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*Research Article*

# The Radius of Starlikeness of the Certain Classes of p-Valent Functions Defined by Multiplier Transformations

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The aim of this paper is to give the radius of starlikeness of the certain classes of $p$-valent functions defined by multiplier transformations. The results are obtained by using techniques of Robertson (1953,1963) which was used by Bernardi (1970), Libera (1971), Livingstone (1966), and Goel (1972).



## 1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclasses of $\mathcal{H}$ consisting of the functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} \cdots$. Let $\mathcal{A}(p, n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D}$. In particular, we set

$$\mathcal{A}(p, 1) := \mathcal{A}_p, \qquad \mathcal{A}(1, 1) := \mathcal{A} = \mathcal{A}_1. \tag{1.2}$$

If $f(z)$ and $g(z)$ are analytic in $\mathbb{D}$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{D}). \tag{1.3}$$

If there exists a Schwarz function $w(z)$ which is analytic in $\mathbb{D}$ with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathbb{D}$.

For two analytic functions $f(z)$ and $F(z)$, we say that $F(z)$ is superordinate to $f(z)$ if $f(z)$ is subordinate to $F(z)$.

For integer $n \geq 1$, let $\Omega(n)$ denote the class of functions $w(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$, and $w(z) = z^n \phi(z)$ for all $z \in \mathbb{D}$, where $\phi(z)$ is regular and analytic in $\mathbb{D}$ and satisfies $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Also, let $\mathcal{P}\{(p,n)$ denote the class of functions $p(z) = p + \sum_{k=n}^{\infty} p_k z^k$ which are regular in $\mathbb{D}$ and satisfy the conditions $p(0) = p$, $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. We note that if $p(z) \in \mathcal{P}(p,n)$, then

$$p(z) = p\frac{1 - w(z)}{1 + w(z)} = \frac{1 - z^n \phi(z)}{1 + z^n \phi(z)} \tag{1.4}$$

for some functions $w(z) \in \Omega(n)$ and every $z \in \mathbb{D}$.

*Definition 1.1.* Let $f(z) \in \mathcal{A}(p,n)$ for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \geq 0$, $l > 0$, one defines the multiplier transformations $\mathcal{I}_p(m,\lambda,l)$ on $\mathcal{A}(p,n)$ by the following infinite series:

$$\mathcal{I}_p(m,\lambda,l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left(\frac{p + \lambda(k-p) + l}{p + l}\right)^m a_k z^k. \tag{1.5}$$

It follows that

$$\begin{gathered} \mathcal{I}_p(0,\lambda,l)f(z) = f(z), \\ (p+l)\mathcal{I}_p(2,\lambda,l)f(z) = \big(p(1-\lambda) + l\big)\mathcal{I}_p(1,\lambda,l)f(z) + \lambda z\big(\mathcal{I}_p(1,\lambda,l)f(z)\big)', \\ \mathcal{I}_p(m_1,\lambda,l)\big(\mathcal{I}_p(m_2,\lambda,l)f(z)\big) = \mathcal{I}_p(m_2,\lambda,l)\big(\mathcal{I}_p(m_1,\lambda,l)f(z)\big) \end{gathered} \tag{1.6}$$

for all integers $m_1$, $m_2$.

*Remark 1.2.* This multiplier transformation was introduced by Cătaş [1]. For $p = 1$, $l = 0$, $\lambda \geq 0$, the operator $\mathfrak{D}_\lambda^m := \mathcal{I}_1(m,\lambda,0)$ was introduced by Al-Oboudi [2] which reduces to the Sălăgean differantial operator [3]. For $\lambda = 1$, the operator $\mathcal{I}_l^m := \mathcal{I}_1(m,1,l)$ was studied recently by Cho and Srivastava [4] and Cho and Kim [5]. The operator $\mathcal{I}_m := \mathcal{I}_1(m,1,1)$ was studied by Uralegaddi and Somanatha [6] and the operator $\mathcal{I}_p(m,l) := \mathcal{I}_p(m,1,l)$ was investigated recently by Sivaprasad Kumar et al. [7].

*Definition 1.3* (see [1]). Let $\varphi(z)$ be analytic in $\mathbb{D}$ and $\varphi(0) = 1$. A function $f(z) \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{A}_p(m,\lambda,l,n;\varphi)$ if it satisfies the following subordination:

$$\frac{\mathcal{I}_p(m+1,\lambda,l)f(z)}{\mathcal{I}_p(m,\lambda,l)f(z)} \prec \varphi(z) \quad (z \in \mathbb{D}). \tag{1.7}$$

*Definition 1.4.* The radius of starlikeness of the class $\mathcal{A}_p(m,\lambda,l,n,\varphi)$ is defined by the following.

For each $f(z) \in \mathcal{A}_p(m,\lambda,l,n;\varphi)$, let $r(f)$ be the supremum of all numbers $r$ such that $f(\mathbb{D}_r)$ is starlike with respect to the origin. Then the radius of starlikeness for $\mathcal{A}_p(m,\lambda,l,n;\varphi)$ is

$$r_{\mathrm{st}}\big(\mathcal{A}_p(m,\lambda,l,n;\varphi)\big) = \inf_{f \in \mathcal{A}_p(m,\lambda,l,n,\varphi)} r(f). \tag{1.8}$$

**Theorem 1.5.** *Let $f(z) \in \mathcal{A}(p,n)$ and $\lambda > 0$, then $f(z)$ belongs to the class $\mathcal{A}_p(m,\lambda,l,n;\chi)$ if and only if $F(z)$, defined by*

$$F(z) = \frac{p+l}{\lambda z^{(p(1-\lambda)+l)/\lambda}} \int_0^z \zeta^{(p(1-\lambda)+l)/\lambda - 1} f(\zeta) d\zeta = z^p + \sum_{k=p+n}^{\infty} \left( \frac{p+l}{p+l+(k-p)\lambda} \right) a_k z^k, \tag{1.9}$$

*belongs to the class $\mathcal{A}_p(m+1,\lambda,l,n;\chi)$.*

This theorem was proved by Cătaş [1].

