

## Research Article

# The Reverse Hölder Inequality for the Solution to $p$ -Harmonic Type System

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Some inequalities to  $A$ -harmonic equation  $A(x, du) = d^*v$  have been proved. The  $A$ -harmonic equation is a particular form of  $p$ -harmonic type system  $A(x, a + du) = b + d^*v$  only when  $a = 0$  and  $b = 0$ . In this paper, we will prove the Poincaré inequality and the reverse Hölder inequality for the solution to the  $p$ -harmonic type system.

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## 1. Introduction

Recently, amount of work about the  $A$ -harmonic equation for the differential forms has been done. In fact, the  $A$ -harmonic equation is an important generalization of the  $p$ -harmonic equation in  $\mathbb{R}^n$ ,  $p > 1$ , and the  $p$ -harmonic equation is a natural extension of the usual Laplace equation (see [1] for the details). The reverse Hölder inequalities have been widely studied and frequently used in analysis and related fields, including partial differential equations and the theory of elasticity (see [2]). In 1999, Nolder gave the reverse Hölder inequality for the solution to the  $A$ -harmonic equation in [3], and different versions of Caccioppoli estimates have been established in [4–6]. In 2004, D’Onofrio and Iwaniec introduced the  $p$ -harmonic type system in [7], which is an important extension of the conjugate  $A$ -harmonic equation. In 2007, Ding proved the following inequality in [8].

**Theorem A.** Let  $(u, v)$  be a pair of solutions to  $A(x, g + du) = h + d^*v$  in a domain  $\Omega \subset \mathbb{R}^n$ . If  $g \in L^p(B, \Lambda^L)$  and  $h \in L^q(B, \Lambda^L)$ , then  $du \in L^p(B, \Lambda^L)$  if and only if  $d^*v \in L^q(B, \Lambda^L)$ . Moreover, there exist constants  $C_1, C_2$  independent of  $u$  and  $v$ , such that

$$\begin{aligned} \|d^*v\|_{q,B}^q &\leq C_1(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|du\|_{p,B}^p), \\ \|du\|_{p,B}^p &\leq C_2(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|d^*v\|_{q,B}^q) \quad \forall B \subset \sigma B \subset \Omega. \end{aligned} \tag{1.1}$$

In this paper, we will prove the Poincaré inequality (see Theorem 2.5) and the reverse Hölder inequality for the solution to the  $p$ -harmonic type system (see Theorem 3.5). Now let us see some notions and definitions about the  $p$ -harmonic type system.

Let  $e_1, e_2, \dots, e_n$  denote the standard orthogonal basis of  $\mathbb{R}^n$ . For  $l = 0, 1, \dots, n$ , we denote by  $\Lambda^l = \Lambda^l(\mathbb{R}^n)$  the linear space of all  $l$ -vectors, spanned by the exterior product  $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_l}$  corresponding to all ordered  $l$ -tuples  $I = (i_1, i_2, \dots, i_l)$ ,  $1 \leq i_1 < i_2 < \dots < i_l \leq n$ . The Grassmann algebra  $\Lambda = \oplus \Lambda^l$  is a graded algebra with respect to the exterior products. For  $\alpha = \sum \alpha_I e_I \in \Lambda$  and  $\beta = \sum \beta_I e_I \in \Lambda$ , then its inner product is obtained by

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I, \quad (1.2)$$

with the summation over all  $I = (i_1, i_2, \dots, i_l)$  and all integers  $l = 0, 1, \dots, n$ . The Hodge star operator  $*$ :  $\Lambda \rightarrow \Lambda$  is defined by the rule

$$\begin{aligned} *1 &= e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}, \\ \alpha \wedge * \beta &= \beta \wedge * \alpha = \langle \alpha, \beta \rangle (*1) \quad \forall \alpha, \beta \in \Lambda. \end{aligned} \quad (1.3)$$

Hence, the norm of  $\alpha \in \Lambda$  can be given by

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge * \alpha) \in \Lambda_0 = \mathbb{R}. \quad (1.4)$$

Throughout this paper,  $\Omega \subset \mathbb{R}^n$  is an open subset, for any constant  $\sigma > 1$ ,  $Q$  denotes a cube such that  $Q \subset \sigma Q \subset \Omega$ , where  $\sigma Q$  denotes the cube whose center is as same as  $Q$  and  $\text{diam}(\sigma Q) = \sigma \text{diam} Q$ . We say  $\alpha = \sum \alpha_I e_I \in \Lambda$  is a differential  $l$ -form on  $\Omega$ , if every coefficient  $\alpha_I$  of  $\alpha$  is Schwartz distribution on  $\Omega$ . The space spanned by differential  $l$ -form on  $\Omega$  is denoted by  $D'(\Omega, \Lambda^l)$ . We write  $L^p(\Omega, \Lambda^l)$  for the  $l$ -form  $\alpha = \sum \alpha_I dx_I$  on  $\Omega$  with  $\alpha_I \in L^p(\Omega)$  for all ordered  $l$ -tuple  $I$ . Thus  $L^p(\Omega, \Lambda^l)$  is a Banach space with the norm

$$\|\alpha\|_{p,\Omega} = \left( \int_{\Omega} |\alpha|^p dx \right)^{1/p} = \left( \int_{\Omega} \left( \sum_I |\alpha_I|^2 \right)^{p/2} dx \right)^{1/p}. \quad (1.5)$$

