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Research Article

On Meromorphic Harmonic Functions with Respect to k-Symmetric Points

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In our previous work in this journal in 2008, we introduced the generalized derivative operator $\mathfrak{D}_m^j$ for $f \in \mathcal{S}_{\mathcal{H}}$. In this paper, we introduce a class of meromorphic harmonic function with respect to k-symmetric points defined by $\mathfrak{D}_m^j$. Coefficient bounds, distortion theorems, extreme points, convolution conditions, and convex combinations for the functions belonging to this class are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we write $f = h + \overline{g}$ where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'| > |g'|$ in D (see [1]). Hengartner and Schober [2] investigated functions harmonic in the exterior of the unit disk $\widetilde{\mathbb{U}} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z|, \tag{1.1}$$

where $h(z)$ and $g(z)$ are defined by

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \qquad g(z) = \beta\overline{z} + \sum_{n=1}^{\infty} b_n z^{-n}, \tag{1.2}$$

for $0 \leq |\beta| < |\alpha|$, $A \in \mathbb{C}$ and $z \in \widetilde{\mathbb{U}}$.

For $z \in \mathbb{U} \setminus \{0\}$, let $\mathcal{M}_{\mathcal{H}}$ denote the class of functions:

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \tag{1.3}$$

which are harmonic in the punctured unit disk $\mathbb{U} \setminus \{0\}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{U} \setminus \{0\}$ and $\mathbb{U}$, respectively, and $h(z)$ has a simple pole at the origin with residue 1 here.

In [3], the authors introduced the operator $\mathfrak{D}_m^j$ for $f \in \mathcal{S}_{\mathcal{H}}$ which is the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For more details about the operator $\mathfrak{D}_m^j$, see [4].

Now, we define $\mathfrak{D}_m^j$ for $f = h + \overline{g}$ given by (1.3) as

$$\mathfrak{D}_m^j f(z) = \mathfrak{D}_m^j h(z) + (-1)^j \overline{\mathfrak{D}_m^j g(z)}, \quad (j, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U} \setminus \{0\}), \tag{1.4}$$

where

$$\begin{aligned} \mathfrak{D}_m^j h(z) &= \frac{(-1)^j}{z} + \sum_{n=1}^{\infty} n^j C(m,n) a_n z^n, \\ \mathfrak{D}_m^j g(z) &= \sum_{n=1}^{\infty} n^j C(m,n) b_n z^n, \\ C(m,n) &= \binom{n+m-1}{m} = \frac{(n+m-1)!}{m!(n-1)!}. \end{aligned} \tag{1.5}$$

A function $f \in \mathcal{M}_{\mathcal{H}}$ is said to be in the subclass $\mathcal{MS}_{\mathcal{H}}^*$ of meromorphically harmonic starlike functions in $\mathbb{U} \setminus \{0\}$ if it satisfies the condition

$$\operatorname{Re}\left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > 0, \quad (z \in \mathbb{U} \setminus \{0\}). \tag{1.6}$$

Note that the class of harmonic meromorphic starlike functions has been studied by Jahangiri and Silverman [5], and Jahangiri [6].

Now, we have the following definition.

Definition 1.1. For $j, m \in \mathbb{N}_0$, $0 \le \alpha < 1$ and $k \ge 1$, let $\mathcal{MHS}_s^{(k)}(j, m, \alpha)$ denote the class of meromorphic harmonic functions f of the form (1.3) such that

$$\operatorname{Re}\left\{ -\frac{\mathfrak{D}_m^{j+1} f(z)}{\mathfrak{D}_m^j f_k(z)} \right\} > 0, \quad (z \in \mathbb{U} \setminus \{0\}), \tag{1.7}$$

where

$$\mathfrak{D}_m^j f_k(z) = \mathfrak{D}_m^j h_k + (-1)^j \overline{\mathfrak{D}_m^j g_k} \quad (j, m \in \mathbb{N}_0, k \ge 1), \tag{1.8}$$

$$h_k(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty} a_n \Phi_n z^n, \qquad g_k(z) = \sum_{n=1}^{\infty} \Phi_n z^n, \tag{1.9}$$

$$\Phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad \left(k \ge 1; \varepsilon = \exp\left(\frac{2\pi i}{k}\right) \right). \tag{1.10}$$

For more details about harmonic functions with respect to k-symmetric points, see papers [7, 8] given by the authors.

