Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 325845, 6 pages
doi:10.1155/2008/325845

Research Article

Recurring Mean Inequality of Random Variables

Mingjin Wang

Department of Applied Mathematics, Jiangsu Polytechnic University, Changzhou, Jiangsu 213164, China

Correspondence should be addressed to Mingjin Wang, wang197913@126.com

Received 16 August 2007; Revised 25 February 2008; Accepted 9 May 2008

Recommended by Jewgeni Dshalalow

A multidimensional recurring mean inequality is shown. Furthermore, we prove some new inequalities, which can be considered to be the extensions of those established inequalities, including, for example, the Polya-Szegö and Kantorovich inequalities .

1. Introduction

The theory of means and their inequalities is fundamental to many fields including mathematics, statistics, physics, and economics.This is certainly true in the area of probability and statistics. There are large amounts of work available in the literature. For example, some useful results have been given by Shaked and Tong [1], Shaked and Shanthikumar [2], Shaked et al. [3], and Tong [4, 5]. Motivated by different concerns, there are numerous ways to introduce mean values. In probability and statistics, the most commonly used mean is expectation. In [6], the author proves the mean inequality of two random variables. The purpose of the present paper is to establish a recurring mean inequality, which generalizes the mean inequality of two random variables to n random variables. This result can, in turn, be extended to establish other new inequalities, which include generalizations of the Polya-Szegö and Kantorovich inequalities [7].

We begin by introducing some preliminary concepts and known results which can also be found in [6].

Definition 1.1. The *supremum and infimum of the random variable* ξ are defined as $\inf_x\{x : P(\xi \le x) = 1\}$ and $\sup_x\{x : P(\xi \ge x) = 1\}$, respectively, and denoted by $\sup \xi$ and $\inf \xi$.

Definition 1.2. If ξ is bounded, the *arithmetic mean of the random variable* ξ, $A(\xi)$, is given by

$$A(\xi) = \frac{\sup \xi + \inf \xi}{2}. \tag{1.1}$$

In addition, if $\inf \xi \geq 0$, one defines the *geometric mean of the random variable* ξ, $G(\xi)$, to be

$$G(\xi) = \sqrt{\sup \xi \cdot \inf \xi}. \tag{1.2}$$

Definition 1.3. If $\xi_1, \dots, \xi_n$ are bounded random variables, the *independent arithmetic mean of the product of random variables* $\xi_1, \dots, \xi_n$, $\overline{A}(\xi_1, \dots, \xi_n)$ is given by

$$\overline{A}(\xi_1, \dots, \xi_n) = \frac{\prod_{i=1}^{n} \sup \xi_i + \prod_{i=1}^{n} \inf \xi}{2}. \tag{1.3}$$

Definition 1.4. If $\xi_1, \dots, \xi_n$ are bounded random variables with $\inf \xi_i \geq 0$, $i = 1, \dots, n$, one defines the *independent geometric mean of the product of random variables* $\xi_1, \dots, \xi_n$ to be

$$\overline{G}(\xi_1, \dots, \xi_n) = \sqrt{\prod_{i=1}^{n} \sup \xi_i \inf \xi_i}. \tag{1.4}$$

Remark 1.5. If $\xi_1, \dots, \xi_n$ are independent, then

$$\overline{A}(\xi_1, \dots, \xi_n) = A\left(\prod_{i=1}^{n} \xi_i\right), \qquad \overline{G}(\xi_1, \dots, \xi_n) = G\left(\prod_{i=1}^{n} \xi_i\right). \tag{1.5}$$

The mean inequality of two random variables [6].

Theorem 1.6. *Let ξ and η be bounded random variables. If* $\inf \xi > 0$ *and* $\inf \eta > 0$, *then*

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leq \frac{\overline{A}^2(\xi, \eta)}{\overline{G}^2(\xi, \eta)}. \tag{1.6}$$

Equality holds if and only if

$$P\left\{\left(\frac{\xi}{\eta} = \frac{a}{B}\right) \cup \left(\frac{\xi}{\eta} = \frac{A}{b}\right)\right\} = 1, \qquad G(\eta^2)E\xi^2 = G(\xi^2)E\eta^2 \tag{1.7}$$

for $A = \sup \xi$, $B = \sup \eta$, $a = \inf \xi$, $b = \inf \eta$.

2. Main results

Our main results are given by the following theorem.