Similarly  $W^{k,p}(\Omega, \Lambda^l)$  denotes those  $l$ -forms on  $\Omega$  with all coefficients in  $W^{k,p}(\Omega)$ . We denote the exterior derivative by

$$d : D'(\Omega, \Lambda^l) \longrightarrow D'(\Omega, \Lambda^{l+1}), \quad \text{for } l = 0, 1, 2, \dots, n, \quad (1.6)$$

and its formal adjoint operator (the Hodge codifferential operator)

$$d^* : D'(\Omega, \Lambda^l) \longrightarrow D'(\Omega, \Lambda^{l-1}). \quad (1.7)$$

The operators  $d$  and  $d^*$  are given by the formulas

$$d\alpha = \sum_I d\alpha_I \wedge dx_I, \quad d^* = (-1)^{nl+1} *d*. \quad (1.8)$$

The following two definitions appear in [7].

*Definition 1.1.* The Hodge system holds:

$$A(x, a + du) = b + d^*v, \quad (1.9)$$

where  $a \in L^p(\Omega, \Lambda^l)$  and  $b \in L^q(\Omega, \Lambda^l)$ , is a  $p$ -harmonic type system if  $A$  is a mapping from  $\Omega \times \Lambda^l$  to  $\Lambda^l$  satisfying

- (1)  $x \rightarrow A(x, \xi)$  is measurable in  $x \in \Omega$  for every  $\xi \in \Lambda^l$ ;
- (2)  $\xi \rightarrow A(x, \xi)$  is continuous in  $\xi \in \Lambda^l$  for almost every  $x \in \Omega$ ;
- (3)  $A(x, t\xi) = t^{p-1}A(x, \xi)$  for every  $t \geq 0$ ;
- (4)  $K \langle A(x, \xi) - A(x, \zeta), \xi - \zeta \rangle \geq |\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}$ ;
- (5)  $|A(x, \xi) - A(x, \zeta)| \leq K|\xi - \zeta|(|\xi| + |\zeta|)^{p-2}$

for almost every  $x \in \Omega$  and all  $\xi, \zeta \in \Lambda^l$ , where  $K \geq 1$  is a constant. It should be noted that  $A(x, *) : \Omega \times \Lambda^l \rightarrow \Lambda^l$  is invertible and its inverse denoted by  $A^{-1}$  satisfies similar conditions as  $A$  but with Hölder conjugate exponent  $q$  in place of  $p$ .

*Definition 1.2.* If (1.9) is a  $p$ -harmonic type system, then we say the equation

$$d^*A(x, a + du) = d^*b \quad (1.10)$$

is a  $p$ -harmonic type equation.

The following definition appears in [9].

*Definition 1.3.* A differential form  $u$  is a weak solution for (1.10) in  $\Omega$  if  $u$  satisfies

$$\int_{\Omega} \langle A(x, a + du), d\varphi \rangle + \langle d^*b, \varphi \rangle \equiv 0 \quad (1.11)$$

for every  $\varphi \in W^{k,p}(\Omega, \Lambda^{l-1})$  with compact support.

We can find that if we let  $a = 0$  and  $b = 0$ , then the  $p$ -harmonic type system

$$A(x, a + du) = b + d^*v \quad (1.12)$$

becomes

$$A(x, du) = d^*v. \quad (1.13)$$

It is the conjugate  $A$ -harmonic equation, where the mapping  $A : \Omega \times \Lambda^l \rightarrow \Lambda^l$  satisfies the following conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p. \quad (1.14)$$

If we let  $A(x, \xi) = |\xi|^{p-2}\xi$ , then the conjugate  $A$ -harmonic equation becomes the form

$$|du|^{p-2}du = d^*v. \quad (1.15)$$

It is the conjugate  $p$ -harmonic equation.

So we can see that the conjugate  $p$ -harmonic equation and the conjugate  $A$ -harmonic equation are the specific  $p$ -harmonic type system.

*Remark 1.4.* It should be noted that the mapping  $A(x, *)$  in  $p$ -harmonic system  $A(x, a + du) = b + d^*v$ , is invertible. If we denote its inverse by  $A^{-1}(x, *)$ , then the mapping  $A^{-1}(x, *) : \Lambda^l \rightarrow \Lambda^l$  satisfies similar conditions as  $A$  but with Hölder conjugate exponent  $q$  in place of  $p$ .

## 2. The Poincaré inequality

In this section, we will introduce the Poincaré inequality for the differential forms.

Now first let us see a lemma, which can be found in [9, Section 4] for the details.

