

Research Article

Existence of Solutions for a Class of Elliptic Systems in \mathbb{R}^N Involving the $(p(x), q(x))$ -Laplacian

S. Ogras, R. A. Mashiyev, M. Avci, and Z. Yucedag

Department of Mathematics, Dicle University, 21280 Diyarbakir, Turkey

Correspondence should be addressed to R. A. Mashiyev, mrabil@dicle.edu.tr

Received 4 April 2008; Accepted 17 July 2008

Recommended by M. Garcia Huidobro

In view of variational approach, we discuss a nonlinear elliptic system involving the $p(x)$ -Laplacian. Establishing the suitable conditions on the nonlinearity, we proved the existence of nontrivial solutions.

Copyright © 2008 S. Ogras et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The paper concerns the existence of nontrivial solutions for the following nonlinear elliptic system:

$$\begin{aligned} -\Delta_{p(x)}u &= \frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N, \\ -\Delta_{q(x)}v &= \frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{P,Q}$$

where $p(x)$ and $q(x)$ are two functions such that $1 < p(x), q(x) < N$ ($N \geq 2$), for every $x \in \mathbb{R}^N$. However, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ and $\Delta_{p(x)}$ is the $p(x)$ -Laplacian operator defined by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Using a variational approach, the authors prove the existence of nontrivial solutions.

Over the last decades, the variable exponent Lebesgue space $L^{p(x)}$ and Sobolev space $W^{1,p(x)}$ [1–5] have been a subject of active research stimulated mainly by the development of the studies of problems in elasticity, electrorheological fluids, image processing, flow in porous media, calculus of variations, and differential equations with $p(x)$ -growth conditions [6–13].

Among these problems, the study of $p(x)$ -Laplacian problems via variational methods is an interesting topic. A lot of researchers have devoted their work to this area [14–22].

The operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$ -Laplacian, where p is a continuous nonconstant function. In particular, if $p(x) \equiv p$ (constant), it is the well-known p -Laplacian operator. However, the $p(x)$ -Laplace operator possesses more complicated nonlinearity than p -Laplace operator due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties, as for example, we cannot use the Lagrange multiplier theorem and Morse theorem in a lot of problems involving this operator.

In literature, elliptic systems with standard and nonstandard growth conditions have been studied by many authors [23–28], where the nonlinear function F have different and mixed growth conditions and assumptions in each paper.

In [29], the authors show the existence of nontrivial solutions for the following p -Laplacian problem:

$$\begin{aligned} -\Delta_p u &= \frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= \frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N, \end{aligned} \quad (1.1)$$

where $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ yields some mixed growth conditions and the primitive F being intimately connected with the first eigenvalue of an appropriate system. Using a weak version of the Palais-Smale condition, that is, Cerami condition, they apply the mountain pass theorem to get the nontrivial solutions of the the system.

In [30], the author obtains the existence and multiplicity of solutions for the following problem:

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) &= \frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) &= \frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \Omega, \\ u &= 0, \quad v = 0 \quad \text{on } \Omega, \end{aligned} \quad (1.2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $N \geq 2$, $(p, q) \in C(\overline{\Omega})^2$, $\underline{p}(x) > 1$, $q(x) > 1$, for every $x \in \Omega$. The function F is assumed to be continuous in $x \in \overline{\Omega}$ and of class C^1 in $u, v \in \mathbb{R}$. Introducing some natural growth hypotheses on the right-hand side of the system which will ensure the mountain pass geometry and Palais-Smale condition for the corresponding Euler-Lagrange functional of the system, the author limits himself to the subcritical case for function F to obtain the existence and multiplicity results.

In the paper [31], Xu and An deal with the following problem:

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u &= \frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + |v|^{q(x)-2}v &= \frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N, \\ (u, v) &\in W^{1,p(x)}(\mathbb{R}^N) \times W^{1,q(x)}(\mathbb{R}^N), \end{aligned} \quad (1.3)$$

where $N \geq 2$, $p(x)$, $q(x)$ are functions on \mathbb{R}^N . The function F is assumed to satisfy Caratheodory conditions and to be L^∞ in $x \in \mathbb{R}^N$ and C^1 in $u, v \in \mathbb{R}$. By the critical point

theory, the authors use the two basic results on the existence of solutions of the system; these results correspond to the sublinear and superlinear cases for $p = 2$, respectively.

Inspired by the above-mentioned papers, we concern the existence of nontrivial solutions of problem (P,Q) . We know that in the study of $p(x)$ -Laplace equations in \mathbb{R}^N , the main difficulty arises from the lack of compactness. So, establishing some growth conditions on the right-hand side of the system which will ensure the mountain pass geometry and Cerami condition for the corresponding Euler-Lagrange functional J and applying a subcritical case for function F , we will overcome this difficulty.

2. Notations and preliminaries

We will investigate our problem (P,Q) in the variable exponent Sobolev space $W_0^{1,p(x)}(\mathbb{R}^N)$, so we need to recall some theories and basic properties on spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$.

Set

$$C_+(\mathbb{R}^N) = \left\{ h \in C(\mathbb{R}^N) : \inf_{x \in \mathbb{R}^N} h(x) > 1 \right\}. \quad (2.1)$$

For every $h \in C_+(\mathbb{R}^N)$, denote

$$h^- := \inf_{x \in \mathbb{R}^N} h(x), \quad h^+ := \sup_{x \in \mathbb{R}^N} h(x). \quad (2.2)$$

Let us define by $\mathcal{U}(\mathbb{R}^N)$ the set of all measurable real-valued functions defined on \mathbb{R}^N . For $p \in C_+(\mathbb{R}^N)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in \mathcal{U}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty \right\}, \quad (2.3)$$

which is equipped with the norm, so-called Luxemburg norm [1, 3, 4]:

$$\|u\|_{p(x)} := |u|_{L^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \quad (2.4)$$

and $(L^{p(x)}(\mathbb{R}^N), |\cdot|_{L^{p(x)}(\mathbb{R}^N)})$ becomes a Banach space, we call it generalized Lebesgue space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\}, \quad (2.5)$$

and it can be equipped with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}} = |u|_{p(x)} + |\nabla u|_{p(x)} \quad \forall u \in W^{1,p(x)}(\mathbb{R}^N). \quad (2.6)$$

The space $W_0^{1,p(x)}(\mathbb{R}^N)$ is denoted by the closure of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p(x)}(\mathbb{R}^N)$ and it is equipped with the norm for all $u \in W_0^{1,p(x)}(\mathbb{R}^N)$:

$$\|u\|_{p(x)} = |\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\mathbb{R}^N). \quad (2.7)$$

If $p^- > 1$, then the spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$, and $W_0^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Proposition 2.1 (see [1, 3, 4]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where $1/p'(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2 \|u\|_{p(x)} \|v\|_{p'(x)}. \quad (2.8)$$

Proposition 2.2 (see [1, 3, 4]). *Denote $\varrho_{p(x)}(u) = \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx$ for all $u \in L^{p(x)}(\mathbb{R}^N)$, one has*

$$\min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\} \leq \varrho_{p(x)}(u) \leq \max \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\}. \quad (2.9)$$

