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Research Article

Strong Convergence of a Modified Iterative Algorithm for Mixed-Equilibrium Problems in Hilbert Spaces

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The purpose of this paper is to study the strong convergence of a modified iterative scheme to find a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of variational inequalities for a relaxed cocoercive mapping, as well as the set of solutions of a mixed-equilibrium problem. Our results extend recent results of Takahashi and Takahashi (2007), Marino and Xu (2006), Combettes and Hirstoaga (2005), Iiduka and Takahashi (2005), and many others.

1. Introduction and preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a nonlinear map. P_C be the projection of H onto the convex subset C. The classical variational inequality problem, denoted by $\mathrm{VI}(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \quad \forall v \in C. \tag{1.1}$$

For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0 \quad \forall v \in C, \tag{1.2}$$

if and only if $u = P_C z$. It is known that the projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{1.3}$$

for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ $\forall y \in C$.

One can see that the variational inequality problem (1.1) is equivalent to some fixed-point problems.

The element $u \in C$ is a solution of the variational inequality problem (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant. The alternative equivalent formulation has played a significant role in the studies of the the variational inequalities and related optimization problems.

Recall the following definitions.

(1) B is called v-strongly monotone if for each $x, y \in C$, we have

$$\langle Bx - By, x - y\rangle \geq v\|x - y\|^2 \tag{1.4}$$

for a constant $v > 0$. This implies that

$$\|Bx - By\| \geq v\|x - y\|, \tag{1.5}$$

that is, B is v-expansive and when $v = 1$, it is expansive.

(2) B is called v-cocoercive [1, 2] if for each $x, y \in C$, we have

$$\langle Bx - By, x - y\rangle \geq v\|Bx - By\|^2 \tag{1.6}$$

for a constant $v > 0$. Clearly, every v-cocoercive map B is $1/v$-Lipschitz continuous.

(3) B is called relaxed u-cocoercive if there exists a constant $u > 0$ such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 \quad \forall x, y \in C. \tag{1.7}$$

(4) B is called relaxed (u, v)-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2 \quad \forall x, y \in C \tag{1.8}$$

for $u = 0$, B is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps. It is easy to see that we have the following implication: v-strongly monotonicity $\Rightarrow$ relaxed (u, v)-cocoercivity.

(5) A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ $\forall x, y \in C$. Next, we denote by $F(T)$ the set of fixed points of T.

(6) A mapping $f : H \to H$ is said to be a contraction if there exists a coefficient $\alpha (0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \quad \forall x, y \in H. \tag{1.9}$$

(7) An operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad \forall x \in H. \tag{1.10}$$

(8) A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, one has that $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ implies that $f \in Tx$ for every $(y, g) \in G(T)$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \{\omega \in H : \langle v-u, \omega\rangle \geq 0 \quad \forall u \in C\} \tag{1.11}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{1.12}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, B)$ (see [3]).

Let F be an equilibrium bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \tag{1.13}$$

The set of solutions of (1.13) is denoted by $\mathrm{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y-x\rangle$ for $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y-z\rangle \geq 0$ for $y \in C$. A number of problems in physics, optimization, and economics can be reduced to finding a solution of (1.13). Equilibrium problems have been studied extensively (see, e.g., [4, 5]). Recently, Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem.

Very recently, S. Takahashi and W. Takahashi [6] introduced an new iterative:

$$\begin{gathered} F(y_n, u) + \frac{1}{r_n}\langle u-y_n, y_n-x_n\rangle \geq 0 \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n \quad \forall n \geq 1 \end{gathered} \tag{1.14}$$

for approximating a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Iterative methods for nonexpansive mapping have recently been applied to solve convex minimization problems (see, e.g., [7–16] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.15}$$

where A is a linear-bounded operator, C is the fixed-point set of a nonexpansive mapping S, and b is a given point in H. In [10, 11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I-\alpha_n A)Sx_n + \alpha_n b, \quad n \geq 0, \tag{1.16}$$

converges strongly to the unique solution of the minimization problem (1.15) provided that the sequence α_n satisfies certain conditions. Recently, Marino and Xu [8] introduced a new iterative scheme by the viscosity approximation

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.17}$$

They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C, \tag{1.18}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.19}$$

where C is the fixed-point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoercive map, Takahashi and Toyoda [17] introduced the following iterative process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.20}$$

for every $n = 0, 1, 2, \ldots$, where A is α-cocoercive, $x_0 = x \in C$, α_n is a sequence in $(0,1)$, and λ_n is a sequence in $(0, 2\alpha)$. They show that if $F(S) \cap \mathrm{VI}(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.20) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, A)$. Recently, Iiduka and Takahashi [18] studied similar scheme as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.21}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, α_n is a sequence in $(0,1)$, and λ_n is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap \mathrm{VI}(C, A)$. Very recently, Chen et al. [19] studied the following iterative process:

$$x_1 \in C, \; x_{n+1} = \alpha_n f(x) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1, \tag{1.22}$$

and also obtained a strong convergence theorem by the so-called viscosity approximation method [20].

Let $T_i : C \to C$, where $i = 1, 2, \ldots, N$ be a a finite family of nonexpansive mappings, let $F(T_i)$ denote the fixed-point set of T_i, that is, $F(T_i) := \{x \in C : T_i x = x\}$. Finding an optimal point in the intersection $\cap_{i=1}^N F(T_i)$ of the fixed-point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed-point sets of a family of nonexpansive mappings (see, e.g., [21, 22]). The problem of finding an optimal point that minimizes a given cost function over $\cap_{i=1}^N F(T_i)$ is of wide interdisciplinary interest and practical importance (see, e.g., [12, 16, 23–25]). A simple algorithmic solution to the problem of minimizing a quadratic function over $\cap_{i=1}^N F(T_i)$ is of extreme value in many applications including set theoretic signal estimation (see, e.g., [12, 26]).

