\mu(s) \tag{2.2}$$

of f and μ is well defined on $\mathbb{R}_+$. It is known (see, e.g., [9]) that for any $f, g \in \mathcal{L}$ the convolution

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds \tag{2.3}$$

of f and g is well defined for λ almost every (shortly a.e.) $t \in \mathbb{R}_+$. It follows that for any $\mu \in \mathcal{M}_c$ and $f \in \mathcal{L}$ the convolution $f * d\mu$ is well defined for a.e. $t \in \mathbb{R}_+$, and

$$(f * d\mu)(t) = (f * g)(t), \quad \text{a.e. } t \in \mathbb{R}_+, \tag{2.4}$$

where $\mu = g\lambda$.

In this paper our major goal is to give conditions—possibly sharp—which guarantee the existence of the finite limit of the ratio

$$\frac{1}{\gamma(t)}(f * d\mu)(t) \tag{2.5}$$

as $t \to +\infty$. The weight function $\gamma : \mathbb{R}_+ \to (0, \infty)$ will belong to some special classes of the functions given in the following definitions.

Definition 2.5. Let Γ be the set of the functions $\gamma : \mathbb{R}_+ \to (0,\infty)$ such that

$$\lim_{t\to+\infty} \frac{\gamma(t-s)}{\gamma(t)} = 1 \quad \text{for any fixed } s \geq 0. \tag{2.6}$$

The set of the functions $\gamma \in \Gamma$ for which the above convergence is uniform on any compact interval $[0,T]$ is denoted by Γ_u.

It is clear that if $\gamma \in \Gamma$, then

$$\lim_{t\to+\infty} \frac{\gamma(t+s)}{\gamma(t)} = \lim_{t\to+\infty} \left(\frac{\gamma(t+s-s)}{\gamma(t+s)}\right)^{-1} = 1 \tag{2.7}$$

holds for $s \geq 0$, and hence it holds for any $s \in \mathbb{R}$. Therefore for all $\beta > 0$, we have

$$\lim_{t\to+\infty} \frac{\gamma(\ln(\beta t))}{\gamma(\ln t)} = \lim_{t\to+\infty} \frac{\gamma(\ln t + \ln \beta)}{\gamma(\ln t)} = 1, \tag{2.8}$$

that is the function $1 \leq t \to \gamma(\ln t)$ is so called regularly varying at infinity. Thus applying the Karamata uniform convergence theorem (see, e.g., [10]) it follows that the convergence in (2.8) is uniform in β on any compact set of $(0,\infty)$, assuming that γ is Lebesgue measurable. From this we get that the convergence in (2.6) is uniform on any compact set of $\mathbb{R}_+$ assuming that γ is Lebesgue measurable. Thus Γ_u contains the Lebesgue measurable members of Γ. On the other hand from [10] we know that there exists a nonmeasurable function $\gamma \in \Gamma$ such that $\gamma \notin \Gamma_u$ and hence Γ_u is a proper subset of Γ.

To give an explicit formula for the weighted limit of the convolution $f * d\mu$ at $+\infty$, we should assume some limit relations between γ and f and between γ and μ.

Definition 2.6. Let $\gamma \in \Gamma$.

(a) F_γ denotes the set of functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that the limit

$$L_\gamma(f) := \lim_{t\to+\infty} \frac{f(t)}{\gamma(t)} \tag{2.9}$$

is finite.

(b) $\mathcal{M}_\gamma$ denotes the set of measures $\mu \in \mathcal{M}$ such that for any fixed $T > 0$ the limit

$$L_\gamma(\mu, T) := \lim_{t\to+\infty} \frac{\mu([t-T,t))}{\gamma(t)} \tag{2.10}$$

is finite.

(c) Let

$$F_\mu := \begin{cases} F_b, & \text{if } \mu \in \mathcal{M} \setminus \mathcal{M}_c \\ \mathcal{L}, & \text{if } \mu \in \mathcal{M}_c. \end{cases} \tag{2.11}$$

Definition 2.7. Let $\gamma \in \Gamma$ and $\mu \in \mathcal{M}_\gamma$. $F_{\gamma,\mu}$ denotes the set of functions $f \in F_\mu$ for which

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{t-T}^{t} f(t-s)d\mu(s) = L_\gamma(\mu,1)\int_0^T f \tag{2.12}$$

holds for any fixed $T > 0$.

Remark 2.8. A measure $\mu \in \mathcal{M}_c$ belongs to $\mathcal{M}_\gamma$ if and only if for any fixed $T > 0$ the limit

$$L_\gamma(g,T) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{t-T}^{t} g \tag{2.13}$$

is finite, where $\mu = g\lambda$. Moreover $L_\gamma(\mu,T) = L_\gamma(g,T)$ for any $T > 0$. It can be shown (see Proposition 7.3) that if $g \in F_\gamma$ and $\gamma \in \Gamma_u$ then $L_\gamma(\mu,T) = L_\gamma(g,T) = L_\gamma(g)T,\ T > 0$.

We close this section with the following definition.

Definition 2.9. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be oscillatory on $\mathbb{R}_+$ if there exist two sequences $t_n, t'_n \geq 0,\ n \geq 1$, such that $t_n \to +\infty$ and $t'_n \to +\infty$ as $n \to +\infty$, moreover $f(t_n) < 0 < f(t'_n),\ n \geq 1$.

3. Main results

In this section we state our main results. Their proofs are relegated to Section 8.

We use the following hypothesis.

(H) $\gamma \in \Gamma_u,\ \mu \in \mathcal{M}_\gamma,\ f \in F_\gamma \cap F_{\gamma,\mu}$, and the improper integral

$$\lim_{T\to\infty} L_\gamma(\mu,1)\int_0^T f \tag{3.1}$$

is finite whenever f is oscillatory.

Note that if $L_\gamma(\mu,1) = 0$, then (H) is satisfied for any $f \in F_\gamma \cap F_{\gamma,\mu}$.

In the next result we give an explicit limit formula for the weighted limit of the convolution of f and μ at $+\infty$.

Theorem 3.1. *Assume (H). Then the following results hold.*

(a) *The following three statements are equivalent.*

(a_1) *The limit*

$$L_\gamma(f*d\mu) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^T f(t-s)d\mu(s) \tag{3.2}$$

is finite.

(a_2) *For some* $T > 0$*, the limit*

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)\,d\mu(s) \tag{3.3}$$

is finite.

(a_3) *The values*

$$\begin{aligned}
&\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T_1,t-T_1)} f(t-s)\,d\mu(s) \quad \text{for a fixed } T_1 > 0,\\
&\limsup_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T_2,t-T_2)} f(t-s)\,d\mu(s) \quad \text{for a fixed } T_2 > 0
\end{aligned} \tag{3.4}$$

are finite, moreover

$$\begin{aligned}
&\lim_{T\to\infty} \left(\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)\,d\mu(s) \right)\\
&\quad = \lim_{T\to\infty} \left(\limsup_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)\,d\mu(s) \right).
\end{aligned} \tag{3.5}$$

(b) *Assume that one of the statements* (a_1)–(a_3) *is true. Then the limit* (3.3) *is finite for any* $T > 0$ *and*

$$L_\gamma(f * d\mu) = L_\gamma(f)\mu([0,\infty)) + l_\gamma(f,\mu) + L_\gamma(\mu,1)\int_0^\infty f, \tag{3.6}$$

where

$$L_\gamma(f)\mu([0,\infty)) := \lim_{T\to+\infty} L_\gamma(f)\mu([0,T]), \tag{3.7}$$

$$l_\gamma(f,\mu) := \lim_{T\to+\infty} \left(\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)\,d\mu(s) \right), \tag{3.8}$$

$$L_\gamma(\mu,1)\int_0^\infty f := \lim_{T\to+\infty} L_\gamma(\mu,1)\int_0^T f \tag{3.9}$$

are finite.

Remark 3.2. Our theorem is applicable for the case when $\mu([0,\infty)) = +\infty$ and also when $f \notin \mathcal{L}^1$. Namely, if $L_\gamma(f) = 0$, then (3.7) yields that $L_\gamma(f)\mu([0,\infty))$ is zero independently on the value of $\mu([0,\infty))$. Similarly if $L_\gamma(\mu,1) = 0$, then from (3.9) it follows that $L_\gamma(\mu,1)\int_0^\infty f$ is independently zero on f. We will see that this character of our theorem is important for getting limit formulas for polynomial-type convolutions (see Corollary 6.2 in Section 6).

Now consider the case $\mu \in \mathcal{M}_c$, that is, $\mu = g\lambda$. In this case we can apply Theorem 3.1 by using the hypothesis.

(H_c) $\gamma \in \Gamma_u$, $f \in F_\gamma \cap \mathcal{L}$, the function $g \in \mathcal{L}$ is nonnegative such that

$$L_\gamma(g,T) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{t-T}^{t} g \tag{3.10}$$

is finite for every $T > 0$, and the improper integral $\lim_{T\to\infty} L_\gamma(g,1)\int_0^T f$ is finite whenever f is oscillatory .

Theorem 3.3. *Assume* (H_c). *Then the following results hold.*

(a) *The following three statements are equivalent.*

(a_1) *The limit*

$$L_\gamma(f*g) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^t f(t-s)g(s)ds \tag{3.11}$$

is finite.

(a_2) *For some $T > 0$, the limit*

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_T^{t-T} f(t-s)g(s)ds \tag{3.12}$$

is finite.

