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Research Article

On a New Weighted Hilbert Inequality

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It is shown that a weighted Hilbert inequality for double series can be established by introducing a proper weight function. Thus, a quite sharp result of the classical Hilbert inequality for double series is obtained. And a similar result for the Hilbert integral inequality is also proved. Some applications are considered.

1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. It is all known that the inequality

$$\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}\right)^2 \leq \pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2 \tag{1.1}$$

is called Hilbert theorem for double series [1], where $\sum_{n=1}^{\infty}a_n^2 < +\infty$, $\sum_{n=1}^{\infty}b_n^2 < +\infty$, and the constant factor π^2 in (1.1) is the best possible value. And the equality in (1.1) holds if and only if $\{a_n\}$ or $\{b_n\}$ is a zero-sequence. The corresponding integral form of (1.1) is that

$$\left\{\iint_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy\right\}^2 \leq \pi^2\int_0^{\infty}f^2(x)dx\int_0^{\infty}g^2(x)dx, \tag{1.2}$$

where $\int_0^{\infty}f^2(x)dx < +\infty$, $\int_0^{\infty}g^2(x)dx < +\infty$, and the constant factor π^2 in (1.2) is the best possible value. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [2–6], etc.). The aim of the present paper is to build some new inequalities by using the weight function method and the technique of analysis, and then to study some applications of them.

First we give some lemmas.

Lemma 1.1. *Let n be a positive integer. Then*

$$\int_0^{\infty}\frac{dx}{(n+x^2)(1+x)}=\frac{1}{n+1}\left(\frac{\pi}{2\sqrt{n}}+\frac{1}{2}\ln n\right). \tag{1.3}$$

Proof. Let a, e, f be real numbers. Then

$$\int\frac{dx}{(a^2+x^2)(e+fx)}=\frac{1}{e^2+a^2f^2}\left\{f\ln|e+fx|-\frac{1}{2}\ln\left(a^2+x^2\right)+\frac{e}{a}\arctan\frac{x}{a}\right\}+C, \tag{1.4}$$

where C is an arbitrary constant. This result is given in the paper (see [7]). Based on this indefinite integral it is easy to deduce that the equality (1.3) holds. □

Lemma 1.2. *If* $f(x)=(1/(x+n))(n/x)^{1/2}(1-(\sqrt{x}/(1+\sqrt{x})-\sqrt{n}/(1+\sqrt{n})))$ *and* $g(x)=(1/(x+n))(n/x)^{1/2}(1+(\sqrt{x}/(1+\sqrt{x})-\sqrt{n}/(1+\sqrt{n})))$, *where* $n\in N$, $x\in(0,+\infty)$, *then*

(1) $f(x)$ *and* $g(x)$ *are monotonously decreasing in interval* $(0,+\infty)$;

(2)

$$\int_0^{\infty}f(x)dx=\pi-\pi\omega(n),\qquad \int_0^{\infty}g(x)dx=\pi+\pi\omega(n), \tag{1.5}$$

where the weight function ω is defined by

$$\omega(n)=\frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}-\frac{\ln n}{\pi}\right). \tag{1.6}$$

Proof. (1) At first, notice that $1-\sqrt{x}/(1+\sqrt{x})=1/(1+\sqrt{x})$, hence we can write $f(x)$ in the form of: $f(x)=f_1(x)+f_2(x)$, where $f_1(x)=(1/(x+n)\sqrt{x})(n/(1+\sqrt{n}))$ and $f_2(x)=\sqrt{n}/(x+n)(1+\sqrt{x})\sqrt{x}$. It is obvious that the functions $f_1(x)$ and $f_2(x)$ are monotonously decreasing in $(0,+\infty)$. So, $f(x)$ is also monotonously decreasing in $(0,+\infty)$. In the next place, notice that $1-\sqrt{n}/(1+\sqrt{n})=1/(1+\sqrt{n})$, therefore we can write $g(x)$ in the form of: $g(x)=g_1(x)+g_2(x)$, where $g_1(x)=\sqrt{n}/(1+\sqrt{n})(x+n)\sqrt{x}$ and $g_2(x)=\sqrt{n}/(x+n)(1+\sqrt{x})$. It is clear that the functions $g_1(x)$ and $g_2(x)$ are monotonously decreasing in $(0,+\infty)$. So, $g(x)$ is also monotonously decreasing in $(0,+\infty)$.