Also, note that $\mathcal{MHS}_s^{(2)}(j,0,\alpha) \subset MHS_S^*(n,\alpha)$ was introduced by Bostancı and Öztürk [9].

Finally, let $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ denote the subclass of $\mathcal{MHS}_s^{(k)}(j,m,\alpha)$ consist of harmonic functions $f_j = h_j + \overline{g_j}$ such that h_j and g_j are of the form

$$h_j(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}|a_n|z^n, \qquad g_j(z) = (-1)^j\sum_{n=1}^{\infty}|b_n|z^n. \tag{1.11}$$

Also, let $f_{k_j} = h_{k_j} + \overline{g_{k_j}}$ where h_{k_j} and g_{k_j} are of the form

$$h_{k_j}(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}\Phi_n|a_n|z^n, \qquad g_{k_j}(z) = (-1)^j\sum_{n=1}^{\infty}\Phi_n|b_n|z^n, \tag{1.12}$$

where Φ_n is given by (1.10).

In this paper, we will give a sufficient condition for functions $f = h + \overline{g}$, where h and g given by (1.3) to be in the class $\mathcal{MHS}_s^{(k)}(j,m,\alpha)$. Indeed, it is shown that this coefficient condition is also necessary for functions to be in the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Also, we obtain distortion bounds and characterize the extreme points for functions in $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Convolution and closure theorems are also obtained.

2. Coefficient bounds

First, we prove a sufficient coefficient bound.

Theorem 2.1. *Let $f = h + \overline{g}$ be of the form* (1.3) *and $f_k = h_k + \overline{g_k}$ where h_k and g_k are given by* (1.9). *If*

$$\begin{aligned}&\sum_{n=1}^{\infty}[[(n-1)k+1+\alpha]|a_{(n-1)k+1}| + [(n-1)k+1-\alpha]|b_{(n-1)k+1}|]\Omega_m^j(n,k)\\ &\qquad + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n^{j+1}C(m,n)[|a_n|+|b_n|] \le 1-\alpha,\end{aligned} \tag{2.1}$$

where $j,m \in \mathbb{N}_0$, $0 \le \alpha < 1$, $k \ge 1$ and $\Omega_m^j(n,k) = ((n-1)k+1)^jC(m,nk+1)$, then f is harmonic univalent, sense preserving in $\mathbb{U}\setminus\{0\}$ and $f \in \mathcal{MHS}_s^{(k)}(j,m,\alpha)$.

Proof. For $0 < |z_1| \le |z_2| < 1$, we have

$$\begin{aligned}&|f(z_1)-f(z_2)|\\ &\quad\ge |h(z_1)-h(z_2)| - |g(z_1)-g(z_2)|\\ &\quad\ge \frac{|z_1-z_2|}{|z_1||z_2|} - |z_1-z_2|\sum_{n=1}^{\infty}(|a_n|+|b_n|)|z_1^{n-1}+\cdots+z_2^{n-1}|\\ &\quad> \frac{|z_1-z_2|}{|z_1||z_2|}\left[1-|z_2|^2\sum_{n=1}^{\infty}n(|a_n|+|b_n|)\right]\end{aligned}$$

$$> \frac{|z_1 - z_2|}{|z_1||z_2|}\left[1 - |z_2|^2\left(\sum_{n=1}^{\infty} n(|a_n| + |b_n|) + \sum_{n=1}^{\infty}[(n-1)k+1](|a_{(n-1)k+1}| + |b_{(n-1)k+1}|)\right)\right]$$

$$> \frac{|z_1 - z_2|}{|z_1||z_2|}\left[1 - \sum_{n=1}^{\infty}[(n-1)k+1+\alpha|a_{(n-1)k+1}| - (n-1)k+1-\alpha|b_{(n-1)k+1}|\right]\Omega_m^j(n,k)$$

$$- \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n^{j+1}C(m,n)[|a_n| + |b_n|]. \tag{2.2}$$