(a_3) *The values*

$$\begin{aligned} &\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_{T_1}^{t-T_1} f(t-s)g(s)ds, \quad \textit{for a fixed } T_1 > 0, \\ &\limsup_{t\to+\infty} \frac{1}{\gamma(t)} \int_{T_2}^{t-T_2} f(t-s)g(s)ds, \quad \textit{for a fixed } T_2 > 0, \end{aligned} \tag{3.13}$$

are finite, moreover

$$\lim_{T\to+\infty}\left(\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_T^{t-T} f(t-s)g(s)ds\right) = \lim_{T\to+\infty}\left(\limsup_{t\to+\infty} \int_T^{t-T} f(t-s)g(s)ds\right). \tag{3.14}$$

(b) *Assume that one of the statements* (a_1)–(a_3) *is true. Then the limit* (3.12) *is finite for any $T > 0$, and*

$$L_\gamma(f*g) = L_\gamma(f)\int_0^\infty g + l_\gamma(f,g) + L_\gamma(g,1)\int_0^\infty f, \tag{3.15}$$

where

$$l_\gamma(f,g) := \lim_{T\to+\infty}\left(\lim_{t\to+\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds\right), \tag{3.16}$$

$$L_\gamma(f)\int_0^\infty g := \lim_{T\to+\infty}\left(L_\gamma(f)\int_0^T g\right), \qquad L_\gamma(g,1)\int_0^\infty f := \lim_{T\to+\infty} L_\gamma(g,1)\int_0^T f \tag{3.17}$$

are finite.

When $g \in \mathcal{L}\cap F_\gamma$, then $L_\gamma(g,T) = L_\gamma(g)T,\ T > 0$ (see Proposition 7.3), and we get the following.

Theorem 3.4. *Let $\gamma \in \Gamma_u,\ f,g \in \mathcal{L}\cap F_\gamma$, and assume that*

(i) *the improper integral*

$$L_\gamma(g)\int_0^\infty f := \lim_{T\to+\infty} L_\gamma(g)\int_0^T f \tag{3.18}$$

is finite, whenever f is oscillatory and g is not oscillatory;

(ii) *the improper integral*

$$L_\gamma(f)\int_0^\infty g := \lim_{T\to+\infty} L_\gamma(f)\int_0^T g \tag{3.19}$$

is finite, whenever f is not oscillatory and g is oscillatory.

Then the statements of Theorem 3.3 are valid and (3.15) *can be written in the form*

$$L_\gamma(f*g) = L_\gamma(f)\int_0^\infty g + l_\gamma(f,g) + L_\gamma(g)\int_0^\infty f. \tag{3.20}$$

Remark 3.5. (a) If $\gamma \in \Gamma_u,\ f \in \mathcal{L}^1\cap F_\gamma$, and $g \in \mathcal{L}$ is nonnegative such that $L_\gamma(g,T)$ defined in (3.10) is finite for every $T > 0$, then the conditions of Theorem 3.3 hold.

(b) If $\gamma \in \Gamma_u$, and $f,g \in \mathcal{L}^1\cap F_\gamma$, then Theorem 3.4 is applicable.

Remark 3.6. The well-known result Theorem 1.1 (see, e.g., [1]) is a straightforward consequence of Theorem 3.3.

4. Applications of the main results to subexponential functions

In this section we concentrate on the so-called subexponential functions which are strongly related to the subexponential distributions. Such distributions play an important role, for instance, in modeling heavy-tailed data. Such appears in the situations where some extremely large values occur in a sample compared to the mean size of data (see, e.g., [11] and the references therein).

First we consider the "density-type" subexponential functions.

Definition 4.1. Assume that a function γ is called subexponential if $\gamma \in \Gamma \cap \mathcal{L}^1$ and

$$L_\gamma(\gamma * \gamma) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^t \gamma(t-s)\gamma(s)ds = 2\int_0^\infty \gamma. \tag{4.1}$$

Remark 4.2. Let $\gamma \in \Gamma \cap \mathcal{L}$ such that $L_\gamma(\gamma*\gamma)$ is finite. Then γ is measurable and hence $\gamma \in \Gamma_u$. Thus

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^t \gamma(t-s)\gamma(s)ds \geq \lim_{t\to+\infty} \int_0^T \frac{\gamma(t-s)}{\gamma(t)}\gamma(s)ds = \int_0^T \gamma(s)ds, \quad T > 0, \tag{4.2}$$

which shows that

$$\int_0^\infty \gamma \leq L_\gamma(\gamma*\gamma) < \infty. \tag{4.3}$$

Therefore $\gamma \in \mathcal{L}^1$ and the normalized function $0 \leq t \to \gamma(t)\left(\int_0^\infty \gamma\right)^{-1}$ is a subexponential density function. This gives the meaning of the "density-type" subexponentiality.

From Theorems 3.4 and 6.1, we get the following.

Theorem 4.3. *If γ is a subexponential function and $f, g \in \mathcal{L}^1 \cap F_\gamma$, then*

$$L_\gamma(f*g) = L_\gamma(f)\int_0^\infty g + L_\gamma(g)\int_0^\infty f. \tag{4.4}$$

It is worth to note that formula (4.4) has been obtained by Appleby et al. [7] in the case when the functions γ, f, and g are continuous on $\mathbb{R}_+$. These types of limit formulas were used effectively for studying the subexponential rate of decay of solutions of integral and integro-differential equations (see, e.g., [6, 12]).

Now we apply our main results to subexponential and long-tailed-type distribution functions.

Definition 4.4. Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a distribution function on $\mathbb{R}_+$ such that $H(0) = 0$ and $H(t) < 1$ for all $t > 0$. Then

(a) H is called subexponential if

$$\lim_{t\to+\infty} \frac{1}{\overline{H}(t)} \left(1 - \int_0^t H(t-s)dH(s)\right) = 2 \tag{4.5}$$

or equivalently

$$L_{\overline{H}}(\overline{H}*dH) := \lim_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_0^t \overline{H}(t-s)dH(s) = 1, \tag{4.6}$$

where $\overline{H}$ denotes the tail of H, that is,

$$\overline{H}(t) = 1 - H(t)(>0), \quad t \geq 0. \tag{4.7}$$

(b) H is called long-tailed if

$$\lim_{t\to+\infty} \frac{\overline{H}(t-s)}{\overline{H}(t)} = 1 \quad \text{for any } s \geq 0. \tag{4.8}$$

The definition of the subexponential distribution was introduced by Chistyakov [13] in 1964 and there are a large number of papers in the literature dealing with them. For the major properties and also for applications, we refer to the nice introduction and review paper by Goldie and Klüppelberg [11] and the references in it.

Now we show the consequences of our main results for the above-defined class of distribution functions. The proofs will be explained in Section 8.

It is noted in [14, 15] (see also [11]) that the set of the subexponential distributions is a proper subset of the set of the long-tailed distributions.

In the first theorem, we give equivalent statements for subexponential distributions; and in the second one, we give a limit formula for the more general long-tailed distributions.

Theorem 4.5. *Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a distribution function such that $H(0) = 0$ and $H(t) < 1,\ t > 0$. Then the following statements are equivalent.*

(a) *H is subexponential.*

(b) *H is long-tailed and there is a $T > 0$ such that the limit*

$$\lim_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_{[T,t-T)} \overline{H}(t-s)\,dH(s) \tag{4.9}$$

is finite and

$$\lim_{T\to+\infty}\left(\lim_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_{[T,t-T)} \overline{H}(t-s)\,dH(s)\right) = 0. \tag{4.10}$$

(c) *H is long-tailed and there is a $T > 0$ such that*

$$\limsup_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_{[T,t-T)} \overline{H}(t-s)\,dH(s) \tag{4.11}$$

is finite and

$$\lim_{T\to+\infty}\left(\limsup_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_{[T,t-T)} \overline{H}(t-s)\,dH(s)\right) = 0. \tag{4.12}$$

Theorem 4.6. *Let $F, G, H : \mathbb{R}_+ \to \mathbb{R}_+$ be distribution functions, $F(0) = G(0) = 0$, and H is long-tailed. If*

$$L_{\overline{H}}(\overline{F}) := \lim_{t\to+\infty} \frac{\overline{F}(t)}{\overline{H}(t)}, \qquad L_{\overline{H}}(\overline{G}) := \lim_{t\to+\infty} \frac{\overline{G}(t)}{\overline{H}(t)} \tag{4.13}$$

are finite, and

$$\lim_{T\to+\infty} \left(\limsup_{t\to+\infty} \frac{1}{\overline{H}(t)} \int_{[T,t-T)} \overline{F}(t-s)\,dG(s) \right) = 0, \tag{4.14}$$

then

$$\lim_{t\to+\infty} \frac{1}{\overline{H}(t)} \left(1 - \int_0^t F(t-s)\,dG(s) \right) = L_{\overline{H}}(\overline{F}) + L_{\overline{H}}(\overline{G}). \tag{4.15}$$

The above theorem can be easily applied for tail-equivalent distributions defined as follows (see [11]).

Definition 4.7 (tail-equivalence). Two distributions $H, F : \mathbb{R}_+ \to \mathbb{R}_+$ with the conditions $H(0) = F(0) = 0$ and $H(t) < 1,\ F(t) < 1,\ t > 0$, are called tail-equivalent if $L_{\overline{H}}(\overline{F})$ is a positive number.

Corollary 4.8. *Let $F, G, H : \mathbb{R}_+ \to \mathbb{R}_+$ be distribution functions, $F(0) = G(0) = 0,\ F(t) < 1,\ G(t) < 1,\ t > 0$, and H is long-tailed. If the conditions of Theorem 4.6 are satisfied, then H and $F*dG$ are tail equivalent if and only if $L_{\overline{H}}(\overline{F}) + L_{\overline{H}}(\overline{G}) > 0$, that is, at least one of the distribution functions F and G is tail equivalent to H.*

5. Further corollaries for an extended set of weight functions

First we consider the extension of the set Γ.