(2) Below, we need only to compute the first integral,

$$\begin{aligned}\int_0^{\infty}f(x)dx&=\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{1/2}\right)\left(1-\left(\frac{\sqrt{x}}{1+\sqrt{x}}-\frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)dx\\&=\left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{1/2}\right)dx-\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{1/2}\right)\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)dx\\&=\pi\left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)-2\sqrt{n}\left(\int_0^{\infty}\frac{1}{(n+t^2)}dt-\int_0^{\infty}\frac{1}{(n+t^2)(1+t)}dt\right)\\&=\pi-\left\{\pi-2\sqrt{n}\int_0^{\infty}\frac{1}{(n+t^2)(1+t)}dt-\frac{\sqrt{n}\pi}{1+\sqrt{n}}\right\}.\end{aligned} \tag{1.7}$$

By Lemma 1.1, we obtain the first integral of (1.5) at once after some simple computations and simplifications.

Similarly, the second integral of (1.5) can be gotten. □

Lemma 1.3. *Let* $\{a_n\}$ *be a sequence of real numbers, and let* $c(x)$ *be a real function and* $1-c(n)+c(m) \geq 0$ $(n, m \in N)$. *If* $\sum_{n=1}^{\infty} a_n^2 < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}(1-c(n)+c(m)). \tag{1.8}$$

Proof. It is obvious that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}(1-c(n)+c(m)) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(n) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(m). \tag{1.9}$$

We need only to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(m). \tag{1.10}$$

Let $h(m) = \sum_{k=1}^{\infty}(a_k/(m+k))$. Then

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(m)\\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)a_m c(m) = \sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{a_k}{m+k}\right)a_m c(m) = \sum_{m=1}^{\infty}h(m)a_m c(m) = \sum_{n=1}^{\infty}h(n)a_n c(n)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{a_k}{n+k}\right)a_n c(n) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{n+m}\right)a_n c(n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m a_n}{n+m}c(n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}c(n).
\end{aligned} \tag{1.11}$$

□

Lemma 1.4. *Let* α *be a real number, and let* $f(x)$ *and* $c(x)$ *be two real functions, and* $\int_{\alpha}^{\infty} f^2(x)dx < +\infty$ *and* $1 - c(x-\alpha) + c(y-\alpha) \geq 0$, *where* $(x, y) \in (\alpha, +\infty) \times (\alpha, +\infty)$. *Then*

$$\iint_{\alpha}^{\infty}\frac{f(x)f(y)}{x+y-2\alpha}\,dx\,dy = \iint_{\alpha}^{\infty}\frac{f(x)f(y)}{x+y-2\alpha}(1-c(x-\alpha)+c(y-\alpha))dx\,dy. \tag{1.12}$$

Its proof is similar to that of Lemma 1.3. Hence, it is omitted.

2. Main results

Theorem 2.1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of real numbers. If* $\sum_{n=1}^{\infty} a_n^2 < +\infty$ *and , then*

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\right)^4 \leq \pi^4\left\{\left(\sum_{n=1}^{\infty}a_n^2\right)^2 - \left(\sum_{n=1}^{\infty}\omega(n)a_n^2\right)^2\right\}\left\{\left(\sum_{n=1}^{\infty}b_n^2\right)^2 - \left(\sum_{n=1}^{\infty}\omega(n)b_n^2\right)^2\right\}, \tag{2.1}$$

where the weight function $\omega(n)$ *is defined by* (1.6).

Proof. Let $c(x)$ be a real function and it satisfies condition $1 - c(n) + c(m) \geq 0$ $(n, m \in N)$. First, we suppose that $b_n = a_n$. By Lemma 1.3 and then applying Cauchy's inequality we have

$$\begin{aligned}
&\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\right)^2 \\
&\quad = \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}(1-c(n)+c(m))\right)^2 \\
&\quad = \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{a_m(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{m}{n}\right)^{1/4}\right)\left(\frac{a_n(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{n}{m}\right)^{1/4}\right)\right)^2 \\
&\quad \leq J_1 J_2,
\end{aligned} \tag{2.2}$$

where

$$J_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{m+n}\left(\frac{m}{n}\right)^{1/2}(1-c(n)+c(m)), \quad J_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n^2}{m+n}\left(\frac{n}{m}\right)^{1/2}(1-c(n)+c(m)). \tag{2.3}$$

It is easy to deduce that

$$J_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{m+n}\left(\frac{m}{n}\right)^{1/2}(1-c(n)+c(m)) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/2}(1-c(m)+c(n))\right)a_n^2. \tag{2.4}$$