Proposition 2.3 (see [1]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\mathbb{R}^N)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. Then,*

$$\begin{aligned} \|u\|_{p(x)q(x)} \leq 1 &\implies \|u\|_{p(x)q(x)}^{p^+} \leq \| |u|^{p(x)} \|_{q(x)} \leq \|u\|_{p(x)q(x)}^{p^-}, \\ \|u\|_{p(x)q(x)} \geq 1 &\implies \|u\|_{p(x)q(x)}^{p^-} \leq \| |u|^{p(x)} \|_{q(x)} \leq \|u\|_{p(x)q(x)}^{p^+}. \end{aligned} \quad (2.10)$$

In particular, if $p(x) = p$ is constant, then

$$\| |u|^p \|_{q(x)} = \|u\|_{pq(x)}^p. \quad (2.11)$$

Proposition 2.4 (see [3, 4]). *If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, \dots$, then the following statements are equivalent to each other:*

- (1) $\lim_{n \rightarrow \infty} \|u_n - u\|_{p(x)} = 0$,
- (2) $\lim_{n \rightarrow \infty} \varrho(u_n - u) = 0$,
- (3) $u_n \rightarrow u$ in measure in \mathbb{R}^N and $\lim_{n \rightarrow \infty} \varrho(u_n) = \varrho(u)$.

Definition 2.5. $1 < p(x) < N$ and for all $x \in \mathbb{R}^N$, let define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

where $p^*(x)$ is the so-called critical Sobolev exponent of $p(x)$.

Proposition 2.6 (see [1, 32]). *Let $p(x) \in C_+^{0,1}(\mathbb{R}^N)$, that is, Lipschitz-continuous function defined on \mathbb{R}^N , then there exists a positive constant c such that*

$$\|u\|_{p^*(x)} \leq c \|u\|_{p(x)}, \quad (2.12)$$

for all $u \in W_0^{1,p(x)}(\mathbb{R}^N)$.

In the following discussions, we will use the product space

$$W_{p(x),q(x)} := W_0^{1,p(x)}(\mathbb{R}^N) \times W_0^{1,q(x)}(\mathbb{R}^N), \quad (2.13)$$

which is equipped with the norm

$$\|(u, v)\|_{p(x),q(x)} := \max \{ \|u\|_{p(x)} + \|v\|_{q(x)} \} \quad \forall (u, v) \in W_{p(x),q(x)}, \quad (2.14)$$

where $\|u\|_{p(x)}$ (resp., $\|u\|_{q(x)}$) is the norm of $W_0^{1,p(x)}(\mathbb{R}^N)$ (resp., $W_0^{1,q(x)}(\mathbb{R}^N)$). The space $W_{p(x),q(x)}^*$ denotes the dual space of $W_{p(x),q(x)}$ and equipped with the norm $\|\cdot\|_{*,p(x),q(x)}$. Thus,

$$\|J'(u, v)\|_{*,p(x),q(x)} = \|D_1 J(u, v)\|_{*,p(x)} + \|D_2 J(u, v)\|_{*,q(x)}, \quad (2.15)$$

where $W^{-1,p'(x)}(\mathbb{R}^N)$ (resp., $W^{-1,q'(x)}(\mathbb{R}^N)$) is the dual space of $W_0^{1,p(x)}(\mathbb{R}^N)$ (resp., $W_0^{1,q(x)}(\mathbb{R}^N)$), and $\|\cdot\|_{*,p(x)}$ (resp., $\|\cdot\|_{*,q(x)}$) is its norm.

For (u, v) and (φ, ψ) in $W_{p(x),q(x)}$, let

$$\mathcal{F}(u, v) = \int_{\mathbb{R}^N} F(x, u(x), v(x)) dx. \quad (2.16)$$

Then,

$$\mathcal{F}'(u, v)(\varphi, \psi) = D_1 \mathcal{F}(u, v)(\varphi) + D_2 \mathcal{F}(u, v)(\psi), \quad (2.17)$$

where

$$\begin{aligned} D_1 \mathcal{F}(u, v)(\varphi) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v) \varphi dx, \\ D_2 \mathcal{F}(u, v)(\psi) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v) \psi dx. \end{aligned} \quad (2.18)$$

The Euler-Lagrange functional associated to (P, Q) is defined by

$$J(u, v) = \int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)} dx - (u, v). \quad (2.19)$$

It is easy to verify that $J \in C^1(W_{p(x),q(x)}, \mathbb{R})$ and that

$$J'(u, v)(\varphi, \psi) = D_1 J(u, v)(\varphi) + D_2 J(u, v)(\psi), \quad (2.20)$$

where

$$\begin{aligned} D_1 J(u, v)(\varphi) &= \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - D_1 \mathcal{F}(u, v)(\varphi), \\ D_2 J(u, v)(\psi) &= \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx - D_2 \mathcal{F}(u, v)(\psi). \end{aligned} \quad (2.21)$$

Definition 2.7. (u, v) is called a weak solution of the system (P, Q) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v) \varphi dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v) \psi dx, \quad (2.22)$$

for all $(\varphi, \psi) \in W_{p(x),q(x)}$.

Definition 2.8. We say that J satisfies the Cerami condition (C) if every sequence $(\omega_n) \in W_{p(x),q(x)}$ such that

$$|J(\omega_n)| \leq c, \quad (1 + \|\omega_n\|)J'(\omega_n) \rightarrow 0 \quad (2.23)$$

contains a convergent subsequence in the norm of $W_{p(x),q(x)}$.

In this paper, we will use the following assumptions:

(F1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x, 0, 0) = 0$;

(F2) for all $(u, v) \in \mathbb{R}^2$ and for a.e. $x \in \mathbb{R}^N$

$$\begin{aligned} \left| \frac{\partial F}{\partial u}(x, u, v) \right| &\leq a_1(x)|u, v|^{p^- - 1} + a_2(x)|u, v|^{p^+ - 1}, \\ \left| \frac{\partial F}{\partial v}(x, u, v) \right| &\leq b_1(x)|u, v|^{q^- - 1} + b_2(x)|u, v|^{q^+ - 1}, \end{aligned} \quad (2.24)$$

where

$$1 < p^-, \quad q^- \leq p^+, \quad q^+ < (p^*)^-, (q^*)^-,$$

$$a_i \in L^{\delta(x)}(\mathbb{R}^N) \cap L^{\beta(x)}(\mathbb{R}^N), \quad b_i \in L^{\gamma(x)}(\mathbb{R}^N) \cap L^{\beta(x)}(\mathbb{R}^N), \quad i = 1, 2,$$

$$\delta(x) = \frac{p(x)}{p(x) - 1}, \quad \gamma(x) = \frac{q(x)}{q(x) - 1}, \quad \tilde{p}(x) = \frac{p^*(x)p(x)}{p^*(x) - p(x)}, \quad (2.25)$$

$$\tilde{q}(x) = \frac{q^*(x)q(x)}{q^*(x) - q(x)}, \quad \beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x) - (p^*(x) + q^*(x))};$$

(F3) $(u, v) \cdot \nabla F(x, u, v) - F(x, u, v) \leq 0$ for all $(x, u, v) \in \mathbb{R}^N \times \mathbb{R}^2 \setminus \{(0, 0)\}$, where $\nabla F = (\partial F / \partial u, \partial F / \partial v)$;