**Lemma 2.1.** *Let  $\mathbb{D}$  be a bounded, convex domain in  $\mathbb{R}^n$ . To each  $\mathbf{y} \in \mathbb{D}$  there corresponds a linear operator  $K_{\mathbf{y}} : C^\infty(\mathbb{D}, \Lambda^l) \rightarrow C^\infty(\mathbb{D}, \Lambda^{l-1})$  defined by*

$$(K_{\mathbf{y}}\omega)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} \omega(tx + \mathbf{y} - t\mathbf{y}; x - \mathbf{y}, \xi_1, \dots, \xi_{l-1}) dt, \quad (2.1)$$

and the decomposition

$$\omega = d(K_{\mathbf{y}}\omega) + K_{\mathbf{y}}(d\omega) \quad (2.2)$$

holds at any point  $\mathbf{y} \in \mathbb{D}$ .

We construct a homotopy operator  $T : C^\infty(\mathbb{D}, \Lambda^l) \rightarrow C^\infty(\mathbb{D}, \Lambda^{l-1})$  by averaging  $K_{\mathbf{y}}$  over all points  $\mathbf{y} \in \mathbb{D}$ :

$$T\omega = \int_{\mathbb{D}} \varphi(\mathbf{y}) K_{\mathbf{y}}\omega d\mathbf{y}, \quad (2.3)$$

where  $\varphi$  form  $C^\infty(\mathbb{D})$  is normalized so that  $\int \varphi(\mathbf{y}) d\mathbf{y} = 1$ . It is obvious that  $\omega = d(K_{\mathbf{y}}\omega) + K_{\mathbf{y}}(d\omega)$  remains valid for the operator  $T$ :

$$\omega = d(T\omega) + T(d\omega). \quad (2.4)$$

We define the  $l$ -forms  $\omega_{\mathbb{D}} \in D'(\mathbb{D}, \Lambda^l)$  by  $\omega_{\mathbb{D}} = |\mathbb{D}|^{-1} \int_{\mathbb{D}} \omega(\mathbf{y}) d\mathbf{y}$  for  $l = 0$  and  $\omega_{\mathbb{D}} = d(T\omega)$  for  $l = 1, 2, \dots, n$ , and all  $\omega \in W^{1,p}(D, \Lambda^l)$ ,  $1 < p < \infty$ .

The following definition can be found in [9, page 34].

**Definition 2.2.** For  $\omega \in D'(\mathbb{D}, \Lambda^l)$ , the vector valued differential form

$$\nabla\omega = \left( \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n} \right) \quad (2.5)$$

consists of differential forms  $\partial\omega/\partial x_i \in D'(\mathbb{D}, \Lambda^l)$ , where the partial differentiation is applied to coefficients of  $\omega$ .

The proof of [9, Proposition 4.1] implies the following inequality.

**Lemma 2.3.** *For any  $\omega \in L^p(\mathbb{D}, \Lambda^l)$ , it holds that*

$$\|\nabla T\omega\|_{p, \mathbb{D}} \leq C(n, p)\|\omega\|_{p, \mathbb{D}} \quad (2.6)$$

for any ball or cube  $\mathbb{D} \in \mathbb{R}^n$ .

The following Poincaré inequality can be found in [2].

**Lemma 2.4.** *If  $u \in W_0^{1,p}(\Omega)$ , then there is a constant  $C = C(n, p) > 0$  such that*

$$\left(\frac{1}{|B|} \int_B |u|^{p\chi} dx\right)^{1/p\chi} \leq Cr \left(\frac{1}{|B|} \int_B |\nabla u|^p dx\right)^{1/p}, \quad (2.7)$$

whenever  $B = B(x_0, r)$  is a ball in  $\mathbb{R}^n$ , where  $n \geq 2$  and  $\chi = 2$  for  $p \geq n$ ,  $\chi = np/(n-p)$  for  $p < n$ .

**Theorem 2.5.** *Let  $u \in D'(\mathbb{D}, \Lambda^l)$ , and  $du \in L^p(\mathbb{D}, \Lambda^{l+1})$ . Then,  $u - u_{\mathbb{D}}$  is in  $L^{p\chi}(\mathbb{D}, \Lambda^l)$  and*

$$\left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} \leq C(n, p, l) \text{diam}(\mathbb{D}) \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |du|^p dx\right)^{1/p} \quad (2.8)$$

for any ball or cube  $\mathbb{D} \in \mathbb{R}^n$ , where  $\chi = 2$  for  $p \geq n$  and  $\chi = np/(n-p)$  for  $1 < p < n$ .

*Proof.* We know  $T(du) = u - u_{\mathbb{D}}$ . Now we suppose  $u - u_{\mathbb{D}} = T(du) = \sum_I u_I dx_I$ , where  $I = (i_1, \dots, i_{l+1})$  take over all  $l+1$ -tuples. So we have

$$\nabla T(du) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = \left(\sum_I \frac{\partial u_I}{\partial x_1} dx_I, \dots, \sum_I \frac{\partial u_I}{\partial x_n} dx_I\right). \quad (2.9)$$

So we have

$$\begin{aligned} \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} &= \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \left|\sum_I u_I dx_I\right|^{p\chi} dx\right)^{1/p\chi} \\ &= \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \left(\sum_I |u_I|^2\right)^{p\chi/2} dx\right)^{1/p\chi}. \end{aligned} \quad (2.10)$$