Let $c(x) = \sqrt{x}/(1+\sqrt{x})$. It is obvious that $1 - \sqrt{x}/(1+\sqrt{x}) + \sqrt{y}/(1+\sqrt{y}) \geq 0$ for $x \geq 0$ and $y \geq 0$. Consider the function $f(x) = ((1/(x+n))(n/x)^{1/2})(1 - (\sqrt{x}/(1+\sqrt{x}) - \sqrt{n}/(1+\sqrt{n})))$. By Lemma 1.2, the function $f(x)$ is monotonously decreasing in $(0, +\infty)$. Hereby, we have

$$\begin{aligned}
J_1 &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/2}\left(1 - \frac{\sqrt{m}}{1+\sqrt{m}} + \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)a_n^2 \\
&\leq \sum_{n=1}^{\infty}\left\{\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{1/2}\right)\left(1 - \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)dx\right\}a_n^2.
\end{aligned} \tag{2.5}$$

Using (1.5), we can obtain immediately

$$J_1 \leq \pi\sum_{n=1}^{\infty}a_n^2 - \pi\sum_{n=1}^{\infty}\omega(n)a_n^2. \tag{2.6}$$

Similarly, we have

$$J_2 \leq \pi\sum_{n=1}^{\infty}a_n^2 + \pi\sum_{n=1}^{\infty}\omega(n)a_n^2. \tag{2.7}$$

It follows from (2.2), (2.6), and (2.7) that

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\right)^2 \le \pi^2\left\{\left(\sum_{n=1}^{\infty}a_n^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)a_n^2\right)^2\right\}, \tag{2.8}$$

where the weight function $\omega(n)$ is defined by (1.6).

Next, consider the case for $b_n \neq a_n$. We can apply Schwarz's inequality to estimate the left-hand side of (2.1) as follows:

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\right)^4 &= \left\{\left(\int_0^1\left(\sum_{m=1}^{\infty}a_m t^{m-1/2}\right)\left(\sum_{n=1}^{\infty}b_n t^{n-1/2}\right)dt\right)^2\right\}^2\\ &\le \left\{\int_0^1\left(\sum_{m=1}^{\infty}a_m t^{m-1/2}\right)^2 dt\int_0^1\left(\sum_{n=1}^{\infty}b_n t^{n-1/2}\right)^2 dt\right\}^2\\ &= \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\right)^2\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_m b_n}{m+n}\right)^2.\end{aligned} \tag{2.9}$$

And then by using the inequality (2.8), the inequality (2.1) follows from (2.9) at once. It is obvious that the inequality (2.1) is a refinement of (1.1). Below, we give an extension of (1.2). □

Theorem 2.2. *Let α be a real number, $x \ge \alpha$ and $y \ge \alpha$, and let $f(x)$ and $g(x)$ be two real functions, and $\int_\alpha^\infty f^2(x)dx < +\infty$ and $\int_\alpha^\infty g^2(x)dx < +\infty$. Then*

$$\begin{aligned}\left\{\iint_\alpha^\infty \frac{f(x)g(x)}{x+y-2\alpha}dx\,dy\right\}^4 &\le \pi^4\left\{\left(\int_\alpha^\infty f^2(x)dx\right)^2-\left(\int_\alpha^\infty \tilde{\omega}(x)f^2(x)dx\right)^2\right\}\\ &\quad\times\left\{\left(\int_\alpha^\infty g^2(x)dx\right)^2-\left(\int_\alpha^\infty \tilde{\omega}(x)g^2(x)dx\right)^2\right\},\end{aligned} \tag{2.10}$$

where the weight function $\tilde{\omega}$ is defined by

$$\tilde{\omega}(x) = \begin{cases} 0, & x=\alpha,\\ \dfrac{x-\alpha}{1+x-\alpha}-\dfrac{\sqrt{x-\alpha}\,\ln(x-\alpha)}{\pi(1+x-\alpha)}-\dfrac{\sqrt{x-\alpha}}{1+\sqrt{x-\alpha}}, & x>\alpha.\end{cases} \tag{2.11}$$

Specially, when $\alpha = 0$, it is a refinement of (1.2).

Proof. Let $c(x)$ be a real function, and $1 - c(x-\alpha) + c(y-\alpha) \geq 0 (x,y) \in (\alpha,+\infty) \times (\alpha,+\infty)$.