Definition 5.1. Let $\alpha \in \mathbb{R}$. By $\Gamma(\alpha)$ one denotes the set of the functions $\gamma : \mathbb{R}_+ \to (0, \infty)$ such that

$$\lim_{t\to+\infty} \frac{\gamma(t-s)}{\gamma(t)} = e^{-\alpha s} \tag{5.1}$$

for all $s \geq 0$. By $\Gamma_u(\alpha)$ one denotes the set of the functions $\gamma \in \Gamma(\alpha)$ for which the convergence in (5.1) is uniform on $0 \leq s \leq T$, for any $T > 0$.

Remark 5.2. It is clear that $\Gamma = \Gamma(0)$, $\Gamma_u = \Gamma_u(0)$, and $\gamma \in \Gamma(\alpha)$ ($\gamma \in \Gamma_u(\alpha)$) if and only if $\gamma_1 \in \Gamma$ ($\gamma_1 \in \Gamma_u$), where γ_1 is defined by $\gamma_1(t) := e^{-\alpha t}\gamma(t),\ t \geq 0$.

Let $\gamma \in \Gamma_u(\alpha)$, $\mu \in \mathcal{M}$, and $f \in F_\mu$. Then

$$\begin{aligned}\frac{1}{\gamma(t)}(f*d\mu)(t) &= \frac{1}{\gamma(t)}\int_0^t f(t-s)d\mu(s)\\ &= \frac{1}{\gamma(t)e^{-\alpha t}}\int_0^t e^{-\alpha(t-s)}f(t-s)e^{-\alpha s}d\mu(s)\\ &= \frac{1}{\gamma_1(t)}\int_0^t f_1(t-s)d\mu_1(s) = \frac{1}{\gamma_1(t)}(f_1*d\mu_1)(t), \quad t\in D_{f*d\mu},\end{aligned} \tag{5.2}$$

where $\gamma_1(t) := \gamma(t)e^{-\alpha t}$, $f_1(t) := f(t)e^{-\alpha t}$, $t \ge 0$, and $\mu_1(B) := \int_B e^{-\alpha s}d\mu(s)$ for $B \in \mathcal{B}$.

Thus our earlier results are applicable if $\gamma_1 \in \Gamma_u$ and $\mu_1 \in \mathcal{M}_{\gamma_1}$. But from Remark 5.2 we have that $\gamma_1 \in \Gamma_u$ if and only if $\gamma \in \Gamma_u(\alpha)$. Moreover, $\mu_1 \in \mathcal{M}_{\gamma_1}$ if and only if the limit

$$L_\gamma(\mu,\alpha,T) := \lim_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[t-T,t)} e^{\alpha(t-s)}d\mu(s) \quad (= L_{\gamma_1}(\mu_1,T)) \tag{5.3}$$

is finite for any $T > 0$.

Remark 5.3. $\mu_1 \in \mathcal{M}_c$ if and only if $\mu \in \mathcal{M}_c$. Namely, let $\mu \in \mathcal{M}_c$, $\mu = g\lambda$. Thus $\mu_1(B) = \int_B e^{-\alpha s}d\mu(s) = \int_B g(s)e^{-\alpha s}d\lambda(s)$, $B \in \mathcal{B}$, and hence $\mu_1 \in \mathcal{M}_c$. Now let $\mu_1 \in \mathcal{M}_c$. Then $\mu_1(B) = \int_B e^{-\alpha s}d\mu(s) = 0$ for any $B \in \mathcal{B}$ such that $\lambda(B) = 0$. Therefore $e^{-\alpha s} = 0$ μ-almost every $s \in B$, and hence $\mu(B) = 0$. From this it follows that μ is absolute continuous with respect to λ, that is, $\mu \in \mathcal{M}_c$.

It can be seen that $f_1 \in F_{\mu_1}$ if and only if $f \in F_\mu$.

Remark 5.4. $f_1 \in F_{\gamma_1,\mu_1}$ if and only if $f \in F_{\gamma,\alpha,\mu}$. Here $F_{\gamma,\alpha,\mu}$ denotes the set of functions $f \in F_\mu$ for which

$$\lim_{t\to+\infty}\frac{1}{\gamma(t)}\int_{t-T}^t f(t-s)d\mu(s) = L_\gamma(\mu,\alpha,1)\int_0^T f \tag{5.4}$$

for any $T > 0$.

The above remarks show that our main results, Theorems 3.1–3.4, can be easily reformulated for the class $\Gamma_u(\alpha)$, assuming that we replace the hypotheses (H) and (H_c) by (H(α)) and ($\mathrm{H}_c(\alpha)$), respectively. In fact we use the following modified hypotheses.

(H(α)) $\alpha \in \mathbb{R}$, $\gamma \in \Gamma_u(\alpha)$, $\mu \in \mathcal{M}$ are such that $L_\gamma(\mu,\alpha,T)$ defined in (5.3) is finite for any $T > 0$, $f \in F_\gamma \cap F_{\gamma,\alpha,\mu}$ and the improper integral

$$\lim_{T\to\infty} L_\gamma(\mu,\alpha,1)\int_0^T f\,d\lambda \tag{5.5}$$

is finite, whenever f is oscillatory.

($\mathrm{H}_c(\alpha)$) $\alpha \in \mathbb{R},\ \gamma \in \Gamma_u(\alpha),\ f \in F_\gamma \cap \mathcal{L},\ g \in \mathcal{L}$ is nonnegative such that

$$L_\gamma(g,\alpha,T) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{t-T}^{t} e^{\alpha(t-s)} g(s)ds \tag{5.6}$$

is finite for every $T > 0$, and the improper integral $\lim_{T\to\infty} L_\gamma(g,\alpha,1)\int_0^T f d\lambda$ is finite, whenever f is oscillatory.

The extended form of Theorem 3.1 is as follows.

Theorem 5.5. *Assume* (H_α). *Then the following results hold.*

(a) *The following three statements are equivalent.*

(a_1) *The limit*

$$L_\gamma(f * d\mu) := \lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^t f(t-s)d\mu(s) \tag{5.7}$$

is finite.

(a_2) *For some $T > 0$ the limit*

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)d\mu(s) \tag{5.8}$$

is finite.

(a_3) *The values*

$$\begin{aligned} &\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T_1,t-T_1)} f(t-s)d\mu(s), \quad \text{for a fixed } T_1 > 0, \\ &\limsup_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T_2,t-T_2)} f(t-s)d\mu(s), \quad \text{for a fixed } T_2 > 0, \end{aligned} \tag{5.9}$$

are finite, moreover

$$\lim_{T\to+\infty} \left(\liminf_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)d\mu(s) \right) = \lim_{T\to+\infty} \left(\limsup_{t\to+\infty} \frac{1}{\gamma(t)} \int_{[T,t-T)} f(t-s)d\mu(s) \right). \tag{5.10}$$

(b) *Assume that one of the statements* (a_1)–(a_3) *is true. Then the limit* (5.8) *is finite for any $T > 0$ and*

$$L_\gamma(f * d\mu) = L_\gamma(f) \int_0^\infty e^{-\alpha s} d\mu(s) + l_\gamma(f,\mu) + L_\gamma(\mu,\alpha,1) \int_0^\infty f, \tag{5.11}$$

where

$$L_\gamma(f)\int_0^\infty e^{-\alpha s}d\mu(s) := \lim_{T\to+\infty} L_\gamma(f)\int_0^T e^{-\alpha s}\,d\mu(s),$$
$$L_\gamma(\mu,\alpha,1)\int_0^\infty f := \lim_{T\to+\infty} L_\gamma(\mu,\alpha,1)\int_0^T f, \tag{5.12}$$

and $l_\gamma(f,\mu)$*, defined in* (3.8)*, are finite.*

The extensions of Theorems 3.3 and 3.4 are similar and are left to the reader.

6. Power-type weight function and the role of the middle term

The introduction of our middle term was motivated by two independent papers [2, 4]. In both papers power-type estimations have been proved for the solutions of functional differential equations and of the wave equations with boundary condition, respectively. The joint idea was to transform the original problems into a convolution-type form. By treating the convolution form, power-type estimations were given without investigating any limit formula.

As a consequence of Theorem 3.4, we prove the next result, and as a corollary of it we give a power-type limit formula.

Theorem 6.1. *Let* $\gamma \in \Gamma_u$*, and let* $p,q \in \mathcal{L}\cap F_\gamma$ *be positive such that the limit* $L_\gamma(p*q)$ *is finite. If* $f \in \mathcal{L}\cap F_p$ *and* $g \in \mathcal{L}\cap F_q$*, then the limit* $L_\gamma(f*g)$ *is finite and*

$$L_\gamma(f*g) = L_\gamma(p)L_p(f)\int_0^\infty g + L_p(f)L_q(g)l_\gamma(p,q) + L_\gamma(q)L_q(g)\int_0^\infty f, \tag{6.1}$$

where

$$L_\gamma(p)L_p(f)\int_0^\infty g \ := \lim_{T\to+\infty} L_\gamma(p)L_p(f)\int_0^T g,$$
$$L_\gamma(q)L_q(g)\int_0^\infty f \ := \lim_{T\to+\infty} L_\gamma(q)L_q(g)\int_0^T f, \tag{6.2}$$

and $l_\gamma(p,q)$*, defined in* (3.16)*, are finite.*

The following corollary is a generalization of Theorem 1.2 and shows the importance of our middle term when γ is a power-type function.