(F4) suppose there exist two positive and bounded functions $a \in L^{N/p(x)}(\mathbb{R}^N)$ and $b \in L^{N/q(x)}(\mathbb{R}^N)$ such that

$$\begin{aligned} \limsup_{|(u,v)| \rightarrow 0} \frac{p(x)q(x)|F(x, u, v)|}{q(x)a(x)|u|^{p(x)} + p(x)b(x)|v|^{q(x)}} \\ < \lambda_1 < \liminf_{|(u,v)| \rightarrow +\infty} \frac{p(x)q(x)|F(x, u, v)|}{q(x)a(x)|u|^{p(x)} + p(x)b(x)|v|^{q(x)}}. \end{aligned} \quad (2.26)$$

Let λ_1 denote the first eigenvalue of the nonlinear eigenvalue problem in \mathbb{R}^N :

$$\begin{aligned} -\Delta_{p(x)} u &= \lambda a(x)|u|^{p(x)-2}u \quad \text{in } \mathbb{R}^N, \\ -\Delta_{q(x)} v &= \lambda b(x)|v|^{q(x)-2}v \quad \text{in } \mathbb{R}^N. \end{aligned} \quad (2.27)$$

It is useful to recall the variational characterization:

$$\lambda_1 = \inf \left\{ \frac{\int_{\mathbb{R}^N} ((1/p(x))|\nabla u|^{p(x)} + (1/q(x))|\nabla v|^{q(x)}) dx}{\int_{\mathbb{R}^N} ((a(x)/p(x))|u|^{p(x)} + (b(x)/q(x))|v|^{q(x)}) dx} : (u, v) \in W_{p(x),q(x)} \setminus \{(0, 0)\} \right\}. \quad (2.28)$$

We will assume that λ_1 is a positive real number for all $(u, v) \in W_{p(x),q(x)} \setminus \{(0, 0)\}$. For more details about the eigenvalue problems, we refer the reader to [17].

3. The main results

We will use the mountain pass theorem together with the following lemmas to get our main results.

Lemma 3.1. *Under the assumptions (F1) and (F2), the functional \mathcal{F} is well defined, and it is of class C^1 on $W_{p(x),q(x)}$. Moreover, its derivative is*

$$\mathcal{F}'(u, v)(\omega, z) = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)\omega \, dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v)z \, dx \quad \forall (u, v), (\omega, z) \in W_{p(x),q(x)}. \quad (3.1)$$

Proof. For all pair of real functions $(u, v) \in W_{p(x),q(x)}$, under the assumptions (F1) and (F2), we can write

$$\begin{aligned} F(x, u, v) &= \int_0^u \frac{\partial F}{\partial s}(x, s, v) \, ds + F(x, 0, v) = \int_0^u \frac{\partial F}{\partial s}(x, s, v) \, ds + \int_0^v \frac{\partial F}{\partial s}(x, 0, s) \, ds + F(x, 0, 0), \\ F(x, u, v) &\leq c_1 [a_1(x)(|u|^{p^-} + |v|^{p^- - 1}|u|) + a_2(x)(|u|^{p^+} + |v|^{p^+ - 1}|u|) + b_1(x)|v|^{q^-} + b_2(x)|v|^{q^+}]. \end{aligned} \quad (3.2)$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, u, v) \, dx &\leq c_2 \left[\int_{\mathbb{R}^N} a_1(x)|u|^{p^-} \, dx + \int_{\mathbb{R}^N} a_1(x)|v|^{p^- - 1}|u| \, dx + \int_{\mathbb{R}^N} a_2(x)|u|^{p^+} \, dx \right. \\ &\quad \left. + \int_{\mathbb{R}^N} a_2(x)|v|^{p^+ - 1}|u| \, dx + \int_{\mathbb{R}^N} b_1(x)|v|^{q^-} \, dx + \int_{\mathbb{R}^N} b_2(x)|v|^{q^+} \, dx \right], \end{aligned} \quad (3.3)$$

if we consider the fact that

$$W_0^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p^-(x)}(\mathbb{R}^N) \implies \|u\|_{p(x)}^{p^-} = \|u\|_{p^-(x)}^{p^-} \leq c \|u\|_{p(x)}^{p^-} \quad \text{for } p^- > 1, \quad (3.4)$$

and if we apply Propositions 2.1, 2.3, and 2.6 and take $a_i \in L^{\delta(x)}(\mathbb{R}^N) \cap L^{\beta(x)}(\mathbb{R}^N)$, $b_i \in L^{\gamma(x)}(\mathbb{R}^N)$, then we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, u, v) \, dx &\leq 2c_1 \left(|a_1|_{\delta(x)} \|u\|_{p(x)}^{p^-} + |a_1|_{\beta(x)} \|v\|_{q^*(x)}^{p^- - 1} \|u\|_{p^*(x)} + |a_2|_{\delta(x)} \|u\|_{p(x)}^{p^+} \right. \\ &\quad \left. + |a_2|_{\beta(x)} \|v\|_{q^*(x)}^{p^+ - 1} \|u\|_{p^*(x)} + |b_1|_{\gamma(x)} \|v\|_{q(x)}^{q^-} + |b_2|_{\gamma(x)} \|v\|_{q(x)}^{q^+} \right) \\ &= 2c_1 \left(|a_1|_{\delta(x)} \|u\|_{p^-(x)}^{p^-} + |a_1|_{\beta(x)} \|v\|_{(p^- - 1)q^*(x)}^{p^- - 1} \|u\|_{p^*(x)} + |a_2|_{\delta(x)} \|u\|_{p^+(x)}^{p^+} \right. \\ &\quad \left. + |a_2|_{\beta(x)} \|v\|_{(p^+ - 1)q^*(x)}^{p^+ - 1} \|u\|_{p^*(x)} + |b_1|_{\gamma(x)} \|v\|_{q^-(x)}^{q^-} + |b_2|_{\gamma(x)} \|v\|_{q^+(x)}^{q^+} \right) \\ &\leq c_3 \left(|a_1|_{\delta(x)} \|u\|_{p(x)}^{p^-} + |a_1|_{\beta(x)} \|v\|_{q(x)}^{p^- - 1} \|u\|_{p(x)} + |a_2|_{\delta(x)} \|u\|_{p(x)}^{p^+} \right. \\ &\quad \left. + |a_2|_{\beta(x)} \|v\|_{q(x)}^{p^+ - 1} \|u\|_{p(x)} + |b_1|_{\gamma(x)} \|v\|_{q(x)}^{q^-} + |b_2|_{\gamma(x)} \|v\|_{q(x)}^{q^+} \right) < \infty. \end{aligned} \quad (3.5)$$