Theorem 2.1. *Suppose that $\xi_1, \dots, \xi_n, \xi_{n+1}$ are bounded random variables,* $\inf \xi_i > 0$, $i = 1, \dots, n+1$. *Let $\{U(n)\}$ be a sequence of real numbers. If*

$$\frac{\prod_{i=1}^{n} E\xi_i^2}{E^2\left(\prod_{i=1}^{n} \xi_i\right)} \leq U(n), \tag{2.1}$$

then

$$\frac{\prod_{i=1}^{n+1} E\xi_i^2}{E^2\left(\prod_{i=1}^{n+1} \xi_i\right)} \leq \frac{\overline{A}^2(\xi_1, \dots, \xi_{n+1})}{\overline{G}^2(\xi_1, \dots, \xi_{n+1})} U(n). \tag{2.2}$$

Proof. Let $A_i = \sup \xi_i$, $a_i = \inf \xi_i$, $i = 1, \dots, n+1$. We have

$$P\{(\xi_1 \cdots \xi_n A_{n+1} - a_1 \cdots a_n \xi_{n+1})(A_1 \cdots A_n \xi_{n+1} - \xi_1 \cdots \xi_n a_{n+1}) \geq 0\} = 1. \tag{2.3}$$

So

$$P\{(A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1})\xi_1 \cdots \xi_{n+1} \geq A_1 a_1 \cdots A_n a_n \xi_{n+1}^2 + A_{n+1} a_{n+1} \xi_1^2 \cdots \xi_n^2\} = 1, \tag{2.4}$$

which implies that

$$(A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1}) E(\xi_1 \cdots \xi_{n+1}) \geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} E(\xi_1^2 \cdots \xi_n^2). \tag{2.5}$$

Using the Jensen inequality [7] and assumption (2.1), we get

$$\begin{aligned}(A_1 \cdots A_{n+1} + a_1 \cdots a_{n+1}) E(\xi_1 \cdots \xi_{n+1}) &\geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} E^2(\xi_1 \cdots \xi_n) \\ &\geq A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) + A_{n+1} a_{n+1} \frac{E\xi_1^2 \cdots E\xi_n^2}{U(n)} \\ &\geq 2\left[A_1 a_1 \cdots A_n a_n E(\xi_{n+1}^2) A_{n+1} a_{n+1} \frac{E\xi_1^2 \cdots E\xi_n^2}{U(n)}\right]^{1/2}.\end{aligned} \tag{2.6}$$

Hence,

$$\left[\frac{\overline{G}^2(\xi_1, \dots, \xi_{n+1}) E\xi_1^2 \cdots E\xi_{n+1}^2}{U(n)}\right]^{1/2} \leq \overline{A}(\xi_1, \dots, \xi_{n+1}) E(\xi_1 \cdots \xi_{n+1}), \tag{2.7}$$

from which (2.2) follows. □

Combining this result with Theorem 1.6, the following recurring inequalities are immediate.

Corollary 2.2. *Let $\xi_1, \dots, \xi_n$ be bounded random variables. If $\inf \xi_i > 0$, $i = 1, \dots, n$, then*

$$\begin{aligned}\frac{E\xi_1^2 E\xi_2^2}{E^2(\xi_1 \xi_2)} &\leq \frac{\overline{A}^2(\xi_1, \xi_2)}{\overline{G}^2(\xi_1, \xi_2)}, \\ \frac{E\xi_1^2 E\xi_2^2 E\xi_3^2}{E^2(\xi_1 \xi_2 \xi_3)} &\leq \frac{\overline{A}^2(\xi_1, \xi_2, \xi_3)}{\overline{G}^2(\xi_1, \xi_2, \xi_3)} \frac{\overline{A}^2(\xi_1, \xi_2)}{\overline{G}^2(\xi_1, \xi_2)}, \\ &\vdots \\ \frac{\prod_{k=1}^n E\xi_k^2}{E^2(\prod_{k=1}^n \xi_k)} &\leq \prod_{k=2}^n \frac{\overline{A}^2(\xi_1, \dots \xi_k)}{\overline{G}^2(\xi_1, \dots \xi_k)}.\end{aligned} \tag{2.8}$$

3. Some applications

In this section, we exhibit some of the applications of the inequalities just obtained. We make use of the following known lemma which we state here without proof.