By the inequality

$$\left(\sum_{i=1}^n (a_i)^2\right)^{1/2} \leq \sum_{i=1}^n a_i \leq n^{1/2} \left(\sum_{i=1}^n (a_i)^2\right)^{1/2} \quad (2.11)$$

for any  $a_i \geq 0$ , and the Minkowski inequality, we have

$$\left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \left(\sum_I |u_I|^2\right)^{pX/2} dx\right)^{1/pX} \leq \sum_I \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u_I|^{pX} dx\right)^{1/pX}. \quad (2.12)$$

According to the Poincaré inequality, we have

$$\sum_I \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u_I|^{pX} dx\right)^{1/pX} \leq C_1(n, p) \text{diam}(\mathbb{D}) \sum_I \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |\nabla u_I|^p dx\right)^{1/p}. \quad (2.13)$$

Combining (2.10), (2.12), and (2.13), we can obtain

$$\left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{pX} dx\right)^{1/pX} \leq C_1(n, p) \text{diam}(\mathbb{D}) \sum_I \left(\frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |\nabla u_I|^p dx\right)^{1/p}. \quad (2.14)$$

By (2.9) we have

$$\begin{aligned} \|\nabla T du\|_{p, \mathbb{D}} &= \left\| \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) \right\|_{p, \mathbb{D}} \\ &= \left( \int_{\mathbb{D}} \left| \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) \right|^p dx \right)^{1/p} \\ &= \left( \int_{\mathbb{D}} \left( \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p} \\ &= \left( \int_{\mathbb{D}} \left( \sum_{i=1}^n \sum_I \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p} \\ &= \left( \int_{\mathbb{D}} \left( \sum_I \sum_{i=1}^n \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p}. \end{aligned} \quad (2.15)$$

Combining (2.11) and (2.15), then we have

$$\begin{aligned}
\|\nabla T du\|_{p,\mathbb{D}} &= \left( \int_{\mathbb{D}} \left( \sum_I \sum_{i=1}^n \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p} \\
&\geq (C_n^{(l+1)})^{-1/2} \left( \int_{\mathbb{D}} \left( \sum_I \left( \sum_{i=1}^n \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{1/2} \right)^p dx \right)^{1/p} \\
&\geq (C_n^{(l+1)})^{-1/2} \left( \int_{\mathbb{D}} \sum_I \left( \sum_{i=1}^n \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p} \\
&\geq (C_n^{(l+1)})^{-1/2} (C_n^{(l+1)})^{-(p-1)/p} \sum_I \left( \int_{\mathbb{D}} \left( \sum_{i=1}^n \left| \frac{\partial u_I}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{1/p} \\
&\geq (C_2(n,p,l))^{-1} \sum_I \left( \int_{\mathbb{D}} |\nabla u_I|^p dx \right)^{1/p},
\end{aligned} \tag{2.16}$$

where  $C_2(n,p,l) = (C_n^{(l+1)})^{1/2+(p-1)/p}$ . Now combining (2.14), (2.16), and (2.6), we can get

$$\begin{aligned}
\left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |u - u_{\mathbb{D}}|^p dx \right)^{1/p} &\leq C_1(n,p) \text{diam}(\mathbb{D}) \sum_I \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |\nabla u_I|^p dx \right)^{1/p} \\
&\leq C_1(n,p,l) C_2(n,p,l) \left( \frac{1}{|\mathbb{D}|} \right)^{1/p} \|\nabla T du\|_{p,\mathbb{D}} \\
&\leq C_3(n,p,l) \text{diam}(\mathbb{D}) \left( \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} |du|^p dx \right)^{1/p}.
\end{aligned} \tag{2.17}$$

□

### 3. The reverse Hölder inequality

In this section, we will prove the reverse Hölder inequality for the solution of the  $p$ -harmonic type system. Before we prove the reverse Hölder inequality, let us first see some lemmas.

**Lemma 3.1.** *If  $f, g \geq 0$  and for any nonnegative  $\eta \in C_0^\infty(\Omega)$ , it holds*

$$\int_{\Omega} \eta f dx \leq \int_{\Omega} g dx, \tag{3.1}$$

then for any  $h \geq 0$ :

$$\int_{\Omega} \eta f h dx \leq \int_{\Omega} g h dx. \tag{3.2}$$

*Proof.* Let  $\mu$  be a measure in  $X$ ,  $f$  be a nonnegative  $\mu$ -measurable function in a measure space  $X$ , using the standard representation theorem, we have

$$\int_X f^q d\mu = q \int_0^\infty t^{q-1} \mu(x : f(x) > t) dt \quad (3.3)$$

for any  $0 < t < q$ . Now, we let  $\mu(E) = \int_E \eta f dx$  and  $\nu(E) = \int_E g dx$  then, we can obtain

$$\int_\Omega \eta f h dx = \int_0^\infty \int_{h>t} \eta f dx dt \leq \int_0^\infty \int_{h>t} g dx dt = \int_\Omega g h dx. \quad (3.4)$$

So Lemma 3.1 is proved.  $\square$

**Lemma 3.2.** *If  $(u, v)$  is a pair of solution to the  $p$ -harmonic type system (1.9), then it holds*

$$\int_\Omega |\eta da|^p dx \leq C \int_\Omega |(a + du)d\eta|^p dx \quad (3.5)$$

for any nonnegative  $\eta \in C_0^\infty(\Omega)$  and where  $C = (C_n^{l+1})^p$ .