Firstly, we suppose that $f = g$. Using Lemma 1.4 and then applying Cauchy's inequality we have

$$\begin{aligned}
\left\{\iint_{\alpha}^{\infty} \frac{f(x)f(y)}{x+y-2\alpha}\,dx\,dy\right\}^2 &= \left\{\iint_{\alpha}^{\infty} \frac{f(x)f(y)}{x+y-2\alpha}(1-c(x-\alpha)+c(y-\alpha))dx\,dy\right\}^2 \\
&= \left\{\iint_{\alpha}^{\infty} \frac{f(x)(1-c(x-\alpha)+c(y-\alpha))^{1/2}}{(x+y-2\alpha)^{1/2}}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/4}\right. \\
&\quad \left.\times \frac{f(y)(1-c(x-\alpha)+c(y-\alpha))^{1/2}}{(x+y-2\alpha)^{1/2}}\left(\frac{y-\alpha}{x-\alpha}\right)^{1/4} dx\,dy\right\}^2 \\
&\leq \widetilde{J}_1\widetilde{J}_2,
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\widetilde{J}_1 &= \iint_{\alpha}^{\infty} \frac{f^2(x)(1-c(x-\alpha)+c(y-\alpha))}{x+y-2\alpha}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2} dx\,dy, \\
\widetilde{J}_2 &= \iint_{\alpha}^{\infty} \frac{f^2(y)(1-c(x-\alpha)+c(y-\alpha))}{x+y-2\alpha}\left(\frac{y-\alpha}{x-\alpha}\right)^{1/2} dx\,dy.
\end{aligned} \tag{2.13}$$

In the first place, we consider $\widetilde{J}_1$:

$$\begin{aligned}
\widetilde{J}_1 &= \int_{\alpha}^{\infty}\left\{\int_{\alpha}^{\infty} \frac{1-c(x-\alpha)+c(y-\alpha)}{(x-\alpha)+(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2} dy\right\} f^2(x)dx \\
&= \int_{\alpha}^{\infty}\left\{\int_{0}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2} du\right\} f^2(x)dx + \int_{\alpha}^{\infty}\left\{\int_{\alpha}^{\infty} \frac{c(y-\alpha)-c(x-\alpha)}{(x-\alpha)+(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2} dy\right\} f^2(x)dx \\
&= \pi\left\{\int_{\alpha}^{\infty} f^2(x)dx + \int_{\alpha}^{\infty} \widetilde{\omega}(x) f^2(x)dx\right\},
\end{aligned} \tag{2.14}$$

where

$$\widetilde{\omega}(x) = \frac{1}{\pi}\int_{\alpha}^{\infty}\left\{\frac{1}{(x-\alpha)\,+\,(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2}(c(y-\alpha)-c(x-\alpha))\right\}dy. \tag{2.15}$$

We need only to compute the weight function $\widetilde{\omega}$.

Let us select still $c(x) = \sqrt{x}/(1+\sqrt{x})$. Then $c(y-\alpha) - c(x-\alpha) = \sqrt{y-\alpha}/(1+\sqrt{y-\alpha}) - \sqrt{x-\alpha}/(1+\sqrt{x-\alpha})$. Using (1.4), we have

$$\begin{aligned}
\widetilde{\omega}(x) &= \frac{1}{\pi}\int_{\alpha}^{\infty}\left\{\frac{1}{(x-\alpha)+(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2}\left(\frac{\sqrt{y-\alpha}}{1+\sqrt{y-\alpha}}-\frac{\sqrt{x-\alpha}}{1+\sqrt{x-\alpha}}\right)\right\}dy \\
&= \frac{1}{\pi}\int_{\alpha}^{\infty}\left\{\frac{1}{(x-\alpha)+(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2}\left(\frac{\sqrt{y-\alpha}}{1+\sqrt{y-\alpha}}\right)\right\}dy \\
&\quad -\frac{1}{\pi}\int_{\alpha}^{\infty}\left\{\frac{1}{(x-\alpha)+(y-\alpha)}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2}\left(\frac{\sqrt{x-\alpha}}{1+\sqrt{x-\alpha}}\right)\right\}dy \\
&= \frac{2}{\pi}(\sqrt{x-\alpha})\left\{\int_{0}^{\infty}\frac{du}{(x-\alpha)+u^2}-\int_{0}^{\infty}\frac{1}{((x-\alpha)+u^2)(1+u)}du\right\}-\frac{\sqrt{x-\alpha}}{1+\sqrt{x-\alpha}} \\
&= \frac{x-\alpha}{1+x-\alpha}-\frac{(\sqrt{x-\alpha})\ln(x-\alpha)}{\pi(1+x-\alpha)}-\frac{\sqrt{x-\alpha}}{1+\sqrt{x-\alpha}}.
\end{aligned} \tag{2.16}$$