Hence, \mathcal{F} is well defined. Moreover, one can see easily that \mathcal{F}' is also well defined on $W_{p(x),q(x)}$. Indeed, using (F2) for all $(\omega, z) \in W_{p(x),q(x)}$, we have

$$\begin{aligned} \mathcal{F}'(u, v)(\omega, z) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)\omega \, dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v)z \, dx, \\ \mathcal{F}'(u, v)(\omega, z) &\leq \int_{\mathbb{R}^N} (a_1(x)|u|^{p^- - 1} + a_2(x)|u|^{p^+ - 1})|\omega| \, dx \\ &\quad + \int_{\mathbb{R}^N} (b_1(x)|u|^{q^- - 1} + b_2(x)|u|^{q^+ - 1})|z| \, dx \\ &\leq \int_{\mathbb{R}^N} a_1(x)|u|^{p^- - 1}|\omega| \, dx + \int_{\mathbb{R}^N} a_1(x)|v|^{p^- - 1}|\omega| \, dx \\ &\quad + \int_{\mathbb{R}^N} a_2(x)|u|^{p^+ - 1}|\omega| \, dx + \int_{\mathbb{R}^N} a_2(x)|v|^{p^+ - 1}|\omega| \, dx \\ &\quad + \int_{\mathbb{R}^N} b_1(x)|u|^{q^- - 1}|z| \, dx + \int_{\mathbb{R}^N} b_1(x)|v|^{q^- - 1}|z| \, dx \\ &\quad + \int_{\mathbb{R}^N} b_2(x)|u|^{q^+ - 1}|z| \, dx + \int_{\mathbb{R}^N} b_2(x)|v|^{q^+ - 1}|z| \, dx, \end{aligned} \quad (3.6)$$

and applying Propositions 2.1, 2.3, and 2.6 and considering the conditions $\tilde{p}(x) > p(x)$ and $\tilde{q}(x) > q(x)$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)\omega \, dx &\leq 2 \left(|a_1|_{\delta(x)} \|u\|_{p^*(x)}^{p^- - 1} |\omega|_{\tilde{p}(x)} + |a_1|_{\beta(x)} \|v\|_{q^*(x)}^{p^- - 1} |\omega|_{p^*(x)} \right. \\ &\quad \left. + |a_2|_{\delta(x)} \|u\|_{p^*(x)}^{p^+ - 1} |\omega|_{\tilde{p}(x)} + |a_2|_{\beta(x)} \|v\|_{q^*(x)}^{p^+ - 1} |\omega|_{p^*(x)} \right) \\ &\leq 2 \left(|a_1|_{\delta(x)} \|u\|_{(p^- - 1)p^*(x)}^{p^- - 1} |\omega|_{\tilde{p}(x)} + |a_1|_{\beta(x)} \|v\|_{(p^- - 1)q^*(x)}^{p^- - 1} |\omega|_{p^*(x)} \right. \\ &\quad \left. + |a_2|_{\delta(x)} \|u\|_{(p^+ - 1)p^*(x)}^{p^+ - 1} |\omega|_{\tilde{p}(x)} + |a_2|_{\beta(x)} \|v\|_{(p^+ - 1)q^*(x)}^{p^+ - 1} |\omega|_{p^*(x)} \right) \\ &\leq c_4 \left(|a_1|_{\delta(x)} \|u\|_{p(x)}^{p^- - 1} + |a_1|_{\beta(x)} \|v\|_{q(x)}^{p^- - 1} + |a_2|_{\delta(x)} \|u\|_{p(x)}^{p^+ - 1} + |a_2|_{\beta(x)} \|v\|_{q(x)}^{p^+ - 1} \right) \|\omega\|_{p(x)} \\ &< \infty, \end{aligned} \quad (3.7)$$

and similarly

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v)z \, dx \\ \leq c_5 \left(|b_1|_{\beta(x)} \|u\|_{p(x)}^{q^- - 1} + |b_1|_{\gamma(x)} \|v\|_{q(x)}^{q^- - 1} + |b_2|_{\beta(x)} \|u\|_{p(x)}^{q^+ - 1} + |b_2|_{\gamma(x)} \|v\|_{q(x)}^{q^+ - 1} \right) \|z\|_{q(x)} < \infty. \end{aligned} \quad (3.8)$$

Now let us show that \mathcal{F} is differentiable in sense of Fréchet, that is, for fixed $(u, v) \in W_{p(x),q(x)}$ and given $\varepsilon > 0$, there must be a $\delta = \delta(\varepsilon, u, v) > 0$ such that

$$|\mathcal{F}(u + \omega, v + z) - \mathcal{F}(u, v) - \mathcal{F}'(u, v)(\omega, z)| \leq \varepsilon(\|\omega\|_{p(x)} + \|z\|_{q(x)}), \quad (3.9)$$

for all $(\omega, z) \in W_{p(x),q(x)}$ with $(\|\omega\|_{p(x)} + \|z\|_{q(x)}) \leq \delta$.

Let B_r be the ball of radius r which is centered at the origin of \mathbb{R}^N and denote $B'_r = \mathbb{R}^N - B_r$. Moreover, let us define the functional \mathcal{F}_r on $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$ as follows:

$$\mathcal{F}_r(u, v) = \int_{B_r} F(x, u(x), v(x)) dx. \quad (3.10)$$

If we consider (F1) and (F2), it is easy to see that $\mathcal{F}_r \in C^1(W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r))$, and in addition for all $(\omega, z) \in W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$, we have

$$\mathcal{F}'_r(u, v)(\omega, z) = \int_{B_r} \frac{\partial F}{\partial u}(x, u, v) \omega dx + \int_{B_r} \frac{\partial F}{\partial v}(x, u, v) z dx. \quad (3.11)$$

Also as we know, the operator $\mathcal{F}'_r : W_{p(x),q(x)} \rightarrow W_{p(x),q(x)}^*$ is compact [3]. Then, for all $(u, v), (\omega, z) \in W_{p(x),q(x)}$, we can write

$$\begin{aligned} & |\mathcal{F}(u + \omega, v + z) - \mathcal{F}(u, v) - \mathcal{F}'(u, v)(\omega, z)| \\ & \leq |\mathcal{F}_r(u + \omega, v + z) - \mathcal{F}_r(u, v) - \mathcal{F}'_r(u, v)(\omega, z)| \\ & \quad + \left| \int_{B'_r} (F(x, u + \omega, v + z) - F(x, u, v) - \frac{\partial F}{\partial u}(x, u, v) \omega - \frac{\partial F}{\partial v}(x, u, v) z) dx \right|. \end{aligned} \quad (3.12)$$

By virtue of the mean-value theorem, there exist $\zeta_1, \zeta_2 \in (0, 1)$ such that

$$F(x, u + \omega, v + z) - F(x, u, v) = \frac{\partial F}{\partial u}(x, u + \zeta_1 \omega, v) \omega + \frac{\partial F}{\partial v}(x, u, v + \zeta_2 z) z. \quad (3.13)$$

Using the condition (F2), we have

$$\begin{aligned} & \left| \int_{B'_r} \left(\frac{\partial F}{\partial u}(x, u + \zeta_1 \omega, v) \omega + \frac{\partial F}{\partial v}(x, u, v + \zeta_2 z) z - \frac{\partial F}{\partial u}(x, u, v) \omega - \frac{\partial F}{\partial v}(x, u, v) z \right) dx \right| \\ & \leq \left| \int_{B'_r} (a_1(x)(|u + \zeta_1 \omega|^{p^- - 1} - |u|^{p^- - 1}) + a_2(x)(|u + \zeta_1 \omega|^{p^+ - 1} - |u|^{p^+ - 1})) |\omega| dx \right. \\ & \quad \left. + \int_{B'_r} (b_1(x)(|v + \zeta_2 z|^{q^- - 1} - |v|^{q^- - 1}) + b_2(x)(|v + \zeta_2 z|^{q^+ - 1} - |v|^{q^+ - 1})) |z| dx \right|. \end{aligned} \quad (3.14)$$