Corollary 6.2. *Let* $f,g \in \mathcal{L}$ *and assume that the limits*

$$a := \lim_{t\to+\infty}\frac{f(t)}{t^{\alpha-1}}, \qquad b := \lim_{t\to+\infty}\frac{g(t)}{t^{\beta-1}} \tag{6.3}$$

are finite, where $\alpha, \beta > 0$ are given constants. Then

$$\lim_{t \to +\infty} \frac{1}{t^{\alpha+\beta-1}} \int_0^t f(t-s)g(s)ds = abB(\alpha, \beta), \tag{6.4}$$

where $B : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is the Beta function, that is, $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)(\Gamma(\alpha+\beta))^{-1}$ (Γ is the well-known Gamma function).

In the above limit formula, $\gamma(t) = t^{\alpha+\beta-1}$, $t > 0$, $L_\gamma(f) = L_\gamma(g) = 0$ and the middle term $l_\gamma(f,g) = abB(\alpha,\beta) \neq 0$, whenever $ab \neq 0$.

7. Preliminary results

In this section we state and prove preliminary and auxiliary results. They will be used in the proofs of our main results in the next section. $\mathbb{N}^+$ denotes the set of the positive integers.

Proposition 7.1. *Let $\gamma \in \Gamma$ and $\mu \in \mathcal{M}$ such that $L_\gamma(\mu, T)$ is finite for any $T \in [0, T_0]$ with a fixed $T_0 > 0$. Then $\mu \in \mathcal{M}_\gamma$.*

Proof. Let $T > T_0$ and $T_1 \in (0, T_0]$ such that $n := T/T_1 \in \mathbb{N}^+$. Then

$$\begin{aligned} \frac{\mu([t-T,t))}{\gamma(t)} &= \frac{1}{\gamma(t)} \sum_{k=1}^{n} \mu([t-kT_1, t-(k-1)T_1)) \\ &= \sum_{k=1}^{n} \frac{\mu([t-kT_1, t-(k-1)T_1))}{\gamma(t-(k-1)T_1)} \cdot \frac{\gamma(t-(k-1)T_1)}{\gamma(t)}, \end{aligned} \tag{7.1}$$

and this yields $L_\gamma(\mu, T) = nL_\gamma(\mu, T_1)$. □

Proposition 7.2. *Let $\gamma \in \Gamma$ and $\mu \in \mathcal{M}_\gamma$. Then the following hold.*

(a) *$L_\gamma(\mu, T) = TL_\gamma(\mu, 1)$ for any $T \geq 0$.*

(b) *$\lim_{t \to +\infty}(\mu(\{t\})/\gamma(t)) = 0$, where t is the only element of the set $\{t\}$.*

Proof. (a) First we show that $L_\gamma(\mu, \cdot)$ is additive. In fact for $T_1, T_2 \geq 0$ we have

$$\frac{\mu([t-(T_1+T_2),t))}{\gamma(t)} = \frac{\mu([t-T_1,t))}{\gamma(t)} + \frac{\mu([(t-T_1)-T_2, t-T_1))}{\gamma(t-T_1)} \cdot \frac{\gamma(t-T_1)}{\gamma(t)} \tag{7.2}$$

for $t > T_1 + T_2$. This yields $L_\gamma(\mu, T_1 + T_2) = L_\gamma(\mu, T_1) + L_\gamma(\mu, T_2)$. Therefore $L_\gamma(\mu, \cdot)$ can be extended in a unique way to $\mathbb{R}$ such that it is additive. Now (a) follows since $L_\gamma(\mu, \cdot)$ is nonnegative on $\mathbb{R}_+$.

(b) For any $T > 0$, we have

$$0 \leq \frac{\mu(\{t\})}{\gamma(t)} \leq \frac{\mu([t,t+T))}{\gamma(t+T)} \frac{\gamma(t+T)}{\gamma(t)}, \quad t \geq 0, \tag{7.3}$$

therefore

$$0 \le \limsup_{t\to+\infty} \frac{\mu(\{t\})}{\gamma(t)} \le L_\gamma(\mu, T) = T L_\gamma(\mu, 1). \tag{7.4}$$

Since $T > 0$ is arbitrarily chosen, statement (b) is proved. □

Proposition 7.3. *Let $\gamma \in \Gamma_u$ and assume that $\mu \in \mathcal{M}_c$, that is, $\mu = p\lambda$. If $p \in F_\gamma$, then $L_\gamma(\mu, T) = L_\gamma(p)T$, $T \ge 0$.*

Proof. γ is a positive function, therefore $L_\gamma(p) \ge 0$. Thus for any $\varepsilon \in (0,1)$ there exists a $t_\varepsilon > 0$ such that

$$(1-\varepsilon)L_\gamma(p) \le \frac{p(t)}{\gamma(t)} \le \varepsilon + L_\gamma(p), \quad \text{for } t > t_\varepsilon. \tag{7.5}$$

From this it follows that

$$(1-\varepsilon)L_\gamma(p) \le \frac{p(t-s)}{\gamma(t-s)} \le \varepsilon + L_\gamma(p), \quad s \in [0,T],\ t > t_\varepsilon + T. \tag{7.6}$$

On the other hand there is a $t'_\varepsilon > t_\varepsilon + T$ such that

$$1 - \varepsilon < \frac{\gamma(t-s)}{\gamma(t)} < 1 + \varepsilon, \quad s \in [0,T],\ t > t'_\varepsilon, \tag{7.7}$$

where we used that $\gamma \in \Gamma_u$.

Thus

$$\begin{aligned}(1-\varepsilon)^2 L_\gamma(p) &\le \frac{p(t-s)}{\gamma(t)} \\ &= \frac{p(t-s)}{\gamma(t-s)} \frac{\gamma(t-s)}{\gamma(t)} \\ &\le (1+\varepsilon)(\varepsilon + L_\gamma(p)), \quad s \in [0,T],\ t > t'_\varepsilon,\end{aligned} \tag{7.8}$$

therefore

$$\begin{aligned}(1-\varepsilon)^2 L_\gamma(p) T &\le \frac{1}{\gamma(t)} \int_0^T p(t-s)\,ds \\ &= \frac{1}{\gamma(t)} \int_{t-T}^t p\,d\lambda \\ &= \frac{\mu([t-T,t))}{\gamma(t)} \le (1+\varepsilon)(\varepsilon + L_\gamma(p))T, \quad t > t'_\varepsilon.\end{aligned} \tag{7.9}$$

From this it follows

$$
\begin{aligned}
(1-\varepsilon)^2 L_\gamma(p)T &\le \liminf_{t\to+\infty}\frac{\mu([t-T,t))}{\gamma(t)} \\
&\le \limsup_{t\to+\infty}\frac{\mu([t-T,t))}{\gamma(t)} \\
&\le (1+\varepsilon)(\varepsilon+L_\gamma(p))T
\end{aligned}
\tag{7.10}
$$

for any fixed $\varepsilon \in (0,1)$. This completes the proof as $\varepsilon \to 0+$. □

Definition 7.4. For any $x \ge 0$ and $B \in \mathcal{B}$, let $\varepsilon_x(B)$ be defined by

$$
\varepsilon_x(B) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases} \tag{7.11}
$$

It is clear that for any fixed $x \ge 0$, ε_x is a measure on $\mathcal{B}$ (the unit mass at x), and $\varepsilon_x \in \mathcal{M}$.

Proposition 7.5. *Let $(t_n)_{n=1}^\infty$ be a given sequence in $\mathbb{R}_+$ such that*

$$
\delta := \inf_{n\in\mathbb{N}^+}(t_{n+1}-t_n) > 0, \tag{7.12}
$$

and let $(\alpha_n)_{n=1}^\infty$ be a sequence of nonnegative numbers. Suppose $\gamma \in \Gamma$.

(a) *If the measure $\mu := \sum_{n=1}^\infty \alpha_n \varepsilon_{t_n}$ belongs to $\mathcal{M}_\gamma$, then $\lim_{n\to+\infty}(\alpha_n/\gamma(t_n)) = 0$ and $L_\gamma(\mu,T) = 0$, $T \ge 0$.*

(b) *If $\gamma \in \Gamma_u$ and $\lim_{n\to+\infty}(\alpha_n/\gamma(t_n)) = 0$, then $\mu \in \mathcal{M}_\gamma$.*

Proof. It is clear that $\mu \in \mathcal{M}$.

(a) Let $T_0 \in (0,\delta)$ be fixed. Since $\mu([t_n - T_0, t_n)) = 0$ for every $n = 2,3,\dots$, we have

$$
\lim_{n\to+\infty}\frac{\mu([t_n-T_0,t_n))}{\gamma(t_n)} = 0, \tag{7.13}
$$

and hence $L_\gamma(\mu,T_0) = 0$. This and statement (a) of Proposition 7.2 imply that $L_\gamma(\mu,T) = 0$, $T \ge 0$. Therefore

$$
\begin{aligned}
0 = L_\gamma(\mu,T_0) &= \lim_{n\to+\infty}\frac{\mu([t_n,t_n+T_0))}{\gamma(t_n+T_0)} \\
&= \lim_{n\to+\infty}\frac{\alpha_n}{\gamma(t_n+T_0)} = \lim_{n\to+\infty}\frac{\alpha_n}{\gamma(t_n)}\frac{\gamma(t_n)}{\gamma(t_n+T_0)}.
\end{aligned}
\tag{7.14}
$$

But $\lim_{n\to+\infty}(\gamma(t_n)/\gamma(t_n+T_0)) = 1$, and hence statement (a) is proved.