Lemma 3.1. *If* $0 < m_2 \le m_1 \le M_1 \le M_2$, *then*

$$\frac{(1/2)(m_1 + M_1)}{\sqrt{m_1 M_1}} \le \frac{(1/2)(m_2 + M_2)}{\sqrt{m_2 M_2}}. \tag{3.1}$$

Theorem 3.2 (the extensions of the inequality of *Polya-Szegö*). *Let* $a_{ij} > 0$, $a_i = \min_j a_{ij}$, $A_i = \max_j a_{ij}$, *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$. *Then,*

$$\prod_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2 \le \frac{m^{n-2}}{4^{n-1}} \prod_{k=2}^{n} \frac{(a_1 \cdots a_k + A_1 \cdots A_k)^2}{a_1 \cdots a_k A_1 \cdots A_k} \left(\sum_{j=1}^{m} \prod_{i=1}^{n} a_{ij} \right)^2. \tag{3.2}$$

Proof. This result is a consequence of inequality (2.8). Let ξ_1 have the distribution

$$P(\xi_1 = a_{1j}) = \frac{1}{m}, \quad j = 1, \ldots, m. \tag{3.3}$$

We define $n-1$ functions as follows:

$$f_i(a_{1j}) = a_{ij}, \quad i = 2, \ldots, n,\ j = 1, \ldots, m. \tag{3.4}$$

Let $\xi_i = f_i(\xi_1)$, $i = 2, \ldots, n$. Then,

$$E\xi_i^2 = \frac{1}{m} \sum_{j=1}^{m} a_{ij}^2, \quad i = 1, \ldots, n,$$
$$E(\xi_1 \cdots \xi_n) = \frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} a_{ij}, \tag{3.5}$$
$$\overline{A}(\xi_1, \ldots, \xi_k) = \frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k), \qquad \overline{G}(\xi_1, \ldots, \xi_k) = \sqrt{a_1 \cdots a_k A_1 \cdots A_k}.$$

Inequality (2.8) then becomes

$$\frac{\prod_{i=1}^{n} (1/m) \sum_{j=1}^{m} a_{ij}^2}{\left((1/m) \sum_{j=1}^{m} \prod_{i=1}^{n} a_{ij}\right)^2} \le \prod_{k=2}^{n} \frac{\left[(1/2)(a_1 \cdots a_k + A_1 \cdots A_k)\right]^2}{\left[\sqrt{a_1 \cdots a_k A_1 \cdots A_k}\right]^2}, \tag{3.6}$$

from which our result follows. □

Remark 3.3. For $n = 2$, we can get the inequality of *Polya-Szegö* [7]:

$$\left(\sum_{k=1}^{m} a_k^2 \right) \left(\sum_{k=1}^{m} b_k^2 \right) \le \frac{1}{4} \left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}} \right)^2 \left(\sum_{k=1}^{m} a_k b_k \right)^2, \tag{3.7}$$

where $a_k, b_k > 0$, $k = 1, \ldots, m$, $a = \min a_k$, $A = \max a_k$, $b = \min b_k$, and $B = \max b_k$.

Theorem 3.4 (the extensions of *Kantorovich's* inequality). *Let A be an $m \times m$ positive Hermitian matrix. Denote by λ_1 and λ_m the maximum and minimum eigenvalues of A, respectively. For real $\beta_1, \ldots, \beta_n$ and $\beta = \beta_1 + \cdots + \beta_n$, and any vector $0 \neq x \in \mathbb{R}^m$,the following inequality is satisfied:*

$$\frac{\prod_{i=1}^{n} x^* A^{\beta_i} x}{\left(x^* A^{\beta/2} x\right)^2} \leq \frac{\left(x^* x\right)^{n-2}}{4^{n-1}} \prod_{k=2}^{n} \frac{\left[l_1 \cdots l_k + L_1 \cdots L_k\right]^2}{l_1 \cdots l_k L_1 \cdots L_k}, \tag{3.8}$$

where

$$l_i = \begin{cases} \lambda_m^{\beta_i/2}, & \beta_i \geq 0, \\ \lambda_1^{\beta_i/2}, & \beta_i < 0, \end{cases} \qquad L_i = \begin{cases} \lambda_1^{\beta_i/2}, & \beta_i \geq 0, \\ \lambda_m^{\beta_i/2}, & \beta_i < 0, \end{cases} \quad i = 1, \ldots, n. \tag{3.9}$$

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_m$ be eigenvalues of A and let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. There is a Hermitian matrix U that satisfies

$$A = U^* \Lambda U. \tag{3.10}$$

Let

$$y = Ux = \left(y_1, y_2, \ldots, y_m\right)^T, \qquad p_i = \frac{\left|y_i\right|^2}{\sum_{i=1}^{m} \left|y_i\right|^2}, \quad i = 1, \ldots, m. \tag{3.11}$$