We study the mapping W_n defined by

$$\begin{aligned}
U_{n0} &= I, \\
U_{n1} &= \lambda_{n1} T_1 U_{n0} + (1 - \lambda_{n1}) I, \\
U_{n2} &= \lambda_{n2} T_2 U_{n1} + (1 - \lambda_{n2}) I, \\
&\vdots \\
U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\
W_n := U_{nN} &= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_n N) I,
\end{aligned} \tag{1.23}$$

where $\{\lambda_{n1}\}, \{\lambda_{n2}\}, \ldots, \{\lambda_{nN}\} \in (0,1]$. Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}$, …, $\{\lambda_{nN}\}$. Nonexpansivity of T_i yields the nonexpansivity of W_n. Moreover, in [27, Lemma 3.1], it is shown that $F(W_n) = \cap_{i=1}^{N} F(T_i)$. In [28], Qin et al. introduce a more general iterative process as follows: $X_1 \in H$

$$\begin{gathered}
F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0 \quad \forall u \in C, \\
x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n A) W_n P_C (I - s_n B) y_n \quad \forall n \geq 1,
\end{gathered} \tag{1.24}$$

where W_n is defined by (1.23), A is a linear-bounded operator, and B is relaxed cocoercive. They prove that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of the variational inequalities for relaxed cocoercive maps, and the set of solutions of the equilibrium problems (1.13), which solves another variational inequality:

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0 \quad \forall p \in F, \tag{1.25}$$

where $F = \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$, and it is also the optimality condition for the minimization problem $\min_{x \in F}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for $\gamma f(h'(x) = \gamma f(x)$ for $x \in H)$.

Recently, Ceng and Yao [14] introduce a mixed-equilibrium problem (MEP) as follows. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\varphi : C \to R$ be a real-valued function and $\Theta : C \times C \to R$ be an equilibrium bifunction, that is, $\Theta(u, u) = 0$ for each $u \in C$, the MEP is given as follows, which is to find $x^* \in C$ such that

$$\mathrm{MEP} : \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C. \tag{1.26}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (EP), which is to find $x^* \in C$ such that

$$\mathrm{EP} : \Theta(x^*, y) \geq 0 \quad \forall y \in C. \tag{1.27}$$

Denote the set of solutions of MEP by Ω and the set of solutions of EP by S-EP. The MEP includes fixed-point problems, optimization problems, variational inequality problems, Nash, EPS, and the EP as special cases (see, e.g., [2, 5, 21, 22, 29]). Some methods have been proposed to solve the EP (see, e.g., [1, 3, 5, 7, 19, 23, 24]).

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\| \quad \forall x, y \in C. \tag{1.28}$$

Recall also that a mapping $S : C \to H$ is said to be nonexpansive if

$$\|Sx - Sy\| \le \|x - y\| \quad \forall x, y \in C. \tag{1.29}$$

Denote the set of fixed points of S by $\mathrm{Fix}(S)$. It is well known that if C is bounded closed convex and $S : C \to C$ is nonexpansive, then $\mathrm{Fix}(S) \neq \emptyset$.

Inspired and motivated by the ongoing research in this field, we investigate the problem of finding a common element of the set of solution of (1.26) and the set of common fixed points of finite many nonexpansive mappings in a Hilbert space. First, we introduce a hybrid iterative scheme for finding a common element of the set of solutions of MEP and the set of common fixed points of finite many nonexpansive mapping. Furthermore, we prove that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of MEP and the set of common fixed points of finite many nonexpansive mapping. Our results extend the recent ones announced by Chen et al. [19], Combettes and Hirstoaga [4], Iiduka and Takahashi [18], Marino and Xu [8], Qin et al. [28], S. Takahashi and W. Takahashi [6], Wittmann [30], and many others.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x - u\| \le \|x - y\| \quad \forall y \in C. \tag{1.30}$$

We denote u by $P_C(x)$, where P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \ge 0 \quad \forall y \in C. \tag{1.31}$$

In this paper, for solving the MPE for an equilibrium bifunction, $\Theta : C \times C \to R$ satisfies the following conditions:

(H1) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \le 0$ $\forall x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto \Theta(x,y)$ is concave and upper semicontinuous;

(H3) for each $x \in C$, $y \mapsto \Theta(x,y)$ is convex.

Let $F : C \to H$ and $\eta : C \times C \to H$ be two mapping. Then, F is called

(i) η-monotone if

$$\langle F(x) - F(y), \eta(x,y)\rangle \ge 0 \quad \forall x, y \in C; \tag{1.32}$$

(ii) η-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), \eta(x,y)\rangle \ge \alpha\|x - y\|^2 \quad \forall x, y \in C; \tag{1.33}$$

(iii) Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\langle F(x) - F(y), \eta(x,y)\rangle \le \beta\|x-y\| \quad \forall x,y \in C \tag{1.34}$$

when $\eta(x,y) = x - y\ \forall x,y \in C$and if there exists a constant $\lambda > 0$ such that

$$\|\eta(x,y)\| \le \lambda\|x-y\| \quad \forall x,y \in C. \tag{1.35}$$

A differentiable function $K : C \to R$ on a convex set C is called

(i) η-convex [14] if

$$K(y) - K(x) \ge \langle K'(x), \eta(y,x)\rangle \quad \forall x,y \in C, \tag{1.36}$$

where $K'(x)$ is the Frechet derivative of K at x;

(ii) η-strongly convex [15] if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y,x)\rangle \ge \frac{\mu}{2}\|x-y\|^2 \quad \forall x,y \in C. \tag{1.37}$$

Let C be a nonempty closed convex subset of real Hilbert space H, $\varphi : C \to R$ be a real-valued function, and $\Theta : C \times C \to R$ be an equilibrium bifunction. Let r be a positive parameter. For a given point $x \in C$, consider the auxiliary problem for MEP (MEP(x,r)) which consists of finding $y \in C$ such that

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \ge 0 \quad \forall z \in C, \tag{1.38}$$

where $\eta : C \times C \to H$ and $K'(x)$ is the Frechet derivative of a functional $K : C \to R$ at x. Let $T_r : C \to C$ be the mapping such that for each $x \in C$, $T_r(x)$ is the solution of MEP(x,r), that is,

$$T_r(x) = \left\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \ge 0,\ \forall z \in C \right\}. \tag{1.39}$$

Lemma 1.1 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to R$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions (H1)–(H3).*

Assume that

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*

  (a) *$\eta(x,y) + \eta(y,x) = 0\ \forall x,y \in C$,*
  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C$, $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to R$ is η-strongly convex with constant $\mu > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each* $x \in C$, *there exist a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$ *such that for any* $y \in C \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{1.40}$$