Notice that $\lim_{x\to\alpha}\widetilde{\omega}(x) = 0$. Hence, the function defined by (2.11) is just. So,we attain

$$\widetilde{J}_1 = \pi\left(\int_{\alpha}^{\infty} f^2(x)dx + \int_{\alpha}^{\infty}\widetilde{\omega}(x)f^2(x)dx\right). \tag{2.17}$$

Similarly, we have

$$\widetilde{J}_2 = \pi\left(\int_{\alpha}^{\infty} f^2(x)dx - \int_{\alpha}^{\infty}\widetilde{\omega}(x)f^2(x)dx\right). \tag{2.18}$$

We obtain from (2.12), (2.17), and (2.18) that

$$\left\{\iint_{\alpha}^{\infty}\frac{f(x)f(y)}{x+y-2\alpha}\,dx\,dy\right\}^2 \le \pi^2\left\{\left(\int_{\alpha}^{\infty} f^2(x)dx\right)^2 - \left(\int_{\alpha}^{\infty}\widetilde{\omega}(x)f^2(x)dx\right)^2\right\}, \tag{2.19}$$

where the weight function $\widetilde{\omega}$ is defined by (2.11).

Secondly, consider the case for $f \neq g$. We can apply Schwarz's inequality to estimate the left-hand side of (2.10) as follows:

$$\begin{aligned}
&\left\{\iint_{\alpha}^{\infty} \frac{f(x)g(x)}{x+y-2\alpha}\,dx\,dy\right\}^4 \\
&= \left\{\left(\int_0^{\infty}\left(\iint_{\alpha}^{\infty} f(x)g(y)dx\,dy\right)e^{-u(x+y-2\alpha)}du\right)^2\right\}^2 \\
&= \left\{\left(\int_0^{\infty}\left(\int_{\alpha}^{\infty} f(x)e^{-u(x-\alpha)}dx\right)\left(\int_{\alpha}^{\infty} g(y)e^{-u(y-\alpha)}dy\right)du\right)^2\right\}^2 \\
&\leq \left\{\int_0^{\infty}\left(\int_{\alpha}^{\infty} f(x)e^{-u(x-\alpha)}dx\right)^2 du\int_0^{\infty}\left(\int_{\alpha}^{\infty} g(y)e^{-u(y-\alpha)}dy\right)^2 du\right\}^2 \\
&= \left\{\int_0^{\infty}\left(\iint_{\alpha}^{\infty} f(x)f(y)dx\,dy\right)e^{-u(x+y-2\alpha)}du\int_0^{\infty}\left(\iint_{\alpha}^{\infty} g(x)g(y)dx\,dy\right)e^{-u(x+y-2\alpha)}du\right\}^2 \\
&= \left\{\left(\iint_{\alpha}^{\infty} \frac{f(x)f(y)}{x+y-2\alpha}\,dx\,dy\right)\left(\iint_{\alpha}^{\infty} \frac{g(x)g(y)}{x+y-2\alpha}\,dx\,dy\right)\right\}^2.
\end{aligned} \tag{2.20}$$

It follows from (2.19) and (2.20) that the inequality (2.10) is valid. □

3. Applications

As applications, we will give some new refinements of Hardy-Littlewood's theorem and Widder's theorem below.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$ for all x. Define a sequence $\{a_n\}$ by

$$a_n = \int_0^1 x^n f(x)dx, \quad n = 0,1,2,\ldots. \tag{3.1}$$

Hardy-Littlewood [1] proved that

$$\sum_{n=0}^{\infty} a_n^2 < \pi\int_0^1 f^2(x)dx, \tag{3.2}$$

where π is the best constant that the inequality (3.2) keeps valid.

Theorem 3.1. *With the assumptions as the above-mentioned, define a sequence* $\{a_n\}$ *by* $a_n = \int_0^1 x^{n-1/2} f(x)dx$, $n = 1,2,\ldots$. *Then*

$$\left(\sum_{n=1}^{\infty} a_n^2\right)^2 < \pi\left\{\left(\sum_{n=1}^{\infty} a_n^2\right)^2 - \left(\sum_{n=1}^{\infty} \omega(n)a_n^2\right)^2\right\}^{1/2}\int_0^1 f^2(x)dx, \tag{3.3}$$

where $\omega(n)$ *is defined by* (1.6).