By help of the elementary inequality $|a + b|^s \leq 2^{s-1}(|a|^s + |b|^s)$ for $a, b \in \mathbb{R}^N$, we can write

$$\begin{aligned} & \leq (2^{p^- - 1} - 1) \int_{B'_r} a_1(x) |u|^{p^- - 1} |\omega| dx + (\zeta_1 2)^{p^- - 1} \int_{B'_r} a_1(x) |\omega|^{p^- - 1} |\omega| dx \\ & \quad + (2^{p^+ - 1} - 1) \int_{B'_r} a_2(x) |u|^{p^+ - 1} |\omega| dx + (\zeta_1 2)^{p^+ - 1} \int_{B'_r} a_2(x) |\omega|^{p^+ - 1} |\omega| dx \\ & \quad + (2^{q^- - 1} - 1) \int_{B'_r} b_1(x) |v|^{q^- - 1} |z| dx + (\zeta_2 2)^{q^- - 1} \int_{B'_r} b_1(x) |z|^{q^- - 1} |z| dx \\ & \quad + (2^{q^+ - 1} - 1) \int_{B'_r} b_2(x) |v|^{q^+ - 1} |z| dx + (\zeta_2 2)^{q^+ - 1} \int_{B'_r} b_2(x) |z|^{q^+ - 1} |z| dx, \end{aligned} \quad (3.15)$$

applying Propositions 2.1, 2.3, and 2.6, then we have

$$\begin{aligned}
&\leq c_6 \left(|a_1|_{\delta(x)} \| |u|^{p^- - 1} |_{p^*(x)} |\omega|_{\tilde{p}(x)} + |a_1|_{\delta(x)} \| |\omega|^{p^- - 1} |_{p^*(x)} |\omega|_{\tilde{p}(x)} \right. \\
&\quad + |a_2|_{\delta(x)} \| |u|^{p^+ - 1} |_{p^*(x)} |\omega|_{\tilde{p}(x)} + |a_2|_{\delta(x)} \| |\omega|^{p^+ - 1} |_{p^*(x)} |\omega|_{\tilde{p}(x)} \\
&\quad + |b_1|_{\gamma(x)} \| |v|^{q^- - 1} |_{q^*(x)} |z|_{\tilde{q}(x)} + |b_1|_{\gamma(x)} \| |z|^{q^- - 1} |_{q^*(x)} |z|_{\tilde{q}(x)} \\
&\quad \left. + |b_2|_{\gamma(x)} \| |v|^{q^+ - 1} |_{q^*(x)} |z|_{\tilde{q}(x)} + |b_2|_{\gamma(x)} \| |z|^{q^+ - 1} |_{q^*(x)} |z|_{\tilde{q}(x)} \right), \tag{3.16} \\
&\leq c_7 \left(\left(|a_1|_{\delta(x)} \| |u|_{p(x)}^{p^- - 1} + |a_1|_{\delta(x)} \| |\omega|_{p(x)}^{p^- - 1} \right) + \left(|a_2|_{\delta(x)} \| |u|_{p(x)}^{p^+ - 1} + |a_2|_{\delta(x)} \| |\omega|_{p(x)}^{p^+ - 1} \right) \right) \| \omega \|_{p(x)} \\
&\quad + \left(\left(|b_1|_{\gamma(x)} \| |v|_{q(x)}^{q^- - 1} + |b_1|_{\gamma(x)} \| |z|_{q(x)}^{q^- - 1} \right) + \left(|b_2|_{\gamma(x)} \| |v|_{q(x)}^{q^+ - 1} + |b_2|_{\gamma(x)} \| |z|_{q(x)}^{q^+ - 1} \right) \right) \| z \|_{q(x)},
\end{aligned}$$

and by the fact that

$$\begin{aligned}
|a_i|_{L^{\delta(x)}(B_r)} &\longrightarrow 0, \\
|b_i|_{L^{\gamma(x)}(B_r)} &\longrightarrow 0
\end{aligned} \tag{3.17}$$

for $i = 1, 2$, as $r \rightarrow \infty$, and for r sufficiently large, it follows that

$$\left| \iint_{B_r'} \left(F(x, u + \omega, v + z) - F(x, u, v) - \frac{\partial F}{\partial u}(x, u, v) \omega - \frac{\partial F}{\partial v}(x, u, v) z \right) dx \right| \leq \varepsilon (\| \omega \|_{p(x)} + \| z \|_{q(x)}). \tag{3.18}$$

It remains only to show that \mathcal{F}' is continuous on $W_{p(x), q(x)}$. Let $(u_n, v_n), (u, v) \in W_{p(x), q(x)}$ such that $(u_n, v_n) \rightarrow (u, v)$. Then, for $(\omega, z) \in W_{p(x), q(x)}$, we have

$$\begin{aligned}
|\mathcal{F}'(u_n, v_n)(\omega, z) - \mathcal{F}'(u, v)(\omega, z)| &\leq |\mathcal{F}'_r(u_n, v_n)(\omega, z) - \mathcal{F}'_r(u, v)(\omega, z)| \\
&\quad + \left| \int_{B_r'} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) + \frac{\partial F}{\partial u}(x, u, v) \right) \omega dx \right| \\
&\quad + \left| \int_{B_r'} \left(\frac{\partial F}{\partial v}(x, u_n, v_n) + \frac{\partial F}{\partial v}(x, u, v) \right) z dx \right|,
\end{aligned} \tag{3.19}$$

then by (F2), we can write

$$\int_{B_r'} a_1(x) (|u_n|^{p^- - 1} + |u|^{p^- - 1} + |v_n|^{p^- - 1} + |v|^{p^- - 1}) |\omega| dx \tag{3.20}$$

$$+ \int_{B_r'} a_2(x) (|u_n|^{p^+ - 1} + |u|^{p^+ - 1} + |v_n|^{p^+ - 1} + |v|^{p^+ - 1}) |\omega| dx \tag{I_1}$$

$$+ \int_{B_r'} b_1(x) (|u_n|^{q^- - 1} + |u|^{q^- - 1} + |v_n|^{q^- - 1} + |v|^{q^- - 1}) |z| dx \tag{I_2}$$

$$+ \int_{B_r'} b_2(x) (|u_n|^{q^+ - 1} + |u|^{q^+ - 1} + |v_n|^{q^+ - 1} + |v|^{q^+ - 1}) |z| dx. \tag{3.21}$$