Then, there exists $y \in C$ *such that*

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0 \quad \forall z \in C. \tag{1.41}$$

Lemma 1.2 (see [14]). *Assume that* Θ *satisfies the same assumptions as Lemma 2.1 for* $r > 0$ *and* $x \in C$, *the mapping* $T_r : C \to C$ *can be defined as follows:*

$$T_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0 \ \forall z \in C \right\} \tag{1.42}$$

for all $y \in C$. *Then, the following hold:*

(i) T_r *is single-valued;*

(ii) (a) $\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle \ \forall (x_1, x_2) \in C \times C$; *where* $u_i = S_r(x_i)$, $i = 1, 2$;

(b) T_r *is nonexpansive if* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that* $\mu \geq \lambda\nu$;

(iii) $F(T_r) = \Omega$;

(iv) Ω *is closed and convex.*

Lemma 1.3 (see [24]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in [0, 1] with*

$$0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq 1. \tag{1.43}$$

Suppose

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{1.44}$$

for all integer $n \geq 0$ *and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{1.45}$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4 (see [23]). *Assume* a_n *is sequence of nonexpansive real number such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \tag{1.46}$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

2. Iterative scheme and strong convergence

Now, we introduced the following hybrid iterative scheme. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$ and let A be a strongly positive bounded linear operator on H with coefficient $\tilde{\gamma} > 0$ such that $0 < \tilde{\gamma} < \gamma/\alpha$, where $\gamma > 0$ is some constant. Given $x_0 \in H$, suppose the sequences $\{x_n\}$ and $\{y_n\}$ are generated iterative by

$$\Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0 \quad \forall x \in C,$$
$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n P_C(I - s_n B)y_n \quad \forall n \geq 1, \tag{2.1}$$

where W_n is defined by (1.23), A is a linear bounded operator, and B is relaxed cocoercive, we prove that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of the variational inequalities for relaxed cocoercive maps, and the set of solutions of the equilibrium problems (1.26), which solves another variational inequality

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0 \quad \forall p \in F, \tag{2.2}$$

where $F = \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, B) \cap \Omega$ and is also the optimality condition for the minimization problem $\min_{x\in F}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for $\gamma f(x)$ (i.e., $h' = \gamma f(x)$ for $c \in H$). The results obtained in this paper improve and extend the recent ones announced by Chen et al. [19], Combettes and Hirstoaga [4], Iiduka and Takahashi [18], Marino and Xu [8], Qin et al. [28], S. Takahashi and W. Takahashi [6], Wittmann [30], and many others.

We will need the following result concerning the W-mapping W_n.

Lemma 2.1 (see [4]). *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{N} Fix(T_i)$ is nonempty, and let $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$ be real numbers such that $0 < \lambda_{ni} \leq a < 1$ for $i = 1, 2, \ldots, N$. For any $n \geq 1$, let W_n be the W-mapping of C into itself generated by $T_N, T_{N-1}, \ldots, 1$ and $\lambda_{nN}, \lambda_{n,N-1}, \ldots, \lambda_{n1}$. If X is strictly convex, then $Fix(W_n) = \cap_{i=1}^{N} Fix(T_i)$.*

Now, we study the strong convergence of the hybrid iterative method (2.1).

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\varphi : C \to R$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions (H1)–(H3), let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings on C into H, and let B be a μ-Lipschitzian, relaxed (u, v)-cocoercive map of C into H such that*

$$\bigcap_{i=1}^{N} Fix(T_i) \cap \Omega \cap VI(C, B) \neq \emptyset. \tag{2.3}$$

Let $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$ be a real number such that $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ $\forall i = 1, 2, \ldots, N$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ are three sequences in (0,1), and r is a positive parameter. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with *coefficient* $\overline{\gamma} > 0$ *such that* $\|A\| \leq 1$. *Assume that* $0 < \widetilde{\gamma} < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_1 \in H$ *and suppose that the following conditions are satisfied:*

(i) $\eta : C \times C \to H$ *is Lipschitz with constant* $\lambda > 0$ *such that*

 (a) $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in C$;

 (b) $\eta(\cdot, \cdot)$ *is affine in the first variable;*

 (c) *for each fixed* $y \in C$, $x \mapsto \eta(y, x)$ *is sequentially continuous from the weak topology to the weak topology;*

(ii) $K : C \to R$ *is* η*-strongly convex with constant* $\mu > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* $\nu > 0$, $\mu \geq \lambda\nu$;

(iii) *for each* $x \in C$, *there exists a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$, *such that, for any* $y \in C \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0; \tag{2.4}$$

(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$; $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$; $\sum_{n=1}^{\infty}|s_{n+1} - s_n| < \infty$; $\{s_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq 2(v - u\mu^2)/\mu^2$.

Given $x_0 \in C$ *arbitrarily, then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated iteratively by* (2.1) *converge strongly to* $q \in Fix(T_i) \cap \Omega \cap VI(C, B)$ *provided that* T_r *is firmly nonexpansive, where* $q = P_{Fix(T_i)\cap\Omega\cap VI(C,B)}(I - A + \gamma f)(q)$ *is a unique solution of variational inequalities:*

$$\langle (A - \gamma f)q, q - x\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{N} Fix(T_i) \cap \Omega \cap VI(C, B), \tag{2.5}$$

which is the optimality condition for the minimization problem

$$\min_{p\in F} \frac{1}{2}\langle Ap, p\rangle - h(p), \tag{2.6}$$

where h *is a potential function for* γf.

Proof. Note that for the control condition (iv), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$.

Since A is linear bounded self-adjoint operator on C, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in C, \|u\| = 1\}. \tag{2.7}$$

Observe that

$$\begin{aligned} \langle ((1 - \beta_n)I - \alpha_n A)u, u\rangle &= 1 - \beta_n - \alpha_n\langle Au, u\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0, \end{aligned} \tag{2.8}$$

that is, $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)u,u\rangle : u\in C, \|u\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Au,u\rangle : u\in C,\|u\|=1\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned} \tag{2.9}$$

Let $Q=P_{\cap_{i=1}^N \mathrm{Fix}(T_i)\cap\Omega}$. Note that f is a contraction with coefficient $\alpha\in(0,1)$. Then, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x)-Q(I-A+\gamma f)(y)\| &\le \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\ &\le \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|\\ &\le (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &= (1-(\overline{\gamma}-\gamma\alpha))\|x-y\|, \quad \forall x,y\in C.\end{aligned} \tag{2.10}$$

Therefore, $Q(I-A+\gamma f)$ is a contraction of C into itself, which implies that there exists a unique element $q\in C$ such that $q=Q(I-A+\gamma f)(q)=P_{\cap_{i=1}^N \mathrm{Fix}(T_i)\cap\Omega}(I-A+\gamma f)(q)$.