(b) Let $\varepsilon > 0$ and $0 < T < \delta$ be fixed. Then

$$\frac{\mu([t-T,t))}{\gamma(t)} = \begin{cases} \dfrac{\alpha_n}{\gamma(t)}, & \text{if } t \in (t_n, t_n+T], \\ 0, & \text{if } t \in (t_n+T, t_{n+1}] \end{cases} \tag{7.15}$$

for $n \geq 2$. Thus

$$\begin{aligned} 0 &\leq \frac{\mu([t-T,t))}{\gamma(t)} \\ &\leq \begin{cases} \dfrac{\alpha_n}{\gamma(t)} = \dfrac{\alpha_n}{\gamma(t_n)} \dfrac{\gamma(t_n)}{\gamma(t)} = \dfrac{\alpha_n}{\gamma(t_n)} \dfrac{\gamma(t-\delta_n(t))}{\gamma(t)}, & t \in (t_n, t_n+T],\ \delta_n(t) \in (0,\delta) \\ 0, & t \in (t_n+T, t_{n+1}] \end{cases} \end{aligned} \tag{7.16}$$

for $n \geq 2$. But $\gamma \in \Gamma_u,\ \lim_{n\to+\infty}(\alpha_n/\gamma(t_n)) = 0$ and $t_n \longrightarrow +\infty$, therefore

$$L_\gamma(\mu,T) = \lim_{t\to+\infty} \frac{\mu([t-T,t))}{\gamma(t)} = 0. \tag{7.17}$$

The proof is complete. □

Definition 7.6. Let $T > 0$ be fixed.

(a) $\mathcal{B}_T$ denotes the σ-algebra of the Borel sets of $[0,T]$.

(b) $\mathcal{M}_{T,e}$ denotes the set of the finite measures on $\mathcal{B}_T$.

(c) A topology defined on $\mathcal{M}_{T,e}$ is said to be the weak topology on $\mathcal{M}_{T,e}$ if it is the weakest one which makes the mapping

$$\nu \longrightarrow \int_0^T f\,d\nu, \quad \nu \in \mathcal{M}_{T,e} \tag{7.18}$$

continuous for all continuous $f : [0,T] \to \mathbb{R}$.

Definition 7.7. For a fixed $T > 0$ and $t \geq T$, define the shift operator

$$S_{T,t} : [t-T,t] \longrightarrow [0,T], \qquad S_{T,t}(s) := t-s. \tag{7.19}$$

Let $\mu \in \mathcal{M}$. For $B \in \mathcal{B}_T$

$$S_{T,t}(\mu)(B) := \mu\left(S_{T,t}^{-1}(B)\right). \tag{7.20}$$

Proposition 7.8. *Let $\gamma \in \Gamma,\ \mu \in \mathcal{M}_\gamma$, and $T > 0$ be fixed. Then*

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} S_{T,t}(\mu) = \lambda_{\gamma,\mu}, \tag{7.21}$$

where the convergence is in the weak topology of $\mathcal{M}_{T,e}$.

Proof. We should prove that for any fixed continuous function $f : [0,T] \to \mathbb{R}$, we have

$$\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^T f\, dS_{T,t}(\mu) = \int_0^T f\, d\lambda_{\gamma,\mu}. \tag{7.22}$$

For any $A \subset [0,T]$, the function $\chi_A : [0,T] \to \mathbb{R}$ denotes the characteristic function of A.

Let

$$p : [0,T] \longrightarrow \mathbb{R}, \quad p = c_1\chi_{[0,t_1]} + \sum_{i=2}^{k} c_i \chi_{(t_{i-1},t_i]}, \tag{7.23}$$

where $k \in \mathbb{N}^+$, $0 = t_0 < \cdots < t_k = T$, and $c_i \in \mathbb{R}$ $(i = 1, \dots, k)$.

Then from the statement (b) of Proposition 7.2, it follows

$$\begin{aligned}
\lim_{t\to+\infty} \frac{1}{\gamma(t)} \int_0^T p dS_{T,t}(\mu) &= \lim_{t\to+\infty} \frac{1}{\gamma(t)} \left(c_1 \cdot S_{T,t}(\mu)([0,t_1]) + \sum_{i=2}^{k} c_i S_{T,t}(\mu)((t_{i-1},t_i]) \right) \\
&= \lim_{t\to+\infty} \frac{1}{\gamma(t)} \left(c_1 \mu([t-t_1,t]) + \sum_{i=2}^{k} c_i \mu([t-t_i,t-t_{i-1})) \right) \\
&= \lim_{t\to+\infty} \left(c_1 \frac{\mu(\{t\})}{\gamma(t)} + \sum_{i=1}^{k} c_i \left(\frac{\mu([t-t_i,t))}{\gamma(t)} - \frac{\mu([t-t_{i-1},t))}{\gamma(t)} \right) \right) \\
&= \sum_{i=1}^{k} c_i \lambda_{\gamma,\mu}((t_{i-1},t_i]) = \int_0^T p d\lambda_{\gamma,\mu}.
\end{aligned} \tag{7.24}$$

It is known that for a fixed continuous function $f : [0,T] \to \mathbb{R}$, there exists a sequence of step functions (p_n) such that it converges to f uniformly on $[0,T]$. Thus for arbitrarily fixed $\varepsilon > 0$, there is an index $n_0 \in \mathbb{N}^+$ such that

$$|p_{n_0}(t) - f(t)| < \varepsilon, \quad t \in [0,T]. \tag{7.25}$$

In that case

$$\begin{aligned}
&\left| \frac{1}{\gamma(t)} \int_0^T f dS_{T,t}(\mu) - \int_0^T f d\lambda_{\gamma,\mu} \right| \\
&\quad \le \frac{1}{\gamma(t)} \int_0^T |f - p_{n_0}| dS_{T,t}(\mu) + \left| \frac{1}{\gamma(t)} \int_0^T p_{n_0} dS_{T,t}(\mu) - \int_0^T p_{n_0} d\lambda_{\gamma,\mu} \right| + \int_0^T |p_{n_0} - f| d\lambda_{\gamma,\mu} \\
&\quad \le \frac{\varepsilon S_{T,t}(\mu)([0,T])}{\gamma(t)} + \left| \frac{1}{\gamma(t)} \int_0^T p_{n_0} dS_{T,t}(\mu) - \int_0^T p_{n_0} d\lambda_{\gamma,\mu} \right| + \varepsilon\lambda_{\gamma,\mu}([0,T]) \\
&\quad \le \varepsilon(\lambda_{\gamma,\mu}([0,T]) + \varepsilon) + \varepsilon + \varepsilon\lambda_{\gamma,\mu}([0,T])
\end{aligned} \tag{7.26}$$

for all t large enough. Here we used the conclusion of the first part of our proof and statement (b) of Proposition 7.2. Since $\varepsilon > 0$ is fixed but arbitrary, the proof is complete. □

Corollary 7.9. *Let $\gamma \in \Gamma$ and $\mu \in \mathcal{M}_\gamma$.*

(a) *If $f : \mathbb{R}_+ \to \mathbb{R}$ is Borel measurable and Riemann integrable on any interval $[0, T]$, $T > 0$, then $f \in F_{\gamma,\mu}$.*

(b) *If $\lambda_{\gamma,\mu} = 0$ and $f \in F_b$, then $f \in F_{\gamma,\mu}$.*

(c) *Let $\mu \in \mathcal{M}_c$, $\mu = p\lambda$. If $\gamma \in \Gamma_u$, $p \in F_\gamma$ and $f \in \mathcal{L}$, then $f \in F_{\gamma,\mu}$.*

Proof. From Proposition 7.8, it follows (see, e.g., [12]) that if $f \in F_b$ is $\lambda_{\gamma,\mu}$-a.e. continuous, then $f \in F_{\gamma,\mu}$. From this we get statements (a) and (b).

(c) Let $T > 0$ and $\varepsilon > 0$ be fixed. Since $p \in F_\gamma$, there is a $t_0 > 0$ such that

$$\left| \frac{p(t)}{\gamma(t)} - L_\gamma(p) \right| < \varepsilon, \quad t > t_0, \tag{7.27}$$

and hence

$$\left| \frac{p(t-s)}{\gamma(t-s)} - L_\gamma(p) \right| < \varepsilon, \quad s \in [0, T],\ t > T + t_0. \tag{7.28}$$

Since $\gamma \in \Gamma_u$, there is a $t_1 > t_0 + T$ such that

$$\left| \frac{\gamma(t-s)}{\gamma(t)} - 1 \right| < \varepsilon, \quad s \in [0, T],\ t > t_1. \tag{7.29}$$

Thus

$$\begin{aligned} \left| f(s)\left(\frac{p(t-s)}{\gamma(t)} - L_\gamma(p) \right) \right| &\le |f(s)| \left| \frac{p(t-s)}{\gamma(t-s)} - L_\gamma(p) \right| \frac{\gamma(t-s)}{\gamma(t)} + |f(s)| L_\gamma(p) \left| \frac{\gamma(t-s)}{\gamma(t)} - 1 \right| \\ &\le |f(s)| (\varepsilon(1+\varepsilon) + \varepsilon L_\gamma(p)), \quad s \in [0, T],\ t > t_1. \end{aligned} \tag{7.30}$$

From the general transformation theorem for integrals (see, e.g., [12]) and from the translation invariance of the Lebesgue measure λ, we get: for any $B \in \mathcal{B}_T$ and $t \ge T$,

$$S_{T,t}(\mu)(B) = \mu\left(S_{T,t}^{-1}(B)\right) = \int_{S_{T,t}^{-1}(B)} p\, d\lambda = \int_B p \circ S_{T,t}^{-1} dS_{T,t}(\lambda) = \int_B p(t-s) ds. \tag{7.31}$$

But Proposition 7.3 shows that $\mu \in \mathcal{M}_\gamma$ and $\lambda_{\gamma,\mu} = L_\gamma(p)\lambda$. So

$$\begin{aligned}\left|\frac{1}{\gamma(t)}\int_0^T f\,dS_{T,t}(\mu) - \int_0^T f\,d\lambda_{\gamma,\mu}\right| &\le \int_0^T |f(s)|\left|\frac{p(t-s)}{\gamma(t)} - L_\gamma(p)\right|ds \\ &\le \left(\varepsilon(1+\varepsilon) + \varepsilon L_\gamma(p)\right)\int_0^T |f|\,d\lambda.\end{aligned} \tag{7.32}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

