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Research Article

Existence of Solutions for Nonconvex and Nonsmooth Vector Optimization Problems

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We consider the weakly efficient solution for a class of nonconvex and nonsmooth vector optimization problems in Banach spaces. We show the equivalence between the nonconvex and nonsmooth vector optimization problem and the vector variational-like inequality involving set-valued mappings. We prove some existence results concerned with the weakly efficient solution for the nonconvex and nonsmooth vector optimization problems by using the equivalence and Fan-KKM theorem under some suitable conditions.

1. Introduction

The concept of vector variational inequality was first introduced by Giannessi [1] in 1980. Since then, existence theorems for solution of general versions of the vector variational inequality have been studied by many authors (see, e.g., [2–9] and the references therein). Recently, vector variational inequalities and their generalizations have been used as a tool to solve vector optimization problems (see [7, 10–14]). Chen and Craven [11] obtained a sufficient condition for the existence of weakly efficient solutions for differentiable vector optimization problems involving differentiable convex functions by using vector variational inequalities for vector valued functions. Kazmi [12] proved a sufficient condition for the existence of weakly efficient solutions for vector optimization problems involving differentiable preinvex functions by using vector variational-like inequalities. For the nonsmooth case, Lee et al. [7] established the existence of the weakly efficient solution for nondifferentiable vector optimization problems by using vector variational-like inequalities for set-valued mappings. Similar results can be found in [10]. It is worth mentioning that Lee et al. [7] and Ansari and Yao [10] obtained their existence results under the assumption that $R_+^m \subset C(x)$ for all $x \in R^n$, where $C(x)$ is a convex cone in R^m. However, this condition is restrict and it does not hold in general.

In this paper, we consider the weakly efficient solution for a class of nonconvex and nonsmooth vector optimization problems in Banach spaces. We show the equivalence between the nonconvex and nonsmooth vector optimization problem and the vector variational-like inequality involving set-valued mappings. We prove some existence results concerned with the weakly efficient solution for the nonconvex and nonsmooth vector optimization problems by using the equivalence and Fan-KKM theorem without the restrict condition $R_+^m \subset C(x)$ for all $x \in R^n$. Our results generalize and improve the results obtained by Lee et al. [7] and Ansari and Yao [10].

2. Preliminaries

Let X be a real Banach space endowed with a norm $\|\cdot\|$ and X^* its dual space, we denote by $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*. Let R^m be the m-dimensional Euclidean space, let $S \subset X$ be a nonempty subset, and let $K \subset R^m$ be a nonempty closed convex cone with $\operatorname{int} K \neq \emptyset$, where int denotes interior.

Definition 2.1. A real valued function $h : X \to R$ is said to be locally Lipschitz at a point $x \in X$ if there exists a number $L > 0$ such that

$$|h(y) - h(z)| \leq L\|y - z\| \tag{2.1}$$

for all y, z in a neighborhood of x. h is said to be locally Lipschitz on X if it is locally Lipschitz at each point of X.

Definition 2.2. Let $h : X \to R$ be a locally Lipschitz function. Clarke [15] generalized directional derivative of h at $x \in X$ in the direction v, denoted by $h^\circ(x; v)$, is defined by

$$h^\circ(x; v) = \limsup_{y \to x,\, t \downarrow 0} \frac{h(y + tv) - h(y)}{t}. \tag{2.2}$$

Clarke [15] generalized gradient of h at $x \in X$, denoted by $\partial h(x)$, is defined by

$$\partial h(x) = \left\{ \xi \in X^* : h^\circ(x; v) \geq \langle \xi, d \rangle \ \forall v \in X \right\}. \tag{2.3}$$

Let $f : X \to R^m$ be a vector valued function given by $f = (f_1, f_2, \ldots, f_m)$, where each f_i, $i = 1, 2, \ldots, m$, is a real valued function defined on X. Then f is said to be locally Lipschitz on X if each f_i is locally Lipschitz on X.

The generalized directional derivative of a locally Lipschitz function $f : X \to R^m$ at $x \in X$ in the direction v is given by

$$f^\circ(x; v) = \left(f_1^\circ(x; v), f_2^\circ(x; v), \ldots, f_m^\circ(x; v) \right). \tag{2.4}$$

The generalized gradient of h at x is the set

$$\partial f(x) = \partial f_1(x) \times \partial f_2(x) \times \cdots \times \partial f_m(x), \tag{2.5}$$

where $\partial f_i(x)$ is the generalized gradient of f_i at x for $i = 1, 2, \ldots, m$.

Every element $A = (\xi_1, \xi_2, \ldots, \xi_m) \in \partial f(x)$ is a continuous linear operator from X to R^m and

$$Ay = \left(\langle \xi_1, y \rangle, \langle \xi_2, y \rangle, \ldots, \langle \xi_m, y \rangle \right) \in R^m, \quad \forall y \in X. \tag{2.6}$$

Definition 2.3. Let $f : X \to R^m$ be a locally Lipschitz function.