First, we show that $I-s_nB$ is nonexpansive. Indeed, from the relaxed (u,v)-cocoercive and μ-Lipschitzian definition on B and condition (iv), we have

$$\begin{aligned}\|(I-s_nB)x-(I-s_nB)x\|^2 &= \|(x-y)-s_n(Bx-By)\|^2\\ &= \|x-y\|^2-2s_n\langle x-y,Bx-By\rangle+s_n^2\|Bx-By\|^2\\ &\le \|x-y\|^2-2s_n[-u\|Bx-By\|^2+v\|(x-y)\|^2]+s_n^2\|Bx-By\|^2\\ &\le \|x-y\|^2+2s_n\mu^2u\|x-y\|^2-2s_nv\|x-y\|^2+\mu^2s_n^2\|x-y\|^2\\ &= (1+2s_n\mu^2u-2s_nv+\mu^2s_n^2)\|x-y\|^2\\ &\le \|x-y\|^2,\end{aligned} \tag{2.11}$$

which implies that the mapping $I-s_nB$ is nonexpansive. Now, we observe that $\{x_n\}$ is bounded. Indeed, pick $p\in F$. Since $y_n=T_rx_n$, we have

$$\|y_n-p\|=\|T_rx_n-T_rp\|\le\|x_n-p\|. \tag{2.12}$$

Putting $\rho_n=P_C(I-s_nB)y_n$, we have

$$\|\rho_n-p\|\le\|(I-s_nB)y_n-p\|\le\|y_n-p\|\le\|x_n-p\|. \tag{2.13}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\gamma f(W_nx_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(W_n\rho_n-p)\|\\ &\le ((1-\beta_n-\alpha_n\overline{\gamma})\|\rho_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(W_nx_n)-Ap\|\\ &\le ((1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n[\gamma\|f(W_nx_n)-f(p)\|+\|\gamma f(p)-Ap\|]\\ &\le [1-(\overline{\gamma}-\gamma\alpha)\alpha_n]\|x_n-p\|+\|\gamma f(p)-Ap\|,\end{aligned} \tag{2.14}$$

which gives that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \ge 0. \tag{2.15}$$

Therefore, we obtain that $\{x_n\}$ is bounded, so is $\{y_n\}\{W_n x_n\}$, $\{W_n \rho_n\}$ and $\{f(W_n x_n)\}$ are all bounded. Now, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.16}$$

Let $p \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \Omega \cap \mathrm{VI}(C, B)$. From the definition of T_r, we note that $y_n = T_r(x_n)$ and $y_{n+1} = T_r(x_{n+1})$. It follows that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|T_r(x_{n+1}) - T_r(x_n)\| \\ &\le \|x_{n+1} - x_n\|. \end{aligned} \tag{2.17}$$

Note that

$$\begin{aligned} \|\rho_{n+1} - \rho_n\| &= \|P_C(I - s_{n+1}B)y_{n+1} - P_C(I - s_n B)y_n\| \\ &\le \|(I - s_{n+1}B)y_{n+1} - (I - s_n B)y_n\| \\ &= \|(I - s_{n+1}B)y_{n+1} - (I - s_{n+1})By_n + (s_n - s_{n+1})By_n\| \\ &\le \|y_{n+1} - y_n\| + |s_n - s_{n+1}|\|By_n\|. \end{aligned} \tag{2.18}$$

Substituting (2.17) into (2.18), we have

$$\|\rho_{n+1} - \rho_n\| \le \|x_{n+1} - x_n\| + |s_n - s_{n+1}|\|By_n\|. \tag{2.19}$$

Set $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ $\forall n \ge 0$. Observe that from the definition of z_n, we obtain

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)W_{n+1}\rho_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n \gamma f(W_n x_n) + ((1 - \beta_n)I - \alpha_n A)W_n \rho_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\gamma f(W_{n+1}x_{n+1}) - \frac{\alpha_n}{1 - \beta_n}\gamma f(W_n x_n) + W_{n+1}\rho_{n+1} \\ &\quad - W_n \rho_n + \frac{\alpha_n}{1 - \beta_n}AW_n \rho_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}AW_{n+1}\rho_{n+1} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}[\gamma f(x_{n+1}) - AW_{n+1}\rho_{n+1}] + \frac{\alpha_n}{1 - \beta_n}[AW_n \rho_n - \gamma f(W_n x_n)] \\ &\quad + W_{n+1}\rho_{n+1} - W_{n+1}\rho_n + W_{n+1}\rho_n - W_n \rho_n. \end{aligned} \tag{2.20}$$

It follows that

$$\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\|+\|AW_{n+1}\rho_{n+1}\|)\\
&\quad+\frac{\alpha_n}{1-\beta_n}(\|AW_n\rho_n\|+\|\gamma f(W_nx_n)\|)\\
&\quad+\|W_{n+1}\rho_{n+1}-W_{n+1}\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\|+\|AW_{n+1}\rho_{n+1}\|)\\
&\quad+\frac{\alpha_n}{1-\beta_n}(\|AW_n\rho_n\|+\|\gamma f(W_nx_n)\|)\\
&\quad+\|W_{n+1}\rho_n-W_n\rho_n\|+\|\rho_{n+1}-\rho_n\|-\|x_{n+1}-x_n\|.
\end{aligned}\tag{2.21}$$