Then,

$$\begin{aligned} \frac{\prod_{i=1}^{n} x^* A^{\beta_i} x}{\left(x^* A^{\beta/2} x\right)^2} &= \frac{\prod_{i=1}^{n} x^* U^* \Lambda^{\beta_i} U x}{\left(x^* U^* \Lambda^{\beta/2} U x\right)^2} \\ &= \frac{\prod_{i=1}^{n} y^* \Lambda^{\beta_i} y}{\left(y^* \Lambda^{\beta/2} y\right)^2} \\ &= \frac{\left(y^* y\right)^{n-2} \prod_{i=1}^{n} \sum_{k=1}^{m} \lambda_k^{\beta_i} p_k}{\left(\sum_{k=1}^{m} \lambda_k^{\beta/2} p_k\right)^2} \\ &= \frac{\left(x^* x\right)^{n-2} \prod_{i=1}^{n} \sum_{k=1}^{m} \lambda_k^{\beta_i} p_k}{\left(\sum_{k=1}^{m} \lambda_k^{\beta/2} p_k\right)^2}. \end{aligned} \tag{3.12}$$

What remains to show is that

$$\frac{\prod_{i=1}^{n} \sum_{k=1}^{m} \lambda_k^{\beta_i} p_k}{\left(\sum_{k=1}^{m} \lambda_k^{\beta/2} p_k\right)^2} \leq \frac{1}{4^{n-1}} \prod_{k=2}^{n} \frac{\left[l_1 \cdots l_k + L_1 \cdots L_k\right]^2}{l_1 \cdots l_k L_1 \cdots L_k}, \quad \forall p_i \geq 0, \sum_{i=1}^{m} p_i = 1. \tag{3.13}$$

We define the random variable ζ, and assign $P(\zeta = \lambda_i) = p_i$, $i = 1, \ldots, m$. Suppose $\xi_i = \zeta^{\beta_i/2}$, $i = 1, \ldots, n$. Notice that λ_1 and λ_n are the upper and lower bounds of the random variable ζ, so l_i and L_i are the lower and upper bounds of ξ_i. According to Lemma 3.1, we know that

$$\frac{\overline{A}^2\left(\xi_1, \ldots, \xi_k\right)}{\overline{G}^2\left(\xi_1, \ldots, \xi_k\right)} \leq \frac{\left[(1/2)\left(l_1 \cdots l_k + L_1 \cdots L_k\right)\right]^2}{\left[\sqrt{l_1 \cdots l_k L_1 \cdots L_k}\right]^2}. \tag{3.14}$$

Noticing that

$$E(\xi_1 \cdots \xi_n) = E\zeta^{\beta/2} = \sum_{k=1}^{m} \lambda_k^{\beta/2} p_k, \tag{3.15}$$

we can use inequality (2.8) to express inequality (3.13) as

$$\frac{E\xi_1^2 \cdots E\xi_n^2}{E^2(\xi_1 \cdots \xi_n)} \leq \prod_{k=2}^{n} \frac{\left[(1/2)(l_1 \cdots l_k + L_1 \cdots L_k)\right]^2}{\left[\sqrt{l_1 \cdots l_k L_1 \cdots L_k}\right]^2}. \tag{3.16}$$

□

Remark 3.5. If $n = 2$, $\beta_1 = 1$, and $\beta_2 = -1$, this inequality takes the form

$$\frac{x^* A x x^* A^{-1} x}{(x^* x)^2} \leq \frac{(\lambda_1 + \lambda_m)^2}{4\lambda_1 \lambda_m} \tag{3.17}$$

which is *Kantorovich's* inequality [7].

References

[1] M. Shaked and Y. L. Tong, "Inequalities for probability contents of convex sets via geometric average," *Journal of Multivariate Analysis*, vol. 24, no. 2, pp. 330–340, 1988.

[2] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and Their Applications*, Probability and Mathematical Statistics, Academic Press, Boston, Mass, USA, 1994.

[3] M. Shaked, J. G. Shanthikumar, and Y. L. Tong, "Parametric Schur convexity and arrangement monotonicity properties of partial sums," *Journal of Multivariate Analysis*, vol. 53, no. 2, pp. 293–310, 1995.

[4] Y. L. Tong, "Some recent developments on majorization inequalities in probability and statistics," *Linear Algebra and Its Applications*, vol. 199, supplement 1, pp. 69–90, 1994.

[5] Y. L. Tong, "Relationship between stochastic inequalities and some classical mathematical inequalities," *Journal of Inequalities and Applications*, vol. 1, no. 1, pp. 85–98, 1997.

[6] M. Wang, "The mean inequality of random variables," *Mathematical Inequalities & Applications*, vol. 5, no. 4, pp. 755–763, 2002.

[7] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1952.