Thus,

$$\begin{aligned}
I_1 &\leq \int_{B'_r} a_1(x) |u_n|^{p^- - 1} |\omega| dx + \int_{B'_r} a_1(x) |u|^{p^- - 1} |\omega| dx + \int_{B'_r} a_1(x) |v_n|^{p^- - 1} |\omega| dx \\
&\quad + \int_{B'_r} a_1(x) |v|^{p^- - 1} |\omega| dx + \int_{B'_r} a_2(x) |u_n|^{p^+ - 1} |\omega| dx + \int_{B'_r} a_2(x) |u|^{p^+ - 1} |\omega| dx \\
&\quad + \int_{B'_r} a_2(x) |v_n|^{p^+ - 1} |\omega| dx + \int_{B'_r} a_2(x) |v|^{p^+ - 1} |\omega| dx \\
&\leq c_9 \left(|a_1|_{\delta(x)} \|u_n\|_{p(x)}^{p^- - 1} + |a_1|_{\delta(x)} \|u\|_{p(x)}^{p^- - 1} + |a_1|_{\beta(x)} \|v_n\|_{q(x)}^{p^- - 1} + |a_1|_{\beta(x)} \|v\|_{q(x)}^{p^- - 1} \right. \\
&\quad \left. + |a_2|_{\delta(x)} \|u_n\|_{p(x)}^{p^+ - 1} + |a_2|_{\delta(x)} \|u\|_{p(x)}^{p^+ - 1} + |a_2|_{\beta(x)} \|v_n\|_{q(x)}^{p^+ - 1} + |a_2|_{\beta(x)} \|v\|_{q(x)}^{p^+ - 1} \right) \|\omega\|_{p(x)}.
\end{aligned} \tag{3.22}$$

Similarly,

$$\begin{aligned}
I_2 &\leq c_{10} \left(|b_1|_{\beta(x)} \|u_n\|_{p(x)}^{q^- - 1} + |b_1|_{\beta(x)} \|u\|_{p(x)}^{q^- - 1} + |b_1|_{\gamma(x)} \|v_n\|_{q(x)}^{q^- - 1} + |b_1|_{\gamma(x)} \|v\|_{q(x)}^{q^- - 1} \right. \\
&\quad \left. + |b_2|_{\beta(x)} \|u_n\|_{p(x)}^{q^+ - 1} + |b_2|_{\beta(x)} \|u\|_{p(x)}^{q^+ - 1} + |b_2|_{\gamma(x)} \|v_n\|_{q(x)}^{q^+ - 1} + |b_2|_{\gamma(x)} \|v\|_{q(x)}^{q^+ - 1} \right) \|z\|_{q(x)}.
\end{aligned} \tag{3.23}$$

Since \mathcal{F}'_r is continuous on $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$, then we have

$$|\mathcal{F}'_r(u_n, v_n)(\omega, z) - \mathcal{F}'_r(u, v)(\omega, z)| \longrightarrow 0, \tag{3.24}$$

as $n \rightarrow \infty$. Moreover, using (3.17), when r sufficiently large, I_1 and I_2 tend also to 0. Hence,

$$|\mathcal{F}'(u_n, v_n)(\omega, z) - \mathcal{F}'(u, v)(\omega, z)| \longrightarrow 0, \tag{3.25}$$

as $(u_n, v_n) \rightarrow (u, v)$, this implies \mathcal{F}' is continuous on $W_{p(x),q(x)}$. \square

Lemma 3.2. *Under the assumptions (F1) and (F2), \mathcal{F}' is compact from $W_{p(x),q(x)}$ to $W_{p(x),q(x)}^*$.*

Proof. Let (u_n, v_n) be a bounded sequence in $W_{p(x),q(x)}$. Then, there exists a subsequence (we denote again as (u_n, v_n)) which converges weakly in $W_{p(x),q(x)}$ to a $(u, v) \in W_{p(x),q(x)}$. Then, if we use the same arguments as above, we have

$$\begin{aligned}
|\mathcal{F}'(u_n, v_n)(\omega, z) - \mathcal{F}'(u, v)(\omega, z)| &\leq |\mathcal{F}'_r(u_n, v_n)(\omega, z) - \mathcal{F}'_r(u, v)(\omega, z)| \\
&\quad + \left| \int_{B'_r} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) - \frac{\partial F}{\partial u}(x, u, v) \right) \omega dx \right| \\
&\quad + \left| \int_{B'_r} \left(\frac{\partial F}{\partial v}(x, u_n, v_n) - \frac{\partial F}{\partial v}(x, u, v) \right) z dx \right|.
\end{aligned} \tag{3.26}$$

Since the restriction operator is continuous, then $(u_n, v_n) \rightharpoonup (u, v)$ in $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$. Because of the compactness of \mathcal{F}'_r , the first expression on the right-hand side of the equation tends to 0, as $n \rightarrow \infty$, and as we did above, when r sufficiently large, I_1 and I_2 tend also to 0. This implies \mathcal{F}' is compact from $W_{p(x),q(x)}$ to $W_{p(x),q(x)}^*$. \square

Lemma 3.3. *If (F1), (F2), and (F3) hold, then J satisfies the condition (C), that is, there exists a sequence $(u_n, v_n) \in W_{p(x), q(x)}$ such that*

$$(i) |J(u_n, v_n)| \leq c,$$

$$(ii) (1 + \|u_n\|_{p(x)} + \|v_n\|_{q(x)}) \|J'(u_n, v_n)\|_{*, p(x), q(x)} \rightarrow 0 \text{ as } n \rightarrow +\infty$$

contains a convergent subsequence.

Proof. By the assumption (ii), it is clear that $J'(u_n, v_n)(\omega, z) \leq \xi_n \rightarrow 0$ as $n \rightarrow \infty$ for all $(\omega, z) \in W_{p(x), q(x)}$. Let us choose $(\omega, z) = (u_n, v_n)$, then we have

$$\begin{aligned} \xi_n &\geq J'(u_n, v_n)(u_n, v_n) \\ &\geq \|u_n\|_{p(x)}^{p^-} + \|v_n\|_{q(x)}^{q^-} - \int_{\mathbb{R}^N} \left(\frac{\partial F}{\partial u}(x, u_n, v_n)u_n + \frac{\partial F}{\partial v}(x, u_n, v_n)v_n \right) dx. \end{aligned} \quad (3.27)$$

Moreover, by the assumption (i), we can write

$$c \geq -J(u_n, v_n) \geq -\frac{1}{p^+} \|u_n\|_{p(x)}^{p^-} - \frac{1}{q^+} \|v_n\|_{q(x)}^{q^-} + \int_{\mathbb{R}^N} F(x, u_n, v_n) dx. \quad (3.28)$$

Using the assumption (F3), it follows that

$$\begin{aligned} \xi_n + c &\geq J'(u_n, v_n)(u_n, v_n) - J(u_n, v_n) \\ &\geq \left(1 - \frac{1}{p^+}\right) \|u_n\|_{p(x)}^{p^-} + \left(1 - \frac{1}{q^+}\right) \|v_n\|_{q(x)}^{q^-} + \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{\partial F}{\partial u}(x, u_n, v_n)u_n + \frac{\partial F}{\partial v}(x, u_n, v_n)v_n \right) dx \\ &\geq \left(1 - \frac{1}{p^+}\right) \|u_n\|_{p(x)}^{p^-} + \left(1 - \frac{1}{q^+}\right) \|v_n\|_{q(x)}^{q^-}. \end{aligned} \quad (3.29)$$

Thus, the sequence (u_n, v_n) is bounded in $W_{p(x), q(x)}$. Then, there exists a subsequence (we denote again as (u_n, v_n)) which converges weakly in $W_{p(x), q(x)}$.