From (1.23), since T_i and $U_{n,i}$ $\forall i=1,2,\dots,N$ are nonexpansive,

$$\begin{aligned}
&\|W_{n+1}\rho_n-W_n\rho_n\|\\
&\quad=\|\lambda_{n+1,N}T_NU_{n+1,N-1}u_n+(1-\lambda_{n+1,N})\rho_n-\lambda_{n,N}T_NU_{n,N-1}\rho_n-(1-\lambda_{n,N})\rho_n\|\\
&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|u_n\|+\|\lambda_{n+1,N}T_NU_{n+1,N-1}\rho_n-\lambda_{n,N}T_NU_{n,N-1}\rho_n\|\\
&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|\rho_n\|+\|\lambda_{n+1,N}(T_NU_{n+1,N-1}\rho_n-T_NU_{n,N-1}\rho_n\|+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}\rho_n\|\\
&\quad\le 2M|\lambda_{n+1,N}-\lambda_{n,N}|+\lambda_{n+1,N}\|(U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\|.
\end{aligned}\tag{2.22}$$

Again, from (1.23),

$$\begin{aligned}
&\|(U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\|\\
&=\|\lambda_{n+1,N}T_{N-1}U_{n+1,N-2}\rho_n+(1-\lambda_{n+1,N-1})\rho_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}\rho_n-(1-\lambda_{n,N-1})\rho_n\|\\
&\le|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|\rho_n\|\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}\rho_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}\rho_n\|\\
&\le|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|\rho_n\|+\lambda_{n+1,N-1}\|(T_{N-1}U_{n+1,N-2}\rho_n-T_{N-1}U_{n+1,N-2}\rho_n\|+|\lambda_{n+1,N-1}-\lambda_{n,N-1}|M\\
&\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\lambda_{n+1,N-1}\|(U_{n+1,N-2}\rho_n-U_{n,N-2}\rho_n\|\\
&\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\|(U_{n+1,N-2}\rho_n-U_{n,N-2}\rho_n\|.
\end{aligned}\tag{2.23}$$

Therefore, we have

$$\begin{aligned}
&\|(U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\|\\
&\quad\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}|+\|(U_{n+1,N-3}\rho_n-U_{n,N-3}\rho_n\|\\
&\quad\le 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|+\|(U_{n+1,1}u_n-U_{n,1}\rho_n\|\\
&\quad=\|\lambda_{n+1,1}T_1u_n+(1-\lambda_{n+1,1})\rho_n-\lambda_{n,1}T_1u_n-(1-\lambda_{n,1})\rho_n\|+2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|,
\end{aligned}\tag{2.24}$$

and then

$$\begin{aligned}\|(U_{n+1,N-1}\rho_n - U_{n,N-1}\rho_n\| &\le |\lambda_{n+1,1} - \lambda_{n,1}|\|\rho_n\| + \|\lambda_{n+1,1}T_1\rho_n - \lambda_{n,1}T_1\rho_n\| \\ &\quad + 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \le 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned} \tag{2.25}$$

Substituting (2.25) into (2.22), we have

$$\begin{aligned}\|W_{n+1}\rho_n - W_n\rho_n\| &\le 2M|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\le 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned} \tag{2.26}$$

Using (2.19) and (2.26) in (2.21), we get

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}x_{n+1}\|) \\ &\quad + \frac{\alpha_n}{1-\beta_n}(\|AW_nx_n\| + \|\gamma f(W_nx_n)\|) + 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|,\end{aligned} \tag{2.27}$$

which implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.28}$$

Hence, by Lemma 1.3, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{2.29}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{2.30}$$

From (2.18), (2.19), (2.30), and condition (iv), we have

$$\lim_{n\to\infty}\|\rho_{n+1} - \rho_n\| = \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{2.31}$$

Since $x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n$, we have

$$\begin{aligned}\|x_n - W_n\rho_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - W_n\rho_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(W_nx_n) - AW_n\rho_n\| + \beta_n\|x_n - W_n\rho_n\|,\end{aligned} \tag{2.32}$$

that is,

$$\|x_n - W_n\rho_n\| \le \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(W_n x_n) - AW_n\rho_n\|. \tag{2.33}$$

It follows that

$$\lim_{n\to\infty}\|x_n - W_n\rho_n\| = 0. \tag{2.34}$$

For $p \in \cap_{i=1}^{N}\mathrm{Fix}(T_i) \cap \Omega \cap \mathrm{VI}(C, B)$, note that S_r is firmly nonexpansive, then we have

$$\begin{aligned}
\|y_n - p\|^2 &\le \|T_r x_n - T_r x_n\|^2 \\
&\le \langle T_r x_n - T_r p, x_n - p\rangle \\
&= \langle y_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2),
\end{aligned} \tag{2.35}$$

and hence

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2. \tag{2.36}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Ap) + \beta_n(x_n - W_n\rho_n) + (I - \alpha_n A)(W_n\rho_n - p)\|^2 \\
&\le \|(I - \alpha_n A)(W_n\rho_n - p) + \beta_n(x_n - W_n\rho_n)\|^2 + 2\alpha_n\langle\gamma f(W_n x_n) - Ap, x_{n+1} - p\rangle \\
&\le [\|(I - \alpha_n A)(W_n\rho_n - p)\| + \beta_n\|x_n - W_n\rho_n\|]^2 + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\| \\
&\le [(1 - \alpha_n\overline{\gamma})\|(\rho_n - p)\| + \beta_n\|x_n - W_n\rho_n\|]^2 + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\| \\
&= (1 - \alpha_n\overline{\gamma})^2\|\rho_n - p\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2 + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\|x_n - W_n\rho_n\| \\
&\quad + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\| \\
&\le (1 - \alpha_n\overline{\gamma})^2\{\|x_n - p\|^2 - \|x_n - y_n\|^2\} + \beta_n\|x_n - W_n\rho_n\|]^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\|x_n - W_n\rho_n\| \\
&\quad + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\| \\
&= (1 - 2\alpha_n\overline{\gamma} + (\alpha_n\overline{\gamma})^2)\|x_n - p\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - y_n\|^2 \\
&\quad + \beta_n\|x_n - W_n\rho_n\|]^2 + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\|x_n - W_n\rho_n\| \\
&\quad + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 + (\alpha_n\overline{\gamma})^2\|x_n - p\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - y_n\|^2 \\
&\quad + \beta_n\|x_n - W_n\rho_n\|^2 + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\|x_n - W_n\rho_n\| \\
&\quad + 2\alpha_n\|\gamma f(W_n x_n) - Ap\|\|x_{n+1} - p\|.
\end{aligned} \tag{2.37}$$