This last expression is nonnegative by (2.1), and so f is univalent in $\mathbb{U} \setminus \{0\}$. To show that f is sense preserving in $\mathbb{U} \setminus \{0\}$, we need to show that $|h'(z)| \geq |g'(z)|$ in $\mathbb{U} \setminus \{0\}$. We have

$$\begin{aligned}
|h'(z)| &\geq \frac{1}{|z|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\
&= \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n|r^{n-1} > 1 - \sum_{n=1}^{\infty} n|a_n| \\
&\geq 1 - \sum_{n=1}^{\infty}[(n-1)k+1+\alpha]|a_{(n-1)k+1}|]\Omega_m^j(n,k) - \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n^{j+1}C(m,n)|a_n| \\
&\geq \sum_{n=1}^{\infty}[(n-1)k+1-\alpha]|b_{(n-1)k+1}|]\Omega_m^j(n,k) + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n^{j+1}C(m,n)|b_n| \\
&\geq \sum_{n=1}^{\infty} 2n|b_{2n}| + \sum_{n=1}^{\infty}(2n-1)|b_{2n-1}| \\
&> \sum_{n=1}^{\infty} n|b_n|r^{n-1} = \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.
\end{aligned} \tag{2.3}$$

Now, we will show that $f \in \mathcal{MHS}_s^{(k)}(j,m,\alpha)$. According to (1.4) and (1.7), for $0 \leq \alpha < 1$, we have

$$\operatorname{Re}\left\{-\frac{\mathfrak{D}_m^{j+1}f(z)}{\mathfrak{D}_m^j f_k(z)}\right\} = \operatorname{Re}\left\{-\frac{\mathfrak{D}_m^{j+1}h(z) - (-1)^j\overline{\mathfrak{D}_m^{j+1}g(z)}}{\mathfrak{D}_m^j h_k(z) + (-1)^j\overline{\mathfrak{D}_m^j g_k(z)}}\right\} \geq \alpha. \tag{2.4}$$

Using the fact that $\operatorname{Re}\{w\} \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|1 - \alpha - \frac{\mathfrak{D}_m^{j+1}f(z)}{\mathfrak{D}_m^j f_k(z)}\right| \geq \left|1 + \alpha + \frac{\mathfrak{D}_m^{j+1}f(z)}{\mathfrak{D}_m^j f_k(z)}\right|, \tag{2.5}$$

which is equivalent to

$$\left|\mathfrak{D}_m^{j+1}f(z) - (1-\alpha)\mathfrak{D}_m^j f_k(z)\right| - \left|\mathfrak{D}_m^{j+1}f(z) + (1+\alpha)\mathfrak{D}_m^j f_k(z)\right| \geq 0. \tag{2.6}$$

Substituting $\mathfrak{D}_m^j f(z)$, $\mathfrak{D}_m^{j+1} f(z)$, and $\mathfrak{D}_m^j f_k(z)$ in (2.6) yields