We recall the elementary inequalities:

$$2^{2-p}|a-b|^p \leq (|a|^{p-2} - |b|^{p-2}) \cdot (a-b) \quad \text{if } p \geq 2, \quad (3.30)$$

$$(p-1)|a-b|^2(|a|+|b|)^{p-2} \leq (|a|^{p-2} - |b|^{p-2}) \cdot (a-b) \quad \text{if } 1 < p < 2, \quad (3.31)$$

for all $a, b \in \mathbb{R}^N$, where \cdot denotes the standard inner product in \mathbb{R}^N . We will show that (u_n, v_n) contains a Cauchy subsequence. Let us define the sets

$$\begin{aligned} U_p &= \{x \in \mathbb{R}^N : p(x) \geq 2\}, & V_p &= \{x \in \mathbb{R}^N : 1 < p(x) < 2\}, \\ U_q &= \{x \in \mathbb{R}^N : q(x) \geq 2\}, & V_q &= \{x \in \mathbb{R}^N : 1 < q(x) < 2\}. \end{aligned} \quad (3.32)$$

For all $x \in \mathbb{R}^N$, we put

$$\begin{aligned} \Phi_{n,k} &= (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u_k|^{p(x)-2} \nabla u_k) \cdot (\nabla u_n - \nabla u_k), \\ \Psi_{n,k} &= (|\nabla u_n| + |\nabla u_k|)^{2-p(x)}. \end{aligned} \quad (3.33)$$

Therefore for $p(x) \geq 2$, using (3.30), we have

$$\begin{aligned}
2^{2-p^+} \int_{U_p} |\nabla u_n - \nabla u_k|^{p(x)} dx &\leq \int_{U_p} [(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u_k|^{p(x)-2} \nabla u_k) \cdot (\nabla u_n - \nabla u_k)] dx \\
&\leq \int_{\mathbb{R}^N} \Phi_{n,k} dx := T_{n,k} \\
&= (D_1 J(u_n, v_n) - D_1 J(u_k, v_k) + D_1 \mathcal{F}(u_n, v_n) - D_1 \mathcal{F}(u_k, v_k))(u_n - u_k) \\
&\leq \left(\|D_1 J(u_n, v_n)\|_{*,p(x),q(x)} + \|D_1 J(u_k, v_k)\|_{*,p(x),q(x)} \right) \|u_n - u_k\|_{p(x)} \\
&\quad + \|D_1 \mathcal{F}(u_n, v_n) - D_1 \mathcal{F}(u_k, v_k)\|_{*,p(x),q(x)} \|u_n - u_k\|_{p(x)}.
\end{aligned} \tag{3.34}$$

When $1 < p(x) < 2$, employing (3.31) and Proposition 2.2, it follows

$$\begin{aligned}
\int_{V_p} |\nabla u_n - \nabla u_k|^{p(x)} dx &\leq \int_{V_p} |\nabla u_n - \nabla u_k|^{p(x)} (|\nabla u_n| + |\nabla u_k|)^{p(x)(p(x)-2)/2} (|\nabla u_n| + |\nabla u_k|)^{p(x)(2-p(x))/2} dx \\
&\leq 2 \left| |\nabla u_n - \nabla u_k|^{p(x)} \cdot \Psi_{n,k}^{-p(x)/2} \right|_{2/p(x)} \times \left| \Psi_{n,k}^{p(x)/2} \right|_{2/(2-p(x))} \\
&\leq 2 \max \left\{ \left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_k|^2 \Psi_{n,k}^{-1} dx \right)^{p^-/2}, \left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_k|^2 \Psi_{n,k}^{-1} dx \right)^{p^+/2} \right\} \\
&\quad \times \max \left\{ \left(\int_{\mathbb{R}^N} \Psi_{n,k}^{p(x)/(2-p(x))} dx \right)^{(2-p^-)/2}, \left(\int_{\mathbb{R}^N} \Psi_{n,k}^{p(x)/(2-p(x))} dx \right)^{(2-p^+)/2} \right\} \\
&\leq 2 \max \left\{ (p^- - 1)^{-p^-/2} \cdot T_{n,k}^{p^-/2}, (p^- - 1)^{-p^+/2} \cdot T_{n,k}^{p^+/2} \right\} \\
&\quad \times \max \left\{ \left(\int_{\mathbb{R}^N} \Psi_{n,k}^{p(x)/(2-p(x))} dx \right)^{(2-p^-)/2}, \left(\int_{\mathbb{R}^N} \Psi_{n,k}^{p(x)/(2-p(x))} dx \right)^{(2-p^+)/2} \right\}.
\end{aligned} \tag{3.35}$$

Since $T_{n,k}$ is uniformly bounded in $W_0^{1,p(x)}(\mathbb{R}^N)$ in accordance with n, k , and by the fact that $\|J'(u_m, v_m)\|_{*,p(x),q(x)} \rightarrow 0$ as $m \rightarrow +\infty$, \mathcal{F}' is compact and by Proposition 2.4, we have

$$\lim_{n,k \rightarrow +\infty} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u_k|^{p(x)} dx = 0. \tag{3.36}$$

Applying the same arguments, we can find a subsequence of (u_n, v_n) such that

$$\lim_{n,k \rightarrow +\infty} \int_{\mathbb{R}^N} |\nabla v_n - \nabla v_k|^{q(x)} dx = 0. \tag{3.37}$$

Therefore by Proposition 2.2, for a convenient subsequence, we have

$$\lim_{n,k \rightarrow +\infty} \|(u_n, v_n) - (u_k, v_k)\|_{*,p(x),q(x)} = 0. \tag{3.38}$$

Hence, (u_n, v_n) contains a Cauchy subsequence and so contains a strongly convergent subsequence. The proof is complete. \square

Lemma 3.4. Under the assumptions (F1)–(F4), the functional J satisfies the following.

- (i) There exists $\rho, \sigma > 0$ such that $\|u\|_{p(x)} + \|v\|_{q(x)} = \rho$ implies $J(u, v) \geq \sigma > 0$.
- (ii) There exists $E \in W_{p(x), q(x)}$ such that $\|E\|_{p(x), q(x)} > \rho$ and $J(E) \leq 0$.

Proof. By (F4), we can find $\rho > 0$ such that $\|u\|_{p(x)} + \|v\|_{q(x)} = \rho$, so we have

$$\begin{aligned} F(x, u, v) &< \lambda_1 \left(\frac{a(x)}{p(x)} |u|^{p(x)} + \frac{b(x)}{q(x)} |v|^{q(x)} \right), \\ \int_{\mathbb{R}^N} F(x, u, v) &< \lambda_1 \int_{\mathbb{R}^N} \left(\frac{a(x)}{p(x)} |u|^{p(x)} + \frac{b(x)}{q(x)} |v|^{q(x)} \right) dx, \end{aligned} \quad (3.39)$$

since $\lambda_1 > 0$, then we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, u, v) &< \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} \right) dx, \\ 0 &< \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} \right) dx - \int_{\mathbb{R}^N} F(x, u, v) = J(u, v). \end{aligned} \quad (3.40)$$

Hence, there exists $\sigma > 0$ such that $J \geq \sigma > 0$.