Then, we have

$$
\begin{aligned}
(1-\alpha_n\overline{\gamma})^2\|x_n-y_n\|^2 \le{}& \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|]^2\\
&+(\alpha_n\overline{\gamma})^2\|x_n-p\|^2+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|\\
&+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\
\le{}& (\|x_n-p\|+\|x_{n+1}-p\|)\times\|x_{n+1}-x_n\|\\
&+(\alpha_n\overline{\gamma})^2\|x_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|]^2\\
&+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.
\end{aligned}
\tag{2.38}
$$

So, from (2.30), (2.34), and condition (iv), we have

$$
\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{2.39}
$$

For $p\in F$, we have

$$
\begin{aligned}
\|\rho_n-p\| &= \|P_C(I-s_nA)y_n-P_C(I-sA)p\|^2\\
&\le \|(y_n-p)-s_n(Ay_n-Ap)\|^2\\
&= \|y_n-p\|^2-2s_n\langle y_n-p,Ay_n-Ap\rangle+s_n^2\|Ay_n-Ap\|^2\\
&\le \|x_n-p\|^2-2s_n[-u\|Ay_n-Ap\|^2+v\|y_n-p\|^2]+s_n^2\|Ay_n-Ap\|^2\\
&\le \|x_n-p\|^2+2s_nu\|Ay_n-Ap\|^2-2s_nv\|y_n-p\|^2+s_n^2\|Ay_n-Ap\|^2\\
&\le \|x_n-p\|^2+\left(2s_n+s_n^2-\frac{2s_nv}{\mu^2}\right)\|Ay_n-Ap\|^2.
\end{aligned}
\tag{2.40}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(W_nx_n)-Ap)+\beta_n(x_n-W_n\rho_n)+(I-\alpha_nA)(W_n\rho_n-p)\|^2\\
&\le \|(I-\alpha_nA)(W_n\rho_n-p)+\beta_n(x_n-W_n\rho_n)\|^2+2\alpha_n\langle\gamma f(W_nx_n)-Ap,x_{n+1}-p\rangle\\
&\le [\|(I-\alpha_nA)(W_n\rho_n-p)\|+\beta_n\|x_n-W_n\rho_n\|]^2+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\
&\le [(1-\alpha_n\overline{\gamma})\|(\rho_n-p)\|+\beta_n\|x_n-W_n\rho_n\|]^2+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\
&= (1-\alpha_n\overline{\gamma})^2\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|\\
&\quad+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\
&= \|\rho_n-p\|^2+(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.
\end{aligned}
\tag{2.41}
$$

Substituting (2.40) into (2.41), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 = \|x_n-p\|^2&+\left(2s_n+s_n^2-\frac{2s_nv}{\mu^2}\right)\|Ay_n-Ap\|^2\\
&+(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.
\end{aligned}
\tag{2.42}
$$

It follows from the condition (iv) that

$$\begin{aligned}\left(\frac{2av}{\mu^2}-2bu-b^2\right)\|Ay_n-Ap\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+\left(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2\right)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|\\ &\quad+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\ &\quad+\left(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2\right)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|\\ &\quad+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.\end{aligned} \tag{2.43}$$

From condition (iv), (2.30), and(2.34), we have

$$\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{2.44}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n-p\|^2 &= \|P_C(I-s_nA)y_n-P_C(I-s_nA)p\|^2\\ &\le \langle (I-s_nA)y_n-(I-s_nA)p,\rho_n-p\rangle\\ &= \frac{1}{2}\left\{\|(I-s_nA)y_n-(I-s_nA)p\|^2+\|\rho_n-p\|^2-\|(I-s_nA)y_n-(I-s_nA)p-(\rho_n-p)\|^2\right\}\\ &\le \frac{1}{2}\left\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|(y_n-\rho_n)-s_n(Ay_n-Ap)\|^2\right\}\\ &= \frac{1}{2}\left\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|y_n-\rho_n\|^2-s_n^2\|Ay_n-Ap\|^2+2s_n\langle y_n-\rho_n,Ay_n-Ap\rangle\right\},\end{aligned} \tag{2.45}$$

which yields that

$$\|\rho_n-p\|^2 \le \|x_n-p\|^2-\|y_n-\rho_n\|^2+2s_n\|y_n-\rho_n\|\|Ay_n-Ap\|. \tag{2.46}$$

Substituting (2.44) into (2.40) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2-\|y_n-\rho_n\|^2+2s_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\ &\quad+\left(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2\right)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.\end{aligned} \tag{2.47}$$

It follows that

$$\begin{aligned}\|y_n-\rho_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2s_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\ &\quad+\left(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2\right)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|\\ &\le (\|x_n-p\|-\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2s_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\ &\quad+\left(2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2\right)\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(W_nx_n)-Ap\|\|x_{n+1}-p\|.\end{aligned} \tag{2.48}$$

From condition (iv), (2.30), (2.34), and (2.41), we have

$$\lim_{n\to\infty}\|y_n-\rho_n\|=0. \tag{2.49}$$

Observe that

$$\begin{aligned}\|y_n-W_ny_n\| &\le \|W_ny_n-W_n\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|+\|y_n-\rho_n\|\\ &\le 2\|y_n-\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|.\end{aligned} \tag{2.50}$$

From (2.34), (2.39), and (2.49), we have

$$\lim_{n\to\infty}\|y_n-W_ny_n\|=0. \tag{2.51}$$

Observe that $P_F(\gamma f+(I-A))$ is a contraction. Indeed, for all $x,y\in H$, we have

$$\begin{aligned}\|P_F(\gamma f+(I-A))(x)-P_F(\gamma f+(I-A))(y)\| &\le \|(\gamma f+(I-A))(x))-(\gamma f+(I-A)(y))\|\\ &\le \gamma\|f(x)-f(y)\|+\|I-A\|\|x-y\|\\ &\le \gamma\alpha\|x-y\|+(1-\overline{\gamma})\|x-y\|\\ &=(\gamma\alpha+1-\overline{\gamma})\|x-y\|.\end{aligned} \tag{2.52}$$

The Banach contraction mapping principle guarantees that $P_F(\gamma f+(I-A))$ has a unique fixed point, say $q\in H$. That is, $q=P_F(\gamma f+(I-A))(q)$. Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(q))-Aq,x_n-q\rangle\le 0. \tag{2.53}$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(q))-Aq,x_n-q\rangle=\limsup_{n\to\infty}\langle\gamma f(q))-Aq,x_{n_i}-q\rangle. \tag{2.54}$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup\omega$.