$$\begin{aligned}
&\left|\mathfrak{D}_m^{j+1}h(z)-(-1)^j\overline{D_m^{j+1}g(z)}-(1-\alpha)\left[\mathfrak{D}_m^j h_k(z)+(-1)^j\overline{\mathfrak{D}_m^j g_k(z)}\right]\right| \\
&\quad-\left|\mathfrak{D}_m^{j+1}h(z)-(-1)^j\overline{D_m^{j+1}g(z)}+(1+\alpha)\left[\mathfrak{D}_m^j h_k(z)+(-1)^j\overline{\mathfrak{D}_m^j g_k(z)}\right]\right| \\
&\quad=\left|\frac{(-1)^j}{z}-\sum_{n=1}^{\infty}n^{j+1}C(m,n)a_nz^n+(-1)^j\sum_{n=1}^{\infty}n^{j+1}C(m,n)\overline{b_nz^n}\right. \\
&\qquad\left.+(1-\alpha)\left[\frac{(-1)^j}{z}+\sum_{n=1}^{\infty}n^jC(m,n)\Phi_na_nz^n+(-1)^j\sum_{n=1}^{\infty}n^jC(m,n)\Phi_n\overline{b_nz^n}\right]\right| \\
&\quad-\left|\frac{(-1)^j}{z}-\sum_{n=1}^{\infty}n^{j+1}C(m,n)a_nz^n+(-1)^j\sum_{n=1}^{\infty}n^{j+1}C(m,n)\overline{b_nz^n}\right. \\
&\qquad\left.-(1+\alpha)\left[\frac{(-1)^j}{z}+\sum_{n=1}^{\infty}n^jC(m,n)\Phi_na_nz^n+(-1)^j\sum_{n=1}^{\infty}n^jC(m,n)\Phi_n\overline{b_nz^n}\right]\right| \\
&\quad=\left|\frac{(2-\alpha)(-1)^j}{z}-\sum_{n=1}^{\infty}n^jC(m,n)[n-(1-\alpha)\Phi_n]a_nz^n+(-1)^j\sum_{n=1}^{\infty}n^jC(m,n)[n+(1-\alpha)\Phi_n]\overline{b_nz^n}\right| \\
&\quad-\left|\frac{\alpha(-1)^j}{z}-\sum_{n=1}^{\infty}n^jC(m,n)[n+(1+\alpha)\Phi_n]a_nz^n+(-1)^j\sum_{n=1}^{\infty}n^jC(m,n)[n-(1+\alpha)\Phi_n]\overline{b_nz^n}\right| \\
&\quad\ge\frac{(2-\alpha)}{|z|}-\sum_{n=1}^{\infty}n^jC(m,n)[n-(1-\alpha)\Phi_n]|a_n||z^n|-\sum_{n=1}^{\infty}n^jC(m,n)[n+(1-\alpha)\Phi_n]|b_n||z^n| \\
&\quad-\frac{\alpha}{|z|}-\sum_{n=1}^{\infty}n^jC(m,n)[n+(1+\alpha)\Phi_n]|a_n||z^n|-\sum_{n=1}^{\infty}n^jC(m,n)[n-(1+\alpha)\Phi_n]|b_n||z^n| \\
&\quad=\frac{2(1-\alpha)}{|z|}\left\{1-\sum_{n=1}^{\infty}\frac{n^jC(m,n)[n+\alpha\Phi_n]}{1-\alpha}|a_n|\left|z^{n+1}\right|-\sum_{n=1}^{\infty}\frac{n^jC(m,n)[n-\alpha\Phi_n]}{1-\alpha}|b_n|\left|z^{n+1}\right|\right\} \\
&\quad\ge 2(1-\alpha)\left\{1-\sum_{n=1}^{\infty}\frac{n^jC(m,n)[n+\alpha\Phi_n]}{1-\alpha}|a_n|-\sum_{n=1}^{\infty}\frac{n^jC(m,n)[n-\alpha\Phi_n]}{1-\alpha}|b_n|\right\}.
\end{aligned} \tag{2.7}$$

From the definition of Φ_n, we know that

$$\Phi_n=\begin{cases}1, & n=lk+1,\\ 0, & n\neq lk+1,\end{cases}\qquad (n\ge 2,\ k,l\ge 1). \tag{2.8}$$

Substituting (2.8) in (2.7), then (2.7) is equivalent to

$$\begin{aligned}
&\left|\mathfrak{D}_m^{j+1} f(z) - (1-\alpha)\mathfrak{D}_m^j f_k(z)\right| - \left|\mathfrak{D}_m^{j+1} f(z) + (1+\alpha)\mathfrak{D}_m^j f_k(z)\right| \\
&\quad \geq 2(1-\alpha)\Bigg\{1 - \sum_{n=1}^{\infty} \frac{(nk+1)^j C(m,nk+1)[nk+1+\alpha]}{1-\alpha}|a_{nk+1}| \\
&\qquad - \sum_{n=1}^{\infty} \frac{(nk+1)^j C(m,nk+1)[nk+1-\alpha]}{1-\alpha}|b_{nk+1}| - \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} \frac{n^j C(m,n)}{1-\alpha}|a_n| \\
&\qquad - \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} \frac{n^j C(m,n)}{1-\alpha}|b_n| - \frac{1+\alpha}{1-\alpha}|a_1| - |b_1|\Bigg\} \\
&\quad = 2(1-\alpha)\Bigg\{1 - \sum_{n=1}^{\infty}\left[\frac{(n-1)k+1+\alpha}{1-\alpha}|a_{(n-1)k+1}| - \frac{(n-1)k+1-\alpha}{1-\alpha}|b_{(n-1)k+1}|\right]\Omega_m^j(n,k) \\
&\qquad - \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} \frac{n^{j+1} C(m,n)}{1-\alpha}[|a_n| + |b_n|]\Bigg\} \geq 0, \quad \text{by (2.6).}
\end{aligned} \tag{2.9}$$