*Proof.* Since  $(u, v)$  is a pair of solutions to  $A(x, a + du) = b + d^*v$ , it is also the solution to  $A^{-1}(x, b + d^*v) = a + du$ , where  $A^{-1}(x, *)$  is the inverse  $A(x, *)$ . Now, we suppose that  $da = \sum_I \omega_I dx_I$  and let  $\varphi_1 = -\sum_I \eta \text{sign}(\omega_I) dx_I$ . By using  $\varphi = \varphi_1$  and  $d\varphi_1 = \sum_I \text{sign}(\omega_I) d\eta \wedge dx_I$  in (1.11), we can obtain

$$\int_\Omega \langle A^{-1}(x, b + d^*v), d\varphi_1 \rangle + \langle da, \varphi_1 \rangle dx \equiv 0. \quad (3.6)$$

That is,

$$\int_\Omega \left\langle da, \sum_I \eta \text{sign}(\omega_I) dx_I \right\rangle dx = \int_\Omega \left\langle A^{-1}(x, b + d^*v), -\sum_I \text{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx. \quad (3.7)$$

In other words,

$$\int_\Omega \sum_I \eta |\omega_I| dx = \int_\Omega \left\langle A^{-1}(x, b + d^*v), -\sum_I \text{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx. \quad (3.8)$$

By the elementary inequality

$$\left( \sum_{i=1}^n a_i^2 \right)^{1/2} \leq \sum_{i=1}^n |a_i|, \quad (3.9)$$



we have

$$\begin{aligned} \int_{\Omega} \eta |da| dx &= \int_{\Omega} \eta \left( \sum_I \omega_I^2 \right)^{1/2} dx \leq \int_{\Omega} \sum_I \eta |\omega_I| dx \\ &= \int_{\Omega} \left\langle A^{-1}(x, b + d^*v), -\sum_I \text{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx. \end{aligned} \quad (3.10)$$

Using the inequality

$$|\langle a, b \rangle| \leq |a| |b|, \quad (3.11)$$

(3.10) becomes

$$\begin{aligned} \int_{\Omega} \eta |da| dx &\leq \int_{\Omega} |A^{-1}(x, b + d^*v)| \left| \sum_I \text{sign}(\omega_I) d\eta \wedge dx_I \right| \\ &\leq \int_{\Omega} |A^{-1}(x, b + d^*v)| \sum_I |\text{sign}(\omega_I)| |d\eta| dx \\ &= C_n^{l+1} \int_{\Omega} |A^{-1}(x, b + d^*v)| |d\eta| dx \\ &= C_n^{l+1} \int_{\Omega} |a + du| |d\eta| dx, \end{aligned} \quad (3.12)$$

where  $I$  takes over all  $(l+1)$ -tuples for  $d\eta \in \Lambda^{l+1}$ , thus it has  $C_n^{l+1}$  numbers at most. Now we let  $f = |da|$  and  $g = C_n^{l+1} |a + du| |d\eta|$ . In the subset  $\{x : f\eta = g\}$ , we have

$$\int_{\{x: f\eta=g\}} |d\eta| dx \leq \int_{\{x: f\eta=g\}} |(a + du)d\eta|^p dx. \quad (3.13)$$

In the subset  $\{x : f\eta \neq g\}$ , let  $h = (|f\eta|^p - |g|^p) / (f\eta - g)$ , then we easily obtain  $h > 0$ . So by Lemma 3.1, we have

$$\int_{\{x: f\eta \neq g\}} h f \eta dx \leq \int_{\{x: f\eta \neq g\}} h g dx. \quad (3.14)$$

That is to say

$$\int_{\{x: f\eta \neq g\}} h (f\eta - g) dx \leq 0, \quad (3.15)$$

that is,

$$\int_{\{x:f\eta \neq g\}} |f\eta|^p dx \leq \int_{f\eta \neq g} |g|^p dx. \quad (3.16)$$

Combining (3.13) and (3.16), we have

$$\int_{\Omega} |f\eta|^p dx \leq \int_{\Omega} |g|^p dx, \quad (3.17)$$

that is,

$$\int_{\Omega} |\eta da|^p dx \leq \int_{\Omega} |C_n^{l+1}(a + du)d\eta|^p dx. \quad (3.18)$$

So Lemma 3.2 is proved.

The following lemma appears in [2]. □

**Lemma 3.3.** *Suppose that  $0 < q < p < s \leq \infty$ ,  $\xi \in \mathbb{R}$ , and that  $B = B(x_0, r)$  is a ball. If a nonnegative function  $v \in L^p(B, d\mu)$  satisfies*

$$\left( \frac{1}{\mu(\lambda B')} \int_{\lambda B'} v^s d\mu \right)^{1/s} \leq C(1 - \lambda)^{\xi} \left( \frac{1}{\mu(B')} \int_{B'} v^p d\mu \right)^{1/p} \quad (3.19)$$

for each ball  $B' = B(x_0, r')$  with  $r' \leq r$  and for all  $0 < \lambda < 1$ , then

$$\left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} v^s d\mu \right)^{1/s} \leq C(1 - \lambda)^{\xi/\theta} \left( \frac{1}{\mu(B)} \int_B v^q d\mu \right)^{1/q} \quad \forall 0 < \lambda < 1. \quad (3.20)$$

Here  $C > 0$  is a constant depending on  $p, q, s$  and  $\theta \in (0, 1)$  is such that  $1/p = \theta/q + (1 - \theta)/s$ .