Proposition 7.10. *Let $\gamma : \mathbb{R}_+ \to (0,\infty)$, $\mu \in \mathcal{M}$, and assume that $f \in F_\mu$ is not oscillatory on $\mathbb{R}_+$. Then the following mappings have limits in $\mathbb{R}_e := \mathbb{R} \cup \{-\infty,\infty\}$ as $T \to +\infty$:*

$$\begin{aligned}&0 \le T \longrightarrow \limsup_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s) \in \mathbb{R}_e, \\ &0 \le T \longrightarrow \liminf_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s) \in \mathbb{R}_e.\end{aligned} \tag{7.33}$$

Proof. Let f be nonnegative on $[t_0,\infty)$, where t_0 is large enough. Then for $t_0 \le T_1 < T_2$ and $t \in (2T_2,\infty) \cap D_{f*\mu}$, we get

$$\begin{aligned}\frac{1}{\gamma(t)}\int_{[T_1,t-T_1)} f(t-s)\,d\mu(s) &= \frac{1}{\gamma(t)}\int_{[T_1,T_2)} f(t-s)\,d\mu(s) \\ &\quad + \frac{1}{\gamma(t)}\int_{[T_2,t-T_2)} f(t-s)\,d\mu(s) + \frac{1}{\gamma(t)}\int_{[T_2,t-T_2)} f(t-s)\,d\mu(s) \\ &\quad + \frac{1}{\gamma(t)}\int_{[t-T_2,t-T_1)} f(t-s)\,d\mu(s) \ge \frac{1}{\gamma(t)}\int_{[T_2,t-T_2)} f(t-s)\,d\mu(s).\end{aligned} \tag{7.34}$$

Thus the above-defined mappings are decreasing, and hence their limits exist in $\mathbb{R}_e$ as $T \to +\infty$. When f is eventually nonpositive, then the above procedure can be applied for $-f$. The proof is complete. □

In the next two results, we give explicit formulas for the limit inferior and limit superior of the weighted convolution of f and μ at $+\infty$.

Theorem 7.11. *Assume (H). Then the following results hold.*

(a) *The following two statements are equivalent.*

(a_1) *The limit inferior*

$$\underline{L}_\gamma(f*d\mu) := \liminf_{t\to+\infty}\frac{1}{\gamma(t)}\int_0^t f(t-s)\,d\mu(s) \tag{7.35}$$

is finite.

(a$_2$) *For some* $T > 0$, *the limit inferior*

$$\liminf_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s) \tag{7.36}$$

is finite.

(b) *If the limit inferior* (7.36) *is finite for a fixed* $T > 0$, *then it is finite for any* $T > 0$ *and*

$$\underline{L}_\gamma(f*d\mu) = L_\gamma(f)\mu([0,\infty)) + \underline{l}_\gamma(f,\mu) + L_\gamma(\mu,1)\int_0^\infty f, \tag{7.37}$$

where

$$\underline{l}_\gamma(f,\mu) := \lim_{T\to+\infty}\left(\liminf_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s)\right), \tag{7.38}$$

$L_\gamma(f)\mu([0,\infty))$ *and* $L_\gamma(\mu,1)\int_0^\infty f$ *(they are defined in* (3.7) *and* (3.9), *resp.) are finite.*

Proof. Let $T > 0$ be fixed. Then for any $t > 2T$, and $t \in D_{f*\mu}$, we get

$$\begin{aligned}&\frac{1}{\gamma(t)}\int_0^t f(t-s)\,d\mu(s)\\ &\quad= \frac{1}{\gamma(t)}\int_{[0,T)} f(t-s)\,d\mu(s) + \frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s) + \frac{1}{\gamma(t)}\int_{t-T}^t f(t-s)\,d\mu(s).\end{aligned} \tag{7.39}$$

First we show that

$$\lim_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[0,T)} f(t-s)\,d\mu(s) = L_\gamma(f)\mu([0,T)). \tag{7.40}$$

In fact for $t \ge T$ and $s \in [0,T)$, we have

$$\left|\frac{f(t-s)}{\gamma(t)} - L_\gamma(f)\right| \le \left|\frac{f(t-s)}{\gamma(t-s)} - L_\gamma(f)\right|\frac{\gamma(t-s)}{\gamma(t)} + \left|\frac{\gamma(t-s)}{\gamma(t)} - 1\right||L_\gamma(f)|. \tag{7.41}$$

But $\gamma \in \Gamma_u$, $f \in F_\gamma$, therefore for $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\begin{gathered}\left|\frac{\gamma(t-s)}{\gamma(t)} - 1\right| < \varepsilon, \quad s \in [0,T],\ t > t_0,\\ \left|\frac{f(t)}{\gamma(t)} - L_\gamma(f)\right| < \varepsilon, \quad t > t_0.\end{gathered} \tag{7.42}$$

Thus (7.41) yields

$$\left|\frac{f(t-s)}{\gamma(t)}-L_{\gamma}(f)\right|<\varepsilon(1+\varepsilon)+\varepsilon|L_{\gamma}(f)|,\quad s\in[0,T],\ t>t_0+T, \tag{7.43}$$

and hence

$$\begin{aligned}&\left|\frac{1}{\gamma(t)}\int_{[0,T)}f(t-s)d\mu(s)-L_{\gamma}(f)\int_{[0,T)}1d\mu\right|\\&\quad\le\int_{[0,T)}\left|\frac{f(t-s)}{\gamma(t)}-L_{\gamma}(f)\right|d\mu(s)\le\varepsilon(1+\varepsilon+|L_{\gamma}(f)|)\mu([0,T)),\quad t>t_0+T,\ t\in D_{f*\mu},\end{aligned} \tag{7.44}$$

which implies (7.40).

Since $f\in F_{\gamma,\mu}$, we have

$$\lim_{t\to+\infty}\frac{1}{\gamma(t)}\int_{t-T}^{t}f(t-s)d\mu(s)=\int_0^T f\,d\lambda_{\gamma,\mu}. \tag{7.45}$$

Assume that (a_2) holds. Then (7.39), (7.40), and (7.45) imply (a_1). On the other hand, from (7.39), (7.40), and (7.45) we get that (a_1) yields (a_2), and hence statement (a) is proved. This also verifies the first part of statement (b).

Now we prove the second part of statement (b). Assume that (7.36) is finite for any $T>0$. Then (7.39) yields

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_0^t f(t-s)d\mu(s)\\&\quad=L_{\gamma}(f)\mu([0,T))+\liminf_{t\to\infty}\int_{[T,t-T)}f(t-s)d\mu(s)+\int_0^T f\,d\lambda_{\gamma,\mu},\quad T>0.\end{aligned} \tag{7.46}$$

Now assume that f is not oscillatory. Then there exists $t_0>0$ such that either $f(t)\ge 0$ for every $t\ge t_0$ or $f(t)\le 0$ for every $t\ge t_0$. We consider the case when $f(t)\ge 0$ for $t\ge t_0$, the other case can be handled similarly.

All the three terms on the right-hand side of (7.46) have limit as $T\to\infty$ in $\mathbb{R}_e:=\mathbb{R}\cup\{-\infty,\infty\}$. In fact

$$\lim_{T\to+\infty}L_{\gamma}(f)\mu([0,T))=L_{\gamma}(f)\mu([0,\infty))\in[0,\infty], \tag{7.47}$$

and by Proposition 7.10, we get

$$\lim_{T\to+\infty}\left(\liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)}f(t-s)d\mu(s)\right)\in[0,\infty], \tag{7.48}$$

moreover

$$\lim_{T\to+\infty}\int_0^T f\,d\lambda_{\gamma,\mu} = \int_0^{t_0} f\,d\lambda_{\gamma,\mu} + \lim_{T\to+\infty}\int_{t_0}^T f\,d\lambda_{\gamma,\mu} \in (-\infty,\infty]. \tag{7.49}$$

Now the second part of (b) is proved, since the left-hand side of (7.46) is finite and independent on T.

Now assume that f is oscillatory on $\mathbb{R}_+$ and as we assumed $\lim_{T\to+\infty}\int_0^T f\,d\lambda_{\gamma,\mu}$ is finite. In that case $L_\gamma(f) = 0$, and hence

$$\lim_{T\to+\infty} L_\gamma(f)\mu([0,T)) = 0. \tag{7.50}$$

Thus by using similar arguments to those we used above, statement (b) is proved again. □

Theorem 7.12. *Assume (H). Then the following results hold.*

(a) *The following two statements are equivalent.*

(a_1) *The limit superior*

$$\overline{L}_\gamma(f*d\mu) := \limsup_{t\to+\infty}\frac{1}{\gamma(t)}\int_0^t f(t-s)\,d\mu(s) \tag{7.51}$$

is finite.

(a_2) *For some $T>0$ the limit superior*

$$\limsup_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)}^t f(t-s)\,d\mu(s) \tag{7.52}$$

is finite .