Next, we show $\omega\in\cap_{i=1}^N\mathrm{Fix}(T_i)\cap\Omega\cap\mathrm{VI}(C,B)$. First, we prove $\omega\in\Omega$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{ij}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup\omega$. From $\|W_ny_n-y_n\|\to 0$, we obtain $W_ny_{n_i}\rightharpoonup\omega$. Now, we show that $\omega\in\Omega$. Since $y_n=T_rx_n$, we derive

$$\Theta(y_n,x)+\varphi(x)-\varphi(y_n)+\frac{1}{r}\langle K'(y_n)-K'(x_n),\eta(x,y_n)\rangle\ge 0,\quad\forall x\in C. \tag{2.55}$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(y_n)-K'(x_n),\eta(x,y_n)\rangle+\varphi(x)-\varphi(y_n)\ge-\Theta(y_n,x)\ge\Theta(x,y_n), \tag{2.56}$$

and hence

$$\left\langle \frac{K'(y_{n_i}) - K'(x_{n_i})}{r}, \eta(x, y_{n_i}) \right\rangle + \varphi(x) - \varphi(y_{n_i}) \geq \Theta(x, y_{n_i}). \tag{2.57}$$

Since $(K'(y_{n_i}) - K'(x_{n_i}))/r \to 0)$, and $y_{n_i} \to \omega$ weakly, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we have

$$\Theta(x, \omega) + \varphi(\omega) - \varphi(x) \leq 0, \quad \forall x \in C \tag{2.58}$$

for $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)\omega$. Since $x \in C$ and $\omega \in C$, we have $x_t \in C$ and hence $\Theta(x_t, \omega) + \varphi(\omega) - \varphi(x_t) \leq 0$. From the convexity of equilibrium $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned} 0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\ &\leq t\Theta(x_t, x) + (1-t)\Theta(x_t, \omega) + t\varphi(x) + (1-t)\varphi(\omega) - \varphi(x_t) \\ &\leq t[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)], \end{aligned} \tag{2.59}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then, we have

$$\Theta(\omega, x) + \varphi(x) - \varphi(\omega) \geq 0, \quad \forall x \in C, \tag{2.60}$$

and hence $\omega \in \Omega$.

We will show $\omega \in \mathrm{Fix}(W_n)$. Assume $\omega \notin \mathrm{Fix}(W_n)$. Since $y_{n_j} \to \omega$ weakly and $\omega \neq W_n\omega$, from *Opial* condition, we have

$$\begin{aligned} \liminf_{j\to\infty} \|y_{n_j} - \omega\| &< \liminf_{j\to\infty} \|y_{n_j} - W_n\omega\| \\ &\leq \liminf_{j\to\infty} \|y_{n_j} - W_n y_{n_j}\| + \|W_n y_{n_j} - W_n\omega\| \\ &\leq \liminf_{j\to\infty} \|y_{n_j} - \omega\|. \end{aligned} \tag{2.61}$$

This is a contradiction, so, we get $\omega \in \mathrm{Fix}(W_n) = \cap_{i=1}^{N} \mathrm{Fix}(T_i)$. Therefore, $\omega \in \mathrm{Fix}(W_n) \cap \Omega$.

Next, let us first show that $\omega \in \cap \mathrm{VI}(C, A)$, put

$$TW_1 = \begin{cases} BW_1 + N_C W_1, & W_1 \in C, \\ \varnothing, & W_1 \notin C. \end{cases} \tag{2.62}$$

Since B is relaxed (u, v)-cocoercive and condition (iv), we have

$$\langle Bx - By, x - y \rangle \geq (-\mu)\|Bx - By\|^2 + v\|x - y\|^2 \geq (v - u\mu^2)\|x - y\|^2 \geq 0, \tag{2.63}$$

which yields B is monotone. Thus, T is maximal monotone. Let $(\omega_1, \omega_2) \in G(T)$. Since $\omega_2 - \omega_1 \in N_C\omega_1$ and $\rho_n \in C$, we have

$$\langle \omega_1 - \rho_n, \omega_2 - B\omega_1 \rangle \geq 0. \tag{2.64}$$

On the other hand, from $\rho_n = P_C(I - s_n B)y_n$, we have

$$\langle \omega_1 - \rho_n, \rho_n - (I - s_n B)y_n \rangle \geq 0, \tag{2.65}$$

and hence

$$\left\langle \omega_1 - \rho_n, \frac{\rho_n - y_n}{s_n} + By_n \right\rangle \geq 0. \tag{2.66}$$

It follows that

$$\begin{aligned}
\langle \omega_1 - \rho_n, \omega_2 \rangle &\geq \langle \omega_1 - \rho_{n_i}, B\omega_1 \rangle \\
&\geq \langle \omega_1 - \rho_{n_i}, B\omega_1 \rangle - \left\langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_n} + By_{n_i} \right\rangle \\
&= \left\langle \omega_1 - \rho_{n_i}, B\omega_1 - \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} - By_{n_i} \right\rangle \\
&= \langle \omega_1 - \rho_{n_i}, B\omega_1 - B\rho_{n_i} \rangle + \langle \omega_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle - \left\langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle \\
&\geq \langle \omega_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle - \left\langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle,
\end{aligned} \tag{2.67}$$

which implies that $\langle \omega_1 - \omega, \omega_2 \rangle \geq 0$. We have $\omega \in T^{-1}0$ and hence $\omega \in \mathrm{VI}(C, A)$. That is, $\omega \in F$. Since $q = P_F(\gamma f + (I - A))(q)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \limsup_{n\to\infty} \langle (\gamma f(q), x_{n_i} - q \rangle \\
&= \langle \gamma f(q) - Aq, \omega - q \rangle \leq 0.
\end{aligned} \tag{2.68}$$