The following lemma appears in [10].

**Lemma 3.4.** *Let  $(u, v)$  be a pair of solutions of the  $p$ -harmonic type system on domain  $\Omega$ , then we have a constant  $C$  only depending on  $K, n, p$ , and  $l$ , such that*

$$\|\eta du\|_{p,\Omega} \leq C(\|(u - c)d\eta\|_{p,\Omega} + \|\eta a\|_{p,\Omega}), \quad (3.21)$$

where  $c$  is any closed form (i.e.,  $dc = 0$ ) and for any  $\eta \in C_0^\infty(\Omega)$ . Also we have a constant  $C'$  only depending on  $K$ ,  $n$ ,  $q$ , such that

$$\|\eta d^*v\|_{q,\Omega} \leq C'(\|(v - c')d\eta\|_{q,\Omega} + \|\eta b\|_{q,\Omega}), \quad (3.22)$$

where  $c'$  is any coclosed form (i.e.,  $d^*c' = 0$ ) and  $q$  is the conjugate exponent of  $p$ .

**Theorem 3.5.** *If  $(u, v)$  is a pair of solutions to the  $p$ -harmonic type system, then there exists a constant  $C > 0$  dependent on  $K$ ,  $p$ ,  $n$ , and  $l$ , such that*

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q (|u - u_Q| + \|a\|_{\infty,Q})^s dx\right)^{1/s} &\leq C(1 - \sigma^{-1})^{-t\chi/p(\chi-1)} (\text{diam } Q + 1)^{\chi/(\chi-1)} \\ &\times \left(\frac{1}{|\sigma Q|} \int_{\sigma Q} (|u - u_{\sigma Q}| + \|a\|_{\infty,\sigma Q})^t dx\right)^{1/t} \end{aligned} \quad (3.23)$$

for any  $0 < s, t < \infty, \sigma > 1$  and all cubes with  $Q \subset \sigma Q \subset \Omega$ , where  $\chi > 1$  is the Poincaré constant.

*Proof.* Suppose that the center of  $Q$  is  $x_0$  and  $\text{diam } Q = r, 0 < \lambda = \sigma^{-1} < 1$ . Let

$$r_m = \lambda + (1 - \lambda)2^{-m}, \quad m = 0, 1, 2, \dots \quad (3.24)$$

Then  $r_m$  is decreasing and  $\lambda < r_m < 1$ . So we have  $u_Q|_{r_m Q} = u_{r_m Q}$ , for any  $m \in 0, 1, 2, \dots$ . Let  $\eta_m \in C_0^\infty(r_m Q)$  be a nonnegative function such that  $\eta_m = 1$  in  $r_{m+1}Q$ ,  $0 \leq \eta_m \leq 1$  in  $r_m Q - r_{m+1}Q$ .  $|d\eta_m| \leq (1 - \lambda)^{-1}2^m r^{-1}$ . Given any  $t \geq 0$  and let  $\omega_m = (|u - u_Q| + \|a\|_{\infty,Q})^{1+t/p} \eta_m$ , then we have

$$du_m = \left(1 + \frac{t}{p}\right) (|u - u_Q| + \|a\|_{\infty,Q})^{t/p} \eta_m d|u - u_Q| + (|u - u_Q| + \|a\|_{\infty,Q})^{1+t/p} d\eta_m. \quad (3.25)$$

By the Minkowski inequality, we can obtain

$$\begin{aligned} \left(\int_{r_m Q} |du_m|^p dx\right)^{1/p} &\leq \left(\int_{r_m Q} (|u - u_Q| + \|a\|_{\infty,Q})^{p+t} |d\eta_m|^p dx\right)^{1/p} \\ &+ \frac{(p+t)}{p} \left(\int_{r_m Q} |d|u - u_Q||^p (|u - u_Q| + \|a\|_{\infty,Q})^t |\eta_m|^p dx\right)^{1/p}. \end{aligned} \quad (3.26)$$