(i) f is said to be K-invex with respect to η at $u \in X$, if there exists $\eta : X \times X \to X$ such that for all $x \in X$ and $A \in \partial f(u)$,

$$f(x) - f(u) - \langle A, \eta(x,u) \rangle \in K. \tag{2.7}$$

(ii) f is said to be K-pseudoinvex with respect to η at $u \in X$ if there exists $\eta : X \times X \to X$ such that for all $x \in X$ and $A \in \partial f(u)$,

$$f(x) - f(u) \in -\operatorname{int} K \Longrightarrow \langle A, \eta(x,u) \rangle \in -\operatorname{int} K. \tag{2.8}$$

In this paper, we consider the following nonsmooth vector optimization problem:

$$\begin{array}{ll} K\text{-minimize} & f(x), \\ \text{subject to} & x \in S, \end{array} \tag{VOP}$$

where $f = (f_1, f_2, \dots, f_m)$, $f_i : X \to R$, $i = 1, 2, \dots, m$, are locally Lipschitz functions.

Definition 2.4. A point $x_0 \in S$ is said to be a weakly efficient solution of f if there exists no $y \in S$ such that

$$f(y) - f(x) \in -\operatorname{int} K. \tag{2.9}$$

In order to prove our main results, we need the following definition and lemmas.

Definition 2.5 (see [16]). A multivalued mapping $G : X \to 2^X$ is called KKM-mapping if for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X, $\operatorname{co}\{x_1, x_2, \dots, x_n\}$ is contained in $\bigcup_{i=1}^n G(x_i)$, where $\operatorname{co} A$ denotes the convex hull of the set A.

Lemma 2.6 (see [16]). *Let M be a nonempty subset of a Hausdorff topological vector space X. Let $G : M \to 2^X$ be a KKM-mapping such that $G(x)$ is closed for any $x \in M$ and is compact for at least one $x \in M$. Then $\bigcap_{y \in M} G(y) \neq \emptyset$.*

Lemma 2.7 (see [2]). *Let K be a convex cone of topological vector space X. If $y - x \in K$ and $x \notin -\operatorname{int} K$, then $y \notin -\operatorname{int} K$ for any $x, y \in X$.*

3. Main results

In order to obtain our main results, we introduce the following vector variational-like inequality problem, which consists in finding $x_0 \in S$ such that for all $A \in \partial f(x_0)$,

$$\langle A, \eta(y, x_0) \rangle \notin -\operatorname{int} K, \quad \forall y \in S. \tag{VVIP}$$

First, we establish the following relations between (VOP) and (VVIP).

Lemma 3.1. *Let* $f : X \to R^m$ *be a locally Lipschitz function and* $\eta : S \times S \to X$*. Then the following arguments hold.*

(i) *Suppose that* f *is* K*-invex with respect to* η*. If* x_0 *is a solution of* (VVIP)*, then* x_0 *is a weakly efficient solution of* (VOP).

(ii) *Suppose that* f *is* K*-pseudoinvex with respect to* η*. If* x_0 *is a solution of* (VVIP)*, then* x_0 *is a weakly efficient solution of* (VOP).

(iii) *Suppose that* f *is* $-K$*-invex with respect to* η*. If* x_0 *is a weakly efficient solution of* (VOP)*, then* x_0 *is a solution of* (VVIP).

Proof. (i) Let x_0 be a solution of (VVIP). Then

$$\langle A, \eta(y, x_0) \rangle \notin -\operatorname{int} K, \quad \forall A \in \partial f(x_0),\ y \in S. \tag{3.1}$$

By the K-invexity of f with respect to η, we get

$$f(y) - f(x_0) - \langle A, \eta(y, x_0) \rangle \in K, \quad \forall A \in \partial f(x_0),\ y \in S. \tag{3.2}$$

From (3.1), (3.2) and Lemma 2.7, we obtain

$$f(y) - f(x_0) \notin -\operatorname{int} K, \quad \forall y \in S. \tag{3.3}$$

Therefore, x_0 is a weakly efficient solution of (VOP).

(ii) Let x_0 be a solution of (VVIP). Suppose that x_0 is not a weakly efficient solution of (VOP). Then, there exists $y \in S$ such that

$$f(y) - f(x_0) \in -\operatorname{int} K. \tag{3.4}$$

Since f is K-pseudoinvex with respect to η, then

$$\langle A, \eta(y, x_0) \rangle \in -\operatorname{int} K, \quad \forall A \in \partial f(x_0), \tag{3.5}$$

which contradicts the fact that x_0 is a solution of (VVIP).

(iii) Assume that x_0 is a weakly efficient solution of (VOP). Then,

$$f(y) - f(x_0) \notin -\operatorname{int} K, \quad \forall y \in S. \tag{3.6}$$

Since f is $-K$-invex with respect to η, then

$$f(y) - f(x_0) - \langle A, \eta(y, x_0) \rangle \in -K, \quad \forall A \in \partial f(x_0),\ y \in S. \tag{3.7}$$

It follows from Lemma 2.7 that

$$\langle A, \eta(y, x_0) \rangle \notin -\operatorname{int} K, \quad \forall A \in \partial f(x_0),\ y \in S. \tag{3.8}$$

Therefore, x_0 is a solution of (VVIP). □

Now we establish the following existence theorem.