Thus, this completes the proof of the theorem. □

We next show that condition (2.1) is also necessary for functions in $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$.

Theorem 2.2. *Let* $f_j = h_j + \overline{g_j}$, *where* h_j *and* g_j *are given by* (1.11), *and* $f_{k_j} = h_{k_j} + \overline{g_{k_j}}$ *where* h_{k_j} *and* g_{k_j} *are given by* (1.12). *Then,* $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$, *if and only if the inequality* (2.1) *holds for the coefficient of* $f_j = h_j + \overline{g_j}$ *and* $f_{k_j} = h_{k_j} + \overline{g_{k_j}}$.

Proof. In view of Theorem 2.1, we need only to show that $f_j \notin \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ if condition (2.1) does not hold. We note that for $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$, then by (1.7) the condition (2.4) must be satisfied for all values of z in $\mathbb{U} \setminus \{0\}$. Substituting for h_j, g_j, h_{k_j}, and g_{k_j} given by (1.11) and (1.12), respectively, in (2.4) and choosing $0 < z = r < 1$, we are required to have $\operatorname{Re}\{\Psi(z)/\Upsilon(z)\} \geq 0$, where

$$\begin{aligned}
\Psi(z) &= -\mathfrak{D}_m^{j+1} h_j(z) + (-1)^n \overline{\mathfrak{D}_m^{j+1} g_j(z)} - \alpha\mathfrak{D}_m^j h_{k_j}(z) - \alpha(-1)^j \overline{\mathfrak{D}_m^j g_{k_j}(z)} \\
&= \frac{1-\alpha}{z} - \sum_{n=1}^{\infty} n^j C(m,n)(n+\alpha\Phi_n)|a_n| z^n + \sum_{n=1}^{\infty} n^j C(m,n)(n-\alpha\Phi_n)|b_n|\overline{z^n}, \\
\Upsilon(z) &= \mathfrak{D}_m^j h_{k_j}(z) + (-1)^j \overline{\mathfrak{D}_m^j g_{k_j}(z)} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} n^j C(m,n)\Phi_n |a_n| z^n + \sum_{n=1}^{\infty} n^j C(m,n)\Phi_n |b_n|\overline{z^n}.
\end{aligned} \tag{2.10}$$

Then, the required condition $\operatorname{Re}\{\Psi(z)/\Upsilon(z)\} \geq 0$ is equivalent to

$$\frac{((1-\alpha)/z) - \sum_{n=1}^{\infty} n^j C(m,n)(n+\alpha\Phi_n)|a_n| r^n + \sum_{n=1}^{\infty} n^j C(m,n)(n-\alpha\Phi_n)|b_n| r^n}{1/z + \sum_{n=1}^{\infty} n^j C(m,n)\Phi_n |a_n| r^n + \sum_{n=1}^{\infty} n^j C(m,n)\Phi_n |b_n| r^n}. \tag{2.11}$$

By using (2.8), and if condition (2.1) does not hold, then the numerator of (2.11) is negative for r sufficiently close to 1. Thus, there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.11) is negative. This contradicts the required condition for $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ and so the proof is complete. □

3. Distortion bounds and extreme points

In this section, we will obtain distortion bounds for functions $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ and also provide extreme points for the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$.