We assume that  $u - u_Q = \sum_I a_I dx_I$ , then we have  $|u - u_Q| = (\sum_I a_I^2)^{1/2}$ . If  $u - u_Q$  is zero, then we have  $|d|u - u_Q|| = 0 = |\nabla T(du)|$ . If  $u - u_Q$  is not equal zero, and the proof of (2.15) implies

that  $|\nabla T du| = (\sum_I \sum_{i=1}^n |\partial a_I / \partial x_i|^2)^{1/2}$

$$\begin{aligned}
 |d|u - u_Q|| &= |\nabla|u - u_Q|| = \left| \left( \frac{\partial|u - u_Q|}{\partial x_1}, \dots, \frac{\partial|u - u_Q|}{\partial x_n} \right) \right| \\
 &= \left( \sum_{i=1}^n \left| \frac{\partial|u - u_Q|}{\partial x_i} \right|^2 \right)^{1/2} = \left( \sum_{i=1}^n \left| \frac{\partial|u - u_Q|}{\partial x_i} \right|^2 \right)^{1/2} \\
 &= \left( \sum_{i=1}^n \left| \frac{\partial(\sum_I a_I^2)^{1/2}}{\partial x_i} \right|^2 \right)^{1/2} = \left( \sum_{i=1}^n \frac{1}{\sum_I a_I^2} \left| \sum_I a_I \frac{\partial a_I}{\partial x_i} \right|^2 \right)^{1/2} \quad (3.27) \\
 &\leq \left( \sum_{i=1}^n \frac{1}{\sum_I a_I^2} \sum_I a_I^2 \sum_I \left( \frac{\partial a_I}{\partial x_i} \right)^2 \right)^{1/2} = \left( \sum_{i=1}^n \sum_I \left( \frac{\partial a_I}{\partial x_i} \right)^2 \right)^{1/2} \\
 &= \left( \sum_{i=1}^n \sum_I \left| \frac{\partial a_I}{\partial x_i} \right|^2 \right)^{1/2} = |\nabla T(du)| = |\nabla(u - u_Q)|.
 \end{aligned}$$

So we have

$$|d|u - u_Q|| \leq |\nabla T(du)|. \quad (3.28)$$

For any  $\eta \in C_0^\infty(\Omega)$ , according to (2.6), we have

$$\|\eta \nabla T dw\|_{p, \mathbb{D}} \leq C(n, p) \max_{x \in \mathbb{D}}(\eta) \|dw\|_{p, \mathbb{D}}. \quad (3.29)$$

By the similar method as Lemma 3.1, we can prove the following inequality:

$$\begin{aligned}
 &\left( \int_{r_m Q} |d|u - u_Q||^p (|u - u_Q| + \|a\|_{\infty, Q})^t |\eta_m|^p dx \right)^{1/p} \\
 &\leq \left( \int_{r_m Q} |\eta_m|^p |\nabla T(du)|^p (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/p} \quad (3.30) \\
 &\leq C(n, p) \max_{x \in \mathbb{D}}(\eta_m^p) \left( \int_{r_m Q} |\eta_m|^p |du|^p (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/p}
 \end{aligned}$$

for any  $\eta \in C_0^\infty(\Omega)$ . By Lemma 3.1 and (3.21), we can obtain

$$\begin{aligned}
& \left( \int_{r_m Q} |\eta_m|^p |du|^p (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/p} \\
& \leq 2C \left( \int_{r_m Q} |\eta_m|^p |a|^p (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/p} \\
& \quad + 2C \left( \int_{r_m Q} |d\eta_m|^p (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p} \\
& \leq 2C \left( \int_{r_m Q} |\eta_m|^p \|a\|_{\infty, Q}^p (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/p} \\
& \quad + 2C \left( \int_{r_m Q} |d\eta_m|^p (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p} \\
& \leq 2C \left( \int_{r_m Q} |\eta_m|^p (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p} \\
& \quad + 2C \left( \int_{r_m Q} |d\eta_m|^p (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p}.
\end{aligned} \tag{3.31}$$

Combining (3.26), (3.30), and (3.31), by the values of  $\eta_m$ , we have

$$\left( \int_{r_m Q} |du_m|^p dx \right)^{1/p} \leq C_1(p+t)(1 + (1-\lambda)^{-1}2^m r^{-1}) \left( \int_{r_m Q} (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p}. \tag{3.32}$$

For  $\eta_m = 1$  in  $r_{m+1}Q$  and  $0 \leq \eta_m \leq 1$  in  $r_m Q - r_{m+1}Q$ , and as we have  $|r_m|/r_{m+1} = |\lambda + (1-\lambda)2^{-m}|/(\lambda + (1-\lambda)2^{-m-1}) \leq 2$ , so we have  $|r_m Q|/|r_{m+1}Q| \leq 2^n$ . By the Poincaré inequality, we know

$$\begin{aligned}
& \left( \frac{1}{|r_{m+1}Q|} \int_{r_{m+1}Q} (|u - u_Q| + \|a\|_{\infty, Q})^{X(p+t)} dx \right)^{1/pX} \\
& \leq \frac{1}{|r_{m+1}Q|} \int_{r_m Q} (\eta_m^{pX} |u - u_Q| + \|a\|_{\infty, Q})^{X(p+t)} dx)^{1/pX} \\
& \leq \left( \frac{1}{|r_{m+1}Q|} \int_{r_m Q} |u_m|^{pX} dx \right)^{1/pX} \\
& \leq 2^n \left( \frac{1}{|r_m Q|} \int_{r_m Q} |u_m|^{pX} dx \right)^{1/pX} \\
& \leq C_2 r_m r \left( \frac{1}{|r_m Q|} \int_{r_m Q} |du_m|^p dx \right)^{1/p} \\
& \leq C_3 r_m r (p+t)(1 + (1-\lambda)^{-1}2^m r^{-1}) \left( \int_{r_m Q} (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p} \\
& \leq C_3(p+t)(1-\lambda)^{-1}2^m(1+r) \left( \int_{r_m Q} (|u - u_Q| + \|a\|_{\infty, Q})^{p+t} dx \right)^{1/p}.
\end{aligned} \tag{3.33}$$