Theorem 3.2. *Let $S \subset X$ be a nonempty convex set and $\eta : S \times S \to X$. Let $f : X \to R^m$ be a locally Lipschitz K-pseudoinvex function. Assume that the following conditions hold*

(i) $\eta(x,x) = 0$ *for any $x \in S$, $\eta(y,x)$ is affine with respect to y and continuous with respect to x;*

(ii) *there exist a compact subset D of S and $y_0 \in D$ such that*

$$\langle A, \eta(y_0,x)\rangle \in -\operatorname{int} K, \quad \forall x \in S \setminus D,\ A \in \partial f(x). \tag{3.9}$$

Then (VOP) *has a weakly efficient solution.*

Proof. By Lemma 3.1(ii), it suffices to prove that (VVIP) has a solution. Define $G : S \to 2^S$ by

$$G(y) = \{x \in S : \langle A, \eta(y,x)\rangle \notin -\operatorname{int} K, \forall A \in \partial f(x)\}, \quad \forall y \in S. \tag{3.10}$$

First we show that G is a KKM-mapping. By condition (i), we get $y \in G(y)$. Hence, $G(y) \neq \varnothing$ for all $y \in S$. Suppose that there exists a finite subset $\{x_1, x_2, \ldots, x_m\} \subseteq S$ and that $\alpha_i \geq 0$, $i = 1,2,\ldots,m$, with $\sum_{i=1}^m \alpha_i = 1$ such that $x = \sum_{i=1}^m \alpha_i x_i \notin \bigcup_{i=1}^m G(x_i)$. Then, $x \notin G(x_i)$ for all $i = 1,2,\ldots,m$. It follows that there exists $A \in \partial f(x)$ such that

$$\langle A, \eta(x_i,x)\rangle \in -\operatorname{int} K, \quad i = 1,2,\ldots,m. \tag{3.11}$$

Since K is a convex cone and η is affine with respect to the first argument,

$$\langle A, \eta(x,x)\rangle \in -\operatorname{int} K. \tag{3.12}$$

which gives $0 \in -\operatorname{int} K$. This is a contradiction since $0 \notin -\operatorname{int} K$. Therefore, G is a KKM-mapping.

Next, we show that $G(y)$ is a closed set for any $y \in S$. In fact, let $\{x_n\}$ be a sequence of $G(y)$ which converges to some $x_0 \in S$. Then for all $A_n \in \partial f(x_n)$, we have

$$\langle A_n, \eta(y,x_n)\rangle \notin -\operatorname{int} K. \tag{3.13}$$

Since f is locally Lipschitz, then there exists a neighborhood $N(x_0)$ of x_0 and $L > 0$ such that for any $x, y \in N(x_0)$,

$$|f(x) - f(y)| \leq L\|x - y\|. \tag{3.14}$$

It follows that for any $x \in N(x_0)$ and any $A \in \partial f(x)$, $\|A\| \leq L$. Without loss of generality, we may assume that A_n converges to A_0. Since the set-valued mapping $x \mapsto \partial f(x)$ is closed (see [15, page 29]) and $A_n \in \partial f(x_n)$, $A_0 \in \partial f(x_0)$. By the continuity of $\eta(y,x)$ with respect to the second argument, we have

$$\langle A_n, \eta(y,x_n)\rangle \longrightarrow \langle A_0, \eta(y,x_0)\rangle. \tag{3.15}$$

Since $R^m \setminus -\operatorname{int} K$ is closed, one has

$$\langle A_0, \eta(y,x_0)\rangle \notin -\operatorname{int} K. \tag{3.16}$$

Hence, $G(y)$ is a closed set for any $y \in S$.

By condition (ii), we have $G(y_0) \subset D$. As $G(y_0)$ is closed and D is compact, $G(y_0)$ is compact. Therefore, by Lemma 2.6, we have that there exists $x^* \in S$ such that

$$x^* \in \bigcap_{y \in S} G(y), \tag{3.17}$$

or equivalently,

$$\langle A, \eta(y, x^*) \rangle \notin -\operatorname{int} K, \quad \forall A \in \partial f(x^*),\ y \in S. \tag{3.18}$$

That is, x^* is a solution of (VVIP). This completes the proof. □

Corollary 3.3. *Let $S \subset X$ be a nonempty convex set and $\eta : S \times S \to X$. Let $f : X \to R^m$ be a locally Lipschitz K-invex function. Assume that the following conditions hold:*

(i) *$\eta(x, x) = 0$ for any $x \in S$, $\eta(y, x)$ is affine with respect to y and continuous with respect to x;*

(ii) *there exist a compact subset D of S and $y_0 \in D$ such that*

$$\langle A, \eta(y_0, x) \rangle \in -\operatorname{int} K, \quad \forall x \in S \setminus D,\ A \in \partial f(x). \tag{3.19}$$

Then (VOP) *has a weakly efficient solution.*

Proof. Since a K-invex function is K-pseudoinvex, by Theorem 3.2, we obtain the result. □

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