(b) *If the limit superior* (7.52) *is finite for a fixed $T>0$, then it is finite for any $T>0$ and*

$$\overline{L}_\gamma(f*d\mu) = L_\gamma(f)\mu([0,\infty)) + \bar{l}_\gamma(f,\mu) + L_\gamma(\mu,1)\int_0^\infty f, \tag{7.53}$$

where

$$\bar{l}_\gamma(f,\mu) := \lim_{T\to+\infty}\left(\limsup_{t\to+\infty}\frac{1}{\gamma(t)}\int_{[T,t-T)} f(t-s)\,d\mu(s)\right), \tag{7.54}$$

$L_\gamma(f)\mu([0,\infty))$ and $L_\gamma(\mu,1)\int_0^\infty f$ (they are defined in (3.7) *and* (3.9)*, resp.) are finite.*

Proof. Its proof is similar to the proof of Theorem 7.11, therefore it is omitted. □

Theorem 7.13. *Let $\gamma \in \Gamma_u$, $f, g \in \mathcal{L} \cap F_\gamma$, and assume that*

(i) *the improper integral*

$$L_\gamma(g)\int_0^\infty f := \lim_{T \to +\infty} L_\gamma(g)\int_0^T f \tag{7.55}$$

is finite, whenever f is oscillatory and g is not oscillatory,

(ii) *the improper integral*

$$L_\gamma(f)\int_0^\infty g := \lim_{T \to +\infty} L_\gamma(f)\int_0^T g \tag{7.56}$$

is finite, whenever f is not oscillatory and g is oscillatory.

Then the following results hold.
(a) *The following two statements are equivalent.*

(a_1) *The limit inferior*

$$\underline{L}_\gamma(f*g) := \liminf_{t \to \infty} \frac{1}{\gamma(t)}\int_0^T f(t-s)g(s)ds \tag{7.57}$$

is finite.

(a_2) *For some $T > 0$ the limit inferior*

$$\liminf_{t \to \infty} \frac{1}{\gamma(t)}\int_0^{t-T} f(t-s)g(s)ds \tag{7.58}$$

is finite.

(b) *If the limit inferior* (7.58) *is finite for a fixed $T > 0$, then it is finite for any $T > 0$ and*

$$\underline{L}_\gamma(f*g) = L_\gamma(f)\int_0^\infty g + \underline{l}_\gamma(f,g) + L_\gamma(g)\int_0^\infty f, \tag{7.59}$$

where

$$\underline{l}_\gamma(f,g) := \lim_{T \to +\infty}\left(\liminf_{t \to +\infty} \frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)d(s)\right), \tag{7.60}$$

$L_\gamma(f)\int_0^\infty g$ *and* $L_\gamma(g)\int_0^\infty f$ *(they are defined in Theorem 3.3) are finite.*

Proof. Let $T > 0$ be fixed. Then for each $t > 2T$ and $t \in D_{f*g}$, we have

$$\begin{aligned}&\frac{1}{\gamma(t)}\int_0^t f(t-s)g(s)ds\\&\quad=\frac{1}{\gamma(t)}\int_0^T f(t-s)g(s)ds+\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds+\frac{1}{\gamma(t)}\int_{t-T}^t f(t-s)g(s)ds.\end{aligned} \tag{7.61}$$

The proof of (7.40) can easily be modified to show that

$$\lim_{t\to\infty}\frac{1}{\gamma(t)}\int_0^T f(t-s)g(s)ds = L_\gamma(f)\int_0^T g. \tag{7.62}$$

Since

$$\int_{t-T}^t f(t-s)g(s)ds = \int_0^T f(s)g(t-s)ds,\quad t > 2T,\ t \in D_{f*g}, \tag{7.63}$$

it follows from (7.62) that

$$\lim_{t\to\infty}\frac{1}{\gamma(t)}\int_{t-T}^t f(t-s)g(s)ds = L_\gamma(g)\int_0^T f. \tag{7.64}$$

By using (7.61), (7.62), and (7.64) instead of (7.39), (7.40), and (7.45), the argument employed in the proof of Theorem 7.11(a) and the first part of (b) extends to give (a) and the first part of (b).

Consider now the proof of (7.59). Suppose that (7.58) is finite for every $T > 0$. By (7.61),

$$\underline{L}_\gamma(f*g) = L_\gamma(f)\int_0^T g + \liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds + L_\gamma(g)\int_0^T f \tag{7.65}$$

for each $T > 0$. We separate the proof into four steps.

(h) Suppose first that f and g are oscillatory. Then $L_\gamma(f) = L_\gamma(g) = 0$, hence (7.65) implies that

$$\underline{L}_\gamma(f*g) = \liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds,\quad T > 0. \tag{7.66}$$

It now follows that $\underline{l}_\gamma(f,g)$ is finite and

$$\underline{L}_\gamma(f*g) = \underline{l}_\gamma(f,g). \tag{7.67}$$

(j) Suppose next that exactly one of the functions f and g is oscillatory. Without loss of generality, we can assume that f is oscillatory and g is not oscillatory. Then $L_\gamma(f) = 0$, hence (7.65) shows that

$$\underline{L}_\gamma(f*g) = \liminf_{t\to\infty} \frac{1}{\gamma(t)} \int_T^{t-T} f(t-s)g(s)ds + L_\gamma(g)\int_0^T f, \quad T > 0. \tag{7.68}$$

By (i), $\underline{l}_\gamma(f,g)$ is finite and

$$\underline{L}_\gamma(f*g) = \underline{l}_\gamma(f,g) + L_\gamma(g)\int_0^\infty f. \tag{7.69}$$

(k) Suppose that there is $T_0 > 0$ such that $f(t) \geq 0$ and $g(t) \geq 0$ for every $t \geq T_0$. Then it follows from $L_\gamma(f) \geq 0$ and

$$L_\gamma(f)\int_0^T g = L_\gamma(f)\int_0^{T_0} g + L_\gamma(f)\int_{T_0}^T g, \quad T > T_0, \tag{7.70}$$

that

$$\lim_{T\to\infty} L_\gamma(f)\int_0^T g \in (-\infty, \infty]. \tag{7.71}$$

A similar argument gives that

$$\lim_{T\to\infty} L_\gamma(g)\int_0^T f \in (-\infty, \infty]. \tag{7.72}$$

Since

$$\liminf_{t\to\infty} \frac{1}{\gamma(t)} \int_T^{t-T} f(t-s)g(s)ds \geq 0, \quad T > T_0, \tag{7.73}$$

we have

$$\liminf_{T\to\infty}\left(\liminf_{t\to\infty} \frac{1}{\gamma(t)} \int_T^{t-T} f(t-s)g(s)ds\right) \geq 0. \tag{7.74}$$

Using (7.65), we deduce from (7.71), (7.72), and (7.74) that the limits in (7.71) and (7.72) are finite, and therefore $\underline{l}_\gamma(f,g)$ exists and is finite. This gives (7.59). If $f(t) \leq 0$ and $g(t) \leq 0$ for every $t \geq T_0$, then a similar proof can be applied.

(l) Suppose finally that there is $T_0 > 0$ such that $f(t) \geq 0$ and $g(t) \leq 0$ for every $t \geq T_0$, or $f(t) \leq 0$ and $g(t) \geq 0$ for every $t \geq T_0$. This case follows by an argument entirely similar to that for the case (k). Here the limits (7.71) and (7.72) are in $[-\infty, \infty)$, and (7.74) is nonpositive. □

Theorem 7.14. *Under the hypotheses of Theorem 7.13 the following results hold.*

(a) *The following two statements are equivalent.*

(a_1) *The limit superior*

$$\overline{L}_\gamma(f*g) := \limsup_{t\to\infty}\frac{1}{\gamma(t)}\int_0^t f(t-s)g(s)ds \tag{7.75}$$

is finite.

(a_2) *For some $T > 0$, the limit superior*

$$\limsup_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds \tag{7.76}$$

is finite.

(b) *If the limit superior* (7.76) *is finite for a fixed $T > 0$, then it is finite for any $T > 0$ and*

$$\overline{L}_\gamma(f*g) = L_\gamma(f)\int_0^\infty g + \bar{l}_\gamma(f,g) + L_\gamma(g)\int_0^\infty f, \tag{7.77}$$

where

$$\bar{l}_\gamma(f,g) := \lim_{T\to+\infty}\left(\limsup_{t\to+\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)d(s)\right), \tag{7.78}$$

$L_\gamma(f)\int_0^\infty g$ and $L_\gamma(g)\int_0^\infty f$ (they are defined in Theorem 3.3) are finite.

Proof. The proof is similar to the proof of the previous theorem, therefore it is omitted. □

8. The proofs of the main results

In this section, we give the proofs of the results stated in Sections 3–6.

Proof of Theorem 3.1. A similar argument employed in the proof of Theorem 7.11 gives the equivalence of (a_1) and (a_2), and part (b). It is clear from (a_2) and (b) that (a_2) implies (a_3). If (a_3) holds, then by Theorems 7.11 and 7.12, the values of $\underline{L}_\gamma(f*d\mu)$ and $\overline{L}_\gamma(f*d\mu)$ are finite. Since $\underline{l}_\gamma(f,\mu) = \bar{l}_\gamma(f,\mu)$, it follows from (7.37) and (7.53) that $\underline{L}_\gamma(f*d\mu) = \overline{L}_\gamma(f*d\mu)$. This shows that ($a_3$) yields ($a_1$). □

Proof of Theorem 3.3. This is an immediate consequence of Theorem 3.1. □

Proof of Theorem 3.4. The argument of Theorem 7.13 can easily be generalized to prove the equivalence of (a_1) and (a_2), and part (b). (a_1) and (b) imply (a_3). If (a_3) holds, then we can apply Theorems 7.13 and 7.14. It now follows from (7.59) and (7.77) that $\underline{L}_\gamma(f*g) = \overline{L}_\gamma(f*g)$, and therefore ($a_2$) is satisfied. □