## 2. Main result

**Theorem 2.1.** *The radius of starlikeness of the class $\mathcal{A}_p(m,\lambda,l,n,\phi)$ is*

$$r_{\rm st} = \left( \frac{p+l}{\lambda(p+n) + \sqrt{\lambda^2(p+n)^2 + (p+l)(p+l-2\lambda p)}} \right)^{1/n}. \tag{2.1}$$

*This radius is sharp because the extremal function is*

$$f_*(z) = \frac{\lambda}{p+l} \frac{z^p(c+p+(c-p)z^n)}{(1+z^n)^{2p/n+1}}, \qquad c = \frac{p(1-\lambda)+l}{\lambda}. \tag{2.2}$$

*Proof.* If we take $c = (p(1-\lambda)+l)/\lambda$, then the function $F(z)$ in Theorem 1.5 can be written in the form

$$F(z) = \frac{p+l}{\lambda z^c} \int_0^z \zeta^{c-1} f(\zeta) d\zeta. \tag{2.3}$$

If we take the logarithmic derivative from (2.3) and after simple calculations, we get

$$z\frac{F'(z)}{F(z)} = \frac{z^c f(z) - c\int_0^z \zeta^{c-1} f(\zeta) d\zeta}{\int_0^z \zeta^{c-1} f(\zeta) d\zeta}. \tag{2.4}$$

Since $F(z)$ is starlike, hence there exists a function $w(z) \in \Omega(n)$ such that

$$z\frac{F'(z)}{F(z)} = \frac{z^c f(z) - c\int_0^z \zeta^{c-1} f(\zeta) d\zeta}{\int_0^z \zeta^{c-1} f(\zeta) d\zeta} = p\frac{1-w(z)}{1+w(z)}. \tag{2.5}$$

Solving for $f(z)$,

$$f(z) = \frac{(c+p)+(c-p)w(z)}{(1+w(z))z^c} \int_0^z \zeta^{c-1} f(\zeta) d\zeta. \tag{2.6}$$

Taking the logarithmic derivative from (2.6), we get

$$z\frac{f'(z)}{f(z)} = p\frac{1-w(z)}{1+w(z)} + (b-1)\frac{zw'(z)}{(1+w(z))(1+bw(z))}, \tag{2.7}$$

where $b = (c-p)/(c+p)$. To show that $f(z)$ is starlike in $|z| < r_0$, we must show that

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0 \tag{2.8}$$

for $|z| < r_0$. This condition is equivalent to

$$(1-b)\operatorname{Re}\left(\frac{zw'(z)}{(1+w(z))(1+bw(z))}\right) \le \operatorname{Re}\left(p\frac{1-w(z)}{1+w(z)}\right). \tag{2.9}$$

On the other hand, we have the following relations:

$$\operatorname{Re}\left(p\frac{1-w(z)}{1+w(z)}\right) = p\frac{1-|w(z)|^2}{|1+w(z)|^2},$$

$$(1-b)\operatorname{Re}\left(\frac{zw'(z)}{(1+w(z))(1+bw(z))}\right) \le \frac{(1-b)|zw'(z)|}{|1+w(z)|\,|1+bw(z)|}, \tag{2.10}$$

$$|zw'(z)| \le \frac{n|z|^n}{1-|z|^{2n}}\left(1-|w(z)|^2\right)$$

(Golusin inequality, [8]). Therefore, the inequality (2.9) will be satisfied if

$$\frac{n(1-b)|z|^n}{|1+w(z)|\,|1+bw(z)|}\frac{1-|w(z)|^2}{1-|z|^{2n}} \le p\frac{1-|w(z)|^2}{|1+w(z)|^2}. \tag{2.11}$$

Simplifying and writing $|z| = r$, we obtain

$$\frac{n(1-b)r^n}{1-r^{2n}} \le p\left|\frac{1+bw(z)}{1+w(z)}\right|. \tag{2.12}$$

Since $|w(z)| \le |z|^n = r^n$, $p|(1+bw(z))/(1+w(z))| \ge p((1+br^n)/(1+r^n))$ so that (2.12) will be satisfied if

$$\frac{n(1-b)r^n}{1-r^{2n}} < p\frac{1+br^n}{1+r^n}. \tag{2.13}$$

The inequality (2.13) can be written in the following form:

$$p-(1-b)(p+n)r^n - bpr^{2n} > 0, \tag{2.14}$$

which gives the required root $r_0$ of the theorem.

To see that the result is sharp, consider the function $F(z) = z^p/(1+z^n)^{2p/n}$. For this function, we have

$$f_*(z) = \frac{\lambda}{p+l}\frac{z^p((c+p)+(c-p)z^n)}{(1+z^n)^{2p/n+1}},$$

$$z\frac{f_*'(z)}{f_*(z)} = \frac{p-(1-b)(p+n)z^n - pbz^{2n}}{(1+z^n)^{2p/n+1}}. \tag{2.15}$$

So that $z(f_*'(z)/f_*(z)) = 0$ for $|z| = r_0$. Thus, $f(z)$ is not starlike in any circle $|z| < r$ if $r > r_0$. □

*Remark 2.2.* If we give special values to $m, \lambda, l, n$, we obtain the radius of starlikeness for the corresponding integral operators.

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