Let (τ, θ) be an eigenfunction relative to λ_1 . Then, using the assumption (F4), we can obtain for $\epsilon > 0$ and t sufficiently large,

$$F(x, t^{1/p(x)}\tau, t^{1/q(x)}\theta) \geq t(\lambda_1 + \epsilon) \left(\frac{a(x)}{p(x)} |\tau|^{p(x)} + \frac{b(x)}{q(x)} |\theta|^{q(x)} \right). \quad (3.41)$$

Thus,

$$\begin{aligned} J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) &= t \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} |\nabla \tau|^{p(x)} + \frac{1}{q(x)} |\nabla \theta|^{q(x)} \right) dx \\ &\quad - \int_{\mathbb{R}^N} F(x, t^{1/p(x)}\tau, t^{1/q(x)}\theta) dx \\ &\leq \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} |\nabla \tau|^{p(x)} + \frac{1}{q(x)} |\nabla \theta|^{q(x)} \right) dx \\ &\quad - t(\lambda_1 + \epsilon) \left(\int_{\mathbb{R}^N} \frac{a(x)}{p(x)} |\tau|^{p(x)} dx + \int_{\mathbb{R}^N} \frac{b(x)}{q(x)} |\theta|^{q(x)} dx \right), \end{aligned} \quad (3.42)$$

then it follows

$$J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) \leq -et \left(\frac{1}{p^+} \int_{\mathbb{R}^N} a(x) |\tau|^{p(x)} dx + \frac{1}{q^+} \int_{\mathbb{R}^N} b(x) |\theta|^{q(x)} dx \right). \quad (3.43)$$

So, we can conclude that $\lim_{t \rightarrow +\infty} J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) = -\infty$. Hence, for t sufficiently large, $J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) \leq 0$. As a consequence, we can say that the functional $J(u, v)$ has a critical point; and as we know, the critical points of $J(u, v)$ are the weak solutions of the system (P,Q). \square

Theorem 3.5. The system (P,Q) has at least one nontrivial solution (u, v) .

Proof. By Lemmas 3.3 and 3.4, we can apply the mountain pass theorem to obtain that the system (P,Q) has a nontrivial weak solution. \square

Acknowledgments

The authors want to express their gratitude to Professor M. G. Huidobro and to anonymous referee for a careful reading and valuable suggestions. This research project was supported by DUAPK -2008-59-74, Dicle University, Turkey.

References

- [1] D. E. Edmunds and J. Rákosník, "Sobolev embeddings with variable exponent," *Studia Mathematica*, vol. 143, no. 3, pp. 267–293, 2000.
- [2] X.-L. Fan, J. Shen, and D. Zhao, "Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 749–760, 2001.
- [3] X.-L. Fan and D. Zhao, "On the spaces $L^{p(x)}$ and $W^{m,p(x)}$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.
- [4] O. Kováčik and J. Rákosník, "On spaces $L^{p(x)}$ and $W^{1,p(x)}$," *Czechoslovak Mathematical Journal*, vol. 41(116), no. 4, pp. 592–618, 1991.
- [5] R. A. Mashiyev, "Some properties of variable sobolev capacity," *Taiwanese Journal of Mathematics*, vol. 12, no. 3, pp. 101–108, 2008.
- [6] E. Acerbi and G. Mingione, "Regularity results for stationary electrorheological fluids," *Archive for Rational Mechanics and Analysis*, vol. 164, no. 3, pp. 213–259, 2002.
- [7] S. N. Antontsev and S. I. Shmarev, "A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 3, pp. 515–545, 2005.
- [8] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.
- [9] L. Diening, *Theoretical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Freiburg, Freiburg, Germany, 2002.
- [10] T. C. Halsey, "Electrorheological fluids," *Science*, vol. 258, no. 5083, pp. 761–766, 1992.
- [11] M. Mihăilescu and V. Rădulescu, "A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids," *Proceedings of The Royal Society of London A*, vol. 462, no. 2073, pp. 2625–2641, 2006.
- [12] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, vol. 1748 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.
- [13] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory," *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, vol. 50, no. 4, pp. 675–710, 1986 (Russian).
- [14] C. O. Alves and M. A. Souto, "Existence of solutions for a class of problems in \mathbb{R}^N involving the $p(x)$ -Laplacian," in *Contributions to Nonlinear Analysis. A Tribute to D. G. de Figueiredo on the Occasion of his 70th Birthday*, T. Cazenave, D. Costa, O. Lopes, et al., Eds., vol. 66 of *Progress in Nonlinear Differential Equations and Their Applications*, pp. 17–32, Birkhäuser, Basel, Switzerland, 2006.
- [15] O. M. Buhrii and R. A. Mashiyev, "Uniqueness of solutions of the parabolic variational inequality with variable exponent of nonlinearity," *Nonlinear Analysis: Theory, Methods & Applications*. In press.
- [16] J. Chabrowski and Y. Fu, "Existence of solutions for $p(x)$ -Laplacian problems on a bounded domain," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 2, pp. 604–618, 2005.
- [17] X.-L. Fan, Q. Zhang, and D. Zhao, "Eigenvalues of $p(x)$ -Laplacian Dirichlet problem," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 306–317, 2005.
- [18] X.-L. Fan, "Solutions for $p(x)$ -Laplacian Dirichlet problems with singular coefficients," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 464–477, 2005.
- [19] X.-L. Fan and Q.-H. Zhang, "Existence of solutions for $p(x)$ -Laplacian Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 8, pp. 1843–1852, 2003.
- [20] P. Hästö, "The $p(x)$ -Laplacian and applications," *The Journal of Analysis*, vol. 15, pp. 53–62, 2007.
- [21] M. Mihăilescu, "Existence and multiplicity of solutions for an elliptic equation with $p(x)$ -growth conditions," *Glasgow Mathematical Journal*, vol. 48, no. 3, pp. 411–418, 2006.
- [22] M. Mihăilescu, "Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$ -Laplace operator," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 5, pp. 1419–1425, 2007.
- [23] K. Ariouch and A. El Hamidi, "The Nehari manifold for systems of nonlinear elliptic equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 10, pp. 2149–2167, 2006.

- [24] S. Antontsev and S. Shmarev, "Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 4, pp. 728–761, 2006.
- [25] D. G. de Figueiredo, "Semilinear elliptic systems: a survey of superlinear problems," *Resenhas*, vol. 2, no. 4, pp. 373–391, 1996.
- [26] K. de Thélin and J. Vélin, "Existence and nonexistence of nontrivial solutions for some nonlinear elliptic systems," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 6, no. 1, pp. 153–194, 1993.
- [27] T. Wu, "Multiple positive solutions for semilinear elliptic systems with nonlinear boundary condition," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1712–1722, 2007.
- [28] Q. Zhang, "Existence of positive solutions for elliptic systems with nonstandard $p(x)$ -growth conditions via sub-supersolution method," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 4, pp. 1055–1067, 2007.
- [29] A. Djellit and S. Tas, "Existence of solutions for a class of elliptic systems in \mathbb{R}^N involving the $p(x)$ -Laplacian," *Electronic Journal of Differential Equations*, vol. 2003, no. 56, pp. 1–8, 2003.
- [30] A. El Hamidi, "Existence results to elliptic systems with nonstandard growth conditions," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 30–42, 2004.
- [31] X. Xu and Y. An, "Existence and multiplicity of solutions for elliptic systems with nonstandard growth condition in \mathbb{R}^N ," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 4, pp. 956–968, 2008.
- [32] L. Diening, "Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$," *Mathematische Nachrichten*, vol. 268, no. 1, pp. 31–43, 2004.