Proof of Theorem 1.1. First we suppose that g is nonnegative. It follows from $g \in \mathcal{L}^1$ that

$$\lim_{t\to\infty}\int_{t-T}^{t} g = 0 \tag{8.1}$$

for every $T \geq 0$. We can see that the hypothesis (H_c) is satisfied with $\gamma : \mathbb{R}_+ \to (0,\infty)$, $\gamma(t) = 1$, and $L_\gamma(g,1) = 0$. This shows that Theorem 3.3 can be applied. Consider now the proof that (a_3) holds, and $l_\gamma(f,g) = 0$. There exists $t_0 > 0$ such that

$$|f(t)| < |f(\infty)| + 1, \quad t > t_0. \tag{8.2}$$

This implies that

$$\int_{T}^{t-T} |f(t-s)| g(s)\,ds \leq (|f(\infty)| + 1)\int_{T}^{t-T} g, \quad T > t_0,\ t > 2T,\ t \in D_{f*g}, \tag{8.3}$$

hence

$$\limsup_{t\to\infty}\int_{T}^{t-T} |f(t-s)| g(s)\,ds \leq (|f(\infty)| + 1)\int_{T}^{\infty} g, \quad T > t_0, \tag{8.4}$$

and therefore, by $g \in \mathcal{L}^1$,

$$\lim_{T\to\infty}\left(\limsup_{t\to\infty}\int_{T}^{t-T} |f(t-s)| g(s)\,ds\right) = 0. \tag{8.5}$$

Since

$$\begin{aligned}
-\limsup_{t\to\infty}\int_{T}^{t-T} |f(t-s)| g(s)\,ds &\leq \liminf_{t\to\infty}\int_{T}^{t-T} f(t-s) g(s)\,ds \\
&\leq \limsup_{t\to\infty}\int_{T}^{t-T} f(t-s) g(s)\,ds \\
&\leq \limsup_{t\to\infty}\int_{T}^{t-T} |f(t-s)| g(s)\,ds,
\end{aligned} \tag{8.6}$$

it follows from (8.5) that (a_3) is true, and $l_\gamma(f,g) = 0$. Now (3.15) gives the result.

In the general case, the preceding can be applied to both g^+ and g^-. □

Proof of Theorem 4.3. γ is a subexponential function, and therefore $\gamma \in \Gamma_u$ and

$$L_\gamma(\gamma * \gamma) = 2\int_{0}^{\infty} \gamma. \tag{8.7}$$

Theorem 3.4 may now be applied with $f = g := \gamma$, and $l_\gamma(\gamma,\gamma) = 0$ is obtained. Thus the result follows from Theorem 6.1 with $p = q := \gamma$. □

Proof of Theorem 4.5. Suppose that H is subexponential. Then H is long-tailed as is well known (details can be found in [11]), and thus $\overline{H} \in \Gamma_u$. The distribution function H generates a distribution $\mu_H \in \mathcal{M}$. Since

$$\frac{\mu_H([t-T,t))}{\overline{H}(t)} = \frac{H(t)-H(t-T)}{\overline{H}(t)} = -1 + \frac{\overline{H}(t-T)}{\overline{H}(t)}, \quad T>0,\ t\geq T, \tag{8.8}$$

we can see that $L_{\overline{H}}(\mu_H, T) = 0$ for every $T \geq 0$, and therefore $\mu \in \mathcal{M}_{\overline{H}}$. It follows that the hypothesis (H) is satisfied with $\gamma = f := \overline{H}$ and $\mu := \mu_H$. This shows that Theorem 3.1 can be applied. Since H is subexponential, the equivalence of (a_1) and (a_2) implies (4.9), and (3.6) gives (4.10). We have proved that (b) comes from (a). On the other hand, (c) obviously follows from (b).

By the equivalence of (a_3) and (a_1) in Theorem 3.1, (c) implies (a). □

Proof of Theorem 4.6. H is long-tailed, hence $\overline{H} \in \Gamma_u$. The distribution function G characterizes a distribution $\mu_G \in \mathcal{M}$. Then

$$\begin{aligned}\frac{\mu_G([t-T,t))}{\overline{H}(t)} &= \frac{G(t)-G(t-T)}{\overline{H}(t)} \\ &= -\frac{\overline{G}(t)}{\overline{H}(t)} + \frac{\overline{G}(t-T)}{\overline{H}(t-T)}\frac{\overline{H}(t-T)}{\overline{H}(t)}, \quad T>0,\ t\geq T,\end{aligned} \tag{8.9}$$

giving

$$\lim_{t\to\infty}\frac{\mu_G([t-T,t))}{\overline{H}(t)} = -L_{\overline{H}}(\overline{G}) + L_{\overline{H}}(\overline{G}) = 0, \quad T\geq 0, \tag{8.10}$$

and therefore $\mu_G \in \mathcal{M}_{\overline{H}}$ and $L_{\overline{H}}(\mu_G, T) = 0,\ T \geq 0$.

It follows that the condition (H) is satisfied with $\gamma := \overline{H},\ \mu := \mu_G$ and $f := \overline{F}$.

According to Theorem 3.1(a_3) and (3.6), we now have

$$\lim_{t\to\infty}\frac{1}{\overline{H}(t)}\int_0^t \overline{F}(t-s)dG(s) = L_{\overline{H}}(\overline{F}). \tag{8.11}$$

Applying this and taking into account Proposition 7.2(b), the result follows, since

$$\begin{aligned}\frac{1}{\overline{H}(t)}\left(1-\int_0^t F(t-s)dG(s)\right) &= \frac{1-G(t+)}{\overline{H}(t)} + \frac{1}{\overline{H}(t)}\int_0^t \overline{F}(t-s)dG(s) \\ &= \frac{\overline{G}(t)}{\overline{H}(t)} - \frac{\mu_G(\{t\})}{\overline{H}(t)} + \frac{1}{\overline{H}(t)}\int_0^t \overline{F}(t-s)dG(s), \quad t>0.\end{aligned} \tag{8.12}$$

□

Proof of Theorem 5.5. By the correspondence between hypotheses (H) and (H(α)), Theorem 3.1 implies the result. □

Proof of Theorem 6.1. By Theorem 3.4, we deduce that

$$L_\gamma(p*q) = L_\gamma(p)\int_0^\infty q + l_\gamma(p,q) + L_\gamma(q)\int_0^\infty p. \tag{8.13}$$

Let $\varepsilon > 0$. Since $f \in F_p$ and $g \in F_q$, we can find $t_0 > 0$ such that

$$L_p(f)L_q(g) - \varepsilon < \frac{f(t)}{p(t)}\frac{g(t)}{q(t)} < L_p(f)L_q(g) + \varepsilon, \quad t > t_0, \tag{8.14}$$

and therefore also

$$L_p(f)L_q(g) - \varepsilon < \frac{f(t-s)}{p(t-s)}\frac{g(t)}{q(t)} < L_p(f)L_q(g) + \varepsilon, \quad s \in [T, t-T],\ T > t_0,\ t > 2T. \tag{8.15}$$

This implies that

$$\begin{aligned}
&(L_p(f)L_q(g) - \varepsilon)\frac{1}{\gamma(t)}\int_T^{t-T} p(t-s)q(s)ds \\
&\quad\le \frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds \\
&\quad= \frac{1}{\gamma(t)}\int_T^{t-T} \frac{f(t-s)}{p(t-s)}\frac{g(s)}{q(s)}p(t-s)q(s)ds \\
&\quad\le (L_p(f)L_q(g) + \varepsilon)\frac{1}{\gamma(t)}\int_T^{t-T} p(t-s)q(s)ds, \quad T > t_0,\ t > 2T.
\end{aligned} \tag{8.16}$$

Equation (8.13) shows that $l_\gamma(p,q)$ is finite, hence the definition of $l_\gamma(p,q)$ and the previous inequality give

$$\begin{aligned}
(L_p(f)L_q(g) - \varepsilon)l_\gamma(p,q) &\le \liminf_{T\to\infty}\left(\liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds\right) \\
&\le \limsup_{T\to\infty}\left(\limsup_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds\right) \\
&\le (L_p(f)L_q(g) + \varepsilon)l_\gamma(p,q),
\end{aligned} \tag{8.17}$$

and therefore

$$\begin{aligned}
L_p(f)L_q(g)l_\gamma(p,q) &= \lim_{T\to\infty}\left(\liminf_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds\right) \\
&= \lim_{T\to\infty}\left(\limsup_{t\to\infty}\frac{1}{\gamma(t)}\int_T^{t-T} f(t-s)g(s)ds\right).
\end{aligned} \tag{8.18}$$

The result now follows from Theorem 3.4 (see Theorem 3.3(a_3)) by applying

$$L_\gamma(f) = \lim_{t\to\infty}\frac{f(t)}{p(t)}\frac{p(t)}{\gamma(t)} = L_p(f)L_\gamma(p) \tag{8.19}$$

and $L_\gamma(g) = L_q(g)L_\gamma(q)$. □

Proof of Corollary 6.2. If

$$\gamma : \mathbb{R}_+ \longrightarrow (0,\infty), \qquad \gamma(t) := \begin{cases} t^{\alpha+\beta-1}, & \text{if } t > 0, \\ 1, & \text{if } t = 0, \end{cases} \tag{8.20}$$

then $\gamma \in \Gamma_u$. Let $p, q : \mathbb{R}_+ \to (0,\infty)$ be defined by

$$p(t) := \begin{cases} t^{\alpha-1}, & \text{if } t > 0, \\ 1, & \text{if } t = 0, \end{cases} \qquad q(t) := \begin{cases} t^{\beta-1}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases} \tag{8.21}$$

Then $L_\gamma(p) = L_\gamma(q) = 0$, and therefore $p, q \in \mathcal{L} \cap F_\gamma$. By (6.3) and Theorem 6.1, it is enough to prove

$$\lim_{t\to\infty}\frac{1}{\gamma(t)}\int_0^t p(t-s)q(s)ds = B(\alpha,\beta), \tag{8.22}$$

which comes from the definition of the Beta function. □

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