Theorem 3.1. *If $f_j = h_j + \overline{g_j} \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ and $0 < |z| = r < 1$, then*

$$\frac{1}{r} - \frac{1-\alpha}{2^j(m+1)(2-\alpha)} r \le |f_j(z)| \le \frac{1}{r} + \frac{1-\alpha}{2^j(m+1)(2-\alpha)} r. \tag{3.1}$$

Proof. We will prove the left side of the inequality. The argument for the right side of the inequality is similar to the left side, and thus the details will be omitted. Let $f_j = h_j + \overline{g_j} \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Taking the absolute value of f, we obtain

$$\begin{aligned} |f_j| &= \left| \frac{(-1)^j}{z} + \sum_{n=1}^{\infty} a_n z^n + (-1)^n \sum_{n=1}^{\infty} \overline{b_n z^n} \right| \\ &\ge \frac{1}{r} - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \\ &\ge \frac{1}{r} - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r \\ &\ge \frac{1}{r} - \frac{1-\alpha}{2^j(m+1)(2-\alpha\Phi_2)} \sum_{n=1}^{\infty} \frac{2^j(m+1)(2-\alpha\Phi_2)}{1-\alpha} (|a_n| + |b_n|) r \\ &\ge \frac{1}{r} - \frac{1-\alpha}{2^j(m+1)(2-\alpha)} \sum_{n=1}^{\infty} \left(\frac{n^j C(m,n)(n+\alpha\Phi_n)}{1-\alpha} |a_n| + \frac{n^j C(m,n)(n-\alpha\Phi_n)}{1-\alpha} |b_n| \right) r \\ &\ge \frac{1}{r} - \frac{1-\alpha}{2^j(m+1)(2-\alpha)} r, \quad \text{by (2.7)}. \end{aligned} \tag{3.2}$$

The bounds given in Theorem 3.1 hold for functions $f_j = h + \overline{g_j}$ of the form (1.11). And it is also discovered that the bounds hold for functions of the form (1.3), if the coefficient condition (2.1) is satisfied. □

The following covering result follows from the left-hand side of the inequality in Theorem 3.1.

Corollary 3.2. *If $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$, then*

$$f_j(\mathbb{U} \setminus \{0\}) \subset \left\{ w : |w| < \frac{2^j(m+1)(2-\alpha) - (1-\alpha)}{2^j(m+1)(2-\alpha)} \right\}. \tag{3.3}$$

Next, we determine the extreme points of closed convex hulls of $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ denoted by $\operatorname{clco}\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$.

Theorem 3.3. *Let $f_j = h_j + \overline{g_j}$ where h_j and g_j are given by (1.11). Then, $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ if and only if*

$$f_{j,n}(z) = \sum_{n=0}^{\infty}\left(x_n h_{j_n}(z) + y_n g_{j_n}(z)\right), \tag{3.4}$$

where $h_{j,0} = g_{j,0}(z) = (-1)^j/z$, $h_{j,n}(z) = (-1)^j/z + ((1-\alpha)/n^jC(m,n)(n+\alpha\Phi_n))z^n$ $(n = 1,2,3,\ldots)$, $g_{j,n}(z) = (-1)^j/z + (-1)^j((1-\alpha)/n^jC(m,n)(n-\alpha\Phi_n))\overline{z}^k$ $(n = 1,2,3,\ldots)$, $\sum_{n=0}^{\infty}(x_n + y_n) = 1$, $x_n \geq 0$, $y_n \geq 0$. In particular, the extreme points of $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ are $\{h_{j,n}\}$ and $\{g_{j,n}\}$.

Proof. For functions $f_j = h_j + \overline{g_j}$, where h_j and g_j are given by (1.11), we have

$$\begin{aligned} f_{j,n}(z) &= \sum_{n=0}^{\infty}\left(x_n h_{j,n}(z) + y_n g_{j,n}(z)\right) \\ &= \sum_{n=0}^{\infty}(x_n + y_n)\frac{(-1)^j}{z} + \sum_{n=1}^{\infty}\frac{1-\alpha}{n^jC(m,n)(n+\alpha\Phi_n)}x_n z^n \\ &\quad + (-1)^j\sum_{n=1}^{\infty}\frac{1-\alpha}{n^jC(m,n)(n-\alpha\Phi_n)}\overline{y_n z^k}. \end{aligned} \tag{3.5}$$