Finally, we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to q. From (2.1), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Aq) + \beta_n(x_n - q)((1 - \beta_n)I - \alpha_n A)(W_n \rho_n - q)\|^2 \\
&\leq \|\beta_n(x_n - q) + ((1 - \beta_n)I - \alpha_n A)(W_n \rho_n - q)\|^2 + 2\alpha_n \langle \gamma f(W_n x_n) - Aq, x_{n+1} - q \rangle \\
&\leq [\|((1 - \beta_n)I - \alpha_n A)(W_n \rho_n - q)\| + \|\beta_n(x_n - q)\|]^2 \\
&\quad + 2\alpha_n \gamma \langle f(W_n x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leq [(1 - \beta_n - \alpha_n \widetilde{\gamma})\|\rho_n - q\| + \beta_n \|x_n - q\|]^2 \\
&\quad + 2\alpha_n \gamma \alpha \|x_n - q\| \|x_{n+1} - q\| + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leq [(1 - \beta_n - \alpha_n \widetilde{\gamma})\|y_n - q\| + \beta_n \|x_n - q\|]^2 \\
&\quad + 2\alpha_n \gamma \alpha \|x_n - p\| \|x_{n+1} - q\| + 2\alpha_n \langle \gamma f(p) - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \alpha_n \widetilde{\gamma})^2 \|x_n - q\|^2 + \alpha_n \gamma \alpha \{\|x_n - q\|^2 + \|x_{n+1} - q\|^2\} + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle.
\end{aligned} \tag{2.69}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \frac{1-2\alpha_n\tilde{\gamma}+(\alpha_n\tilde{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Aq,x_{n+1}-\alpha\rangle\\
&= \left[1-\frac{2(\tilde{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right]\|x_n-q\|^2+\frac{(\alpha_n\tilde{\gamma})^2}{1-\alpha_n\gamma\alpha}\|x_n-p\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(p)-Aq,x_{n+1}-q\rangle\\
&\le \left[1-\frac{2(\tilde{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right]\|x_n-q\|^2+\frac{2(\tilde{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\\
&\quad\times\left\{\frac{(\alpha_n\tilde{\gamma})^2M_1}{2(\tilde{\gamma}-\gamma\alpha)}+\frac{1}{\tilde{\gamma}-\gamma\alpha}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\right\}\\
&= (1-\delta_n)\|x_n-q\|^2+\delta_n\sigma_n,
\end{aligned}\tag{2.70}$$

where $M_1=\sup\{\|x_n-q\|^2 : n\ge 1\}$, $\delta_n=2(\tilde{\gamma}-\gamma\alpha)\alpha_n/(1-\alpha_n\gamma\alpha)$, and $\beta_n=(\alpha_n\tilde{\gamma})^2M_1/2(\tilde{\gamma}-\gamma\alpha)+(1/(\tilde{\gamma}-\gamma\alpha))\langle\gamma f(q)-Aq,x_{n+1}-q\rangle$. It is easy to see that $\delta_n\to 0$, $\sum_{n=1}^{\infty}\delta_n=\infty$, and $\limsup_{n\to\infty}\beta_n/\delta_n\le 0$. Hence, by Lemma 1.4, the sequence $\{x_n\}$ converges strongly to q. Consequently, we can obtain that $\{y_n\}$ also converges strongly to q. This completes the proof. □

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C\to R$ be a lower semicontinuous and convex functional. Let $\Theta : C\times C\to R$ be an equilibrium bifunction satisfying conditions (H1)–(H3) such that $\Omega\neq\varnothing$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1) and r is a positive parameter. Suppose that the following conditions are satisfied:*

(i) *$\eta : C\times C\to H$ is Lipschitz with constant $\lambda>0$ such that*

  (a) *$\eta(x,y)+\eta(y,x)=0$ $\forall x\cdot y\in C$;*
  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable;*
  (c) *for each fixed $y\in C$, $x\mapsto\eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C\to R$ is η-strongly convex with constant $\mu>0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu>0$, $\mu\ge\lambda\nu$;*

(iii) *for each $x\in C$, there exist a bounded subset $D_x\subseteq C$ and $z_x\in H$, such that for any $y\in C\setminus D_x$,*

$$\Theta(y,z_x)+\varphi(z_x)-\varphi(y)+\frac{1}{r}\langle K'(y)-K'(x),\eta(z_x,y)\rangle\le 0;\tag{2.71}$$

(iv) *$\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$; $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.*

Given $x_0\in C$ arbitrarily, then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by

$$\begin{aligned}
&\Theta(y_n,x)+\varphi(x)-\varphi(y_n)+\frac{1}{r}\langle K'(y_n)-K'(x_n),\eta(x,y_n)\rangle\ge 0,\quad\forall x\in C,\\
&x_{n+1}=\alpha_n\gamma f(W_nx_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)W_ny_n,\quad\forall n\ge 1.
\end{aligned}\tag{-11}$$

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in Fix(T_i) \cap \Omega$ *provided that* T_r *is firmly nonexpansive, where* $q = P_{Fix(T_i)\cap\Omega}(I - A + \gamma f)(q)$ *is a unique solution of variational inequalities*

$$\langle (A - \gamma f)q, q - x \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{N} Fix(T_i) \cap \Omega, \tag{-11}$$

which is the optimality condition for the minimization problem

$$\min_{p \in Fix(T_i)} \frac{1}{2} \langle Ap, p \rangle - h(p), \tag{2.72}$$

where h *is a potential function for* γf.

Proof. Take $T_i x = x$ $\forall i = 1, 2, \ldots, N$ and for all $x \in C$ in (2.1). Then, $W_n x = x$ $\forall x \in C$. The conclusion follows immediately from Theorem 2.2. This completes the proof. □

Corollary 2.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into itself such that* $\cap_{i=1}^N Fix(T_i) \neq \emptyset$. *Let* $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$ *be real number such that* $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ $\forall i = 1, 2, \ldots, N$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0, 1)$ *and* r *is a positive parameter. Assume that the following hold:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Given $x_0 \in H$ *arbitrarily, then the sequences* $\{x_n\}$*are generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n x_n, \quad \forall n \geq 1. \tag{2.73}$$

Proof. Set $\varphi(x) = 0$ and $\Theta(x, y) = 0$ $\forall x, y \in C$ and put $r = 1$. Take $K(x) = \|x\|^2/2$ and $\eta(y, x) = y - x$ $\forall x, y \in C$, we get $y_n = x_n$ in Theorem 2.2. Therefore, the conclusion follows. □

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