Now we set  $\kappa = p + t$ , then by computation, we obtain

$$\begin{aligned} \left( \frac{1}{|r_{m+1}Q|} \int_{r_{m+1}Q} (|u - u_Q| + \|a\|_{\infty, Q})^{\kappa \chi} dx \right)^{1/\kappa \chi} &\leq (C_3)^{p/\kappa} \kappa^{p/\kappa} (1 - \lambda)^{-p/\kappa} 2^{pm/\kappa} (r + 1)^{p/\kappa} \\ &\times \left( \frac{1}{|r_m Q|} \int_{r_m Q} (|u - u_Q| + \|a\|_{\infty, Q})^{\kappa} dx \right)^{1/\kappa}. \end{aligned} \quad (3.34)$$

Since this inequality holds for all  $\kappa > p$ , it can be applied with  $\kappa = \kappa_m = p\chi^m$ . And we can easily prove  $((1/|Q|) \int_Q |f|^p dx)^{1/p}$  is increasing with  $p$  and its limit is  $\text{ess sup}_Q |f|$ . So by iterating we arrive at the desired inequality for  $q = p$ :

$$\begin{aligned} &\text{ess sup}_{\lambda Q} (|u - u_Q| + \|a\|_{\infty, Q}) \\ &\leq \lim_{m \rightarrow \infty} \left( \frac{1}{|r_m Q|} \int_{r_m Q} (|u - u_Q| + \|a\|_{\infty, Q})^{\kappa_m \chi} dx \right)^{1/\kappa_m \chi} \\ &\leq C_4 ((1 - \lambda)^{-1} (r + 1))^{\sum_{i=0}^{\infty} \chi^{-m}} \prod_{m=0}^{\infty} 2^{m\chi^{-m}} \prod_{m=0}^{\infty} (p\chi^m)^{\chi^{-m}} \\ &\quad \times \left( \frac{1}{|Q|} \int_Q (|u - u_Q| + \|a\|_{\infty, Q})^p dx \right)^{1/p} \\ &\leq C_5 (1 - \lambda)^{-\chi/(\chi-1)} (r + 1)^{\chi/(\chi-1)} \left( \frac{1}{|Q|} \int_Q (|u - u_Q| + \|a\|_{\infty, Q})^p dx \right)^{1/p}. \end{aligned} \quad (3.35)$$

We can observe that the constants  $C_5$  and  $\chi$  are independent of  $x_0$  and  $r$  in (3.35), thus (3.35) holds not only in the cube  $Q = Q(x_0, r)$  but also in each ball inside  $Q$ . By Lemma (3.5) we can obtain

$$\begin{aligned} \left( \frac{1}{|\lambda Q|} \int_{\lambda Q} (|u - u_Q| + \|a\|_{\infty, Q})^s dx \right)^{1/s} &\leq C_5 (1 - \lambda)^{-\theta \chi/(\chi-1)} (r + 1)^{\chi/(\chi-1)} \\ &\quad \times \left( \frac{1}{|Q|} \int_Q (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/t} \end{aligned} \quad (3.36)$$

for any  $0 < t < p < s \leq \infty$ , where  $\theta = t(s - p)/p(s - t)$ . So we have  $\theta \leq t/p$  for any  $0 < t < p < s \leq \infty$ . Since  $((1/|Q|) \int_Q |f|^p dx)^{1/p}$  is increasing with  $p$ ,

$$\begin{aligned} \left( \frac{1}{|\lambda Q|} \int_{\lambda Q} (|u - u_Q| + \|a\|_{\infty, Q})^s dx \right)^{1/s} &\leq C_5 (1 - \lambda)^{-t\chi/p(\chi-1)} (r + 1)^{\chi/(\chi-1)} \\ &\quad \times \left( \frac{1}{|Q|} \int_Q (|u - u_Q| + \|a\|_{\infty, Q})^t dx \right)^{1/t} \end{aligned} \quad (3.37)$$

for any  $0 < s < \infty$  and  $1 < p < t < \infty$ . Combining (3.36) and (3.37), we have

$$\begin{aligned} \left( \frac{1}{|\Omega|} \int_{\Omega} (|u - u_Q| + \|a\|_{\infty, \Omega})^s dx \right)^{1/s} &\leq C_6 (1 - \lambda)^{-tX/p(X-1)} (r + 1)^{X/(X-1)} \\ &\times \left( \frac{1}{|\sigma Q|} \int_{\sigma Q} (|u - u_{\sigma Q}| + \|a\|_{\infty, \sigma Q})^t dx \right)^{1/t} \end{aligned} \quad (3.38)$$

for any  $0 < s, t < \infty$  and  $\sigma > 1$  such that  $\sigma Q \subset \Omega$ . Theorem 3.5 is proved.  $\square$

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