Proof. By our assumptions, we may write a_n^2 in the form of: $a_n^2 = \int_0^1 a_n x^{n-1/2} f(x)dx$.

Apply Schwarz's inequality to estimate the left-hand side of (3.3) as follows:

$$\left(\sum_{n=1}^{\infty} a_n^2\right)^2 = \left(\sum_{n=1}^{\infty}\int_0^1 a_n x^{n-1/2} f(x)dx\right)^2$$
$$= \left\{\int_0^1\left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right) f(x)dx\right\}^2 \le \int_0^1\left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right)^2 dx \int_0^1 f^2(x)dx \tag{3.4}$$
$$= \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n x^{m+n-1} dx \int_0^1 f^2(x)dx = \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\right)\int_0^1 f^2(x)dx.$$

It is known from (2.8) and (3.4) that the inequality (3.3) is valid. Theorem is therefore proved. □

Let $a_n \ge 0$ $(n = 0, 1, 2, \ldots)$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, and $A^*(x) = \sum_{n=0}^{\infty}(a_n x^n/n!)$. Then

$$\int_0^1 A^2(x)dx \le \pi \int_0^{\infty}\left(e^{-x}A^*(x)\right)^2 dx. \tag{3.5}$$

This is the famous Widder theorem (see [1]).

Theorem 3.2. *With the assumptions as the above-mentioned, then*

$$\left(\int_0^1 A^2(x)dx\right)^2 \le \pi^2\left\{\left(\int_\alpha^{\infty}\left(e^{-(x-\alpha)}A^*(x-\alpha)\right)^2 dx\right)^2 - \left(\int_\alpha^{\infty}\tilde{\omega}(x)\left(e^{-(x-\alpha)}A^*(x-\alpha)\right)^2 dx\right)^2\right\}, \tag{3.6}$$

where $\tilde{\omega}$ *is defined by* (2.11).

Proof. First, we have the following relation:

$$\int_0^{\infty} e^{-t}A^*(tx)dt = \int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}\frac{a_n(xt)^n}{n!}dt = \sum_{n=0}^{\infty}\frac{a_n x^n}{n!}\int_0^{\infty} t^n e^{-t}dt = \sum_{n=0}^{\infty} a_n x^n = A(x). \tag{3.7}$$

Let $tx = s - \alpha$. Then we have

$$\int_0^1 A^2(x)dx = \int_0^1\left\{\int_0^{\infty} e^{-t}A^*(tx)dt\right\}^2 dx = \int_0^1\left(\int_\alpha^{\infty} e^{-(s-\alpha)/x}A^*(s-\alpha)ds\right)^2\frac{1}{x^2}dx$$
$$= \int_1^{\infty}\left(\int_\alpha^{\infty} e^{-(s-\alpha)y}A^*(s-\alpha)ds\right)^2 dy = \int_0^{\infty}\left(\int_\alpha^{\infty} e^{-(s-\alpha)u-(s-\alpha)}A^*(s-\alpha)ds\right)^2 du$$
$$= \int_0^{\infty}\left(\int_\alpha^{\infty} e^{-(s-\alpha)u} f(s)ds\right)^2 du = \iint_\alpha^{\infty}\frac{f(s)f(t)}{s+t-2\alpha}ds\,dt, \tag{3.8}$$

where $f(x) = e^{-(x-\alpha)}A^*(x-\alpha)$. By using (2.19), the inequality (3.6) follows from (3.8) at once. □

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References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1952.
[2] M. Gao and L. C. Hsu, "A survey of various refinements and generalizations of Hilbert's inequalities," *Journal of Mathematical Research and Exposition*, vol. 25, no. 2, pp. 227–243, 2005.
[3] M. Gao and B. Yang, "On the extended Hilbert's inequality," *Proceedings of the American Mathematical Society*, vol. 126, no. 3, pp. 751–759, 1998.
[4] B. Yang and L. Debnath, "On a new generalization of Hardy-Hilbert's inequality and its applications," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 484–497, 1999.
[5] K. Jichang and L. Debnath, "On new generalizations of Hilbert's inequality and their applications," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 1, pp. 248–265, 2000.
[6] L. He, M. Gao, and S. Wei, "A note on Hilbert's inequality," *Mathematical Inequalities and Applications*, vol. 6, no. 2, pp. 283–288, 2003.
[7] Y. Jin, *Applied Integral Tables*, Chinese Science and Technology University Press, Hefei, China, 2006.