Now, the first part of the proof is complete, and Theorem 2.2 gives

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{1-\alpha}{n^jC(m,n)(n+\alpha\Phi_n)}\,\frac{n^jC(m,n)(n+\alpha\Phi_n)}{1-\alpha}x_n \\ &\quad + \sum_{n=1}^{\infty}\frac{1-\alpha}{n^jC(m,n)(n-\alpha\Phi_n)}\,\frac{n^jC(m,n)(n-\alpha\Phi_n)}{1-\alpha}y_n \\ &= \sum_{n=0}^{\infty}x_n + y_n - (x_0 + y_0) = 1 - (x_0 + y_0) \leq 1. \end{aligned} \tag{3.6}$$

Conversely, suppose that $f_j \in \operatorname{clco}\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. For $n = 1,2,3,\ldots$, set

$$\begin{aligned} x_n &= \frac{n^jC(m,n)(n+\alpha\Phi_n)}{1-\alpha}|a_n| \quad 0 \leq x_n \leq 1, \\ y_n &= \frac{n^jC(m,n)(n-\alpha\Phi_n)}{1-\alpha}|b_n| \quad 0 \leq y_n \leq 1, \end{aligned} \tag{3.7}$$

$x_0 = 1 - \sum_{n=1}^{\infty}(x_n + y_n)$. Therefore, f can be written as

$$\begin{aligned} f_{j,n}(z) &= \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}|a_n|z^n + (-1)^j\sum_{n=1}^{\infty}|b_n|\overline{z}^n \\ &= \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}\frac{(1-\alpha)x_n}{n^jC(m,n)(n+\alpha\Phi_n)}z^n + (-1)^j\sum_{n=1}^{\infty}\frac{(1-\alpha)y_n}{n^jC(m,n)(n-\alpha\Phi_n)}\overline{z}^n \\ &= \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}\left(h_{j,n}(z) - \frac{(-1)^j}{z}\right)x_n + \sum_{n=1}^{\infty}\left(g_{j,n}(z) - \frac{(-1)^j}{z}\right)y_n \qquad (3.8) \\ &= \sum_{n=1}^{\infty}h_{j,n}(z)x_n + \sum_{n=1}^{\infty}g_{j,n}(z)y_n + \frac{(-1)^j}{z}\left(1 - \sum_{n=1}^{\infty}x_n - \sum_{n=1}^{\infty}y_n\right) \\ &= \sum_{n=0}^{\infty}(h_{j,n}(z)x_n + g_{j,n}(z)y_n), \text{ as required.} \end{aligned}$$

□

4. Convolution and convex combination

In this section, we show that the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ is invariant under convolution and convex combination of its member.

For harmonic functions $f_j(z) = (-1)^j/z + \sum_{n=1}^{\infty}|a_n|z^n + (-1)^j\sum_{n=1}^{\infty}|b_n|\overline{z}^n$ and $F_j(z) = (-1)^j/z + \sum_{n=1}^{\infty}|A_n|z^n + (-1)^j\sum_{n=1}^{\infty}|B_n|\overline{z}^n$, the convolution of f_j and F_j is given by

$$(f_j * F_j)(z) = f_j(z) * F_j(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}|a_n||A_n|z^n + (-1)^j\sum_{n=1}^{\infty}|b_n||B_n|\overline{z}^n. \qquad (4.1)$$

Theorem 4.1. *For $0 \le \beta \le \alpha < 1$, let $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ and $F_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\beta)$. Then, $f_j * F_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha) \subset \overline{\mathcal{MHS}_s^{(k)}}(j,m,\beta)$.*

Proof. We wish to show that the coefficients of $f_j * F_j$ satisfy the required condition given in Theorem 2.2. For $F_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\beta)$, we note that $|A_n| \le 1$ and $|B_n| \le 1$. Now, for the convolution function $f_j * F_j$, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{n^jC(m,n)(n+\beta\Phi_n)}{1-\beta}|a_n||A_n| + \sum_{n=1}^{\infty}\frac{n^jC(m,n)(n-\beta\Phi_n)}{1-\beta}|b_n||B_n| \\ &\quad\le \sum_{n=1}^{\infty}\frac{n^jC(m,n)(n+\beta\Phi_n)}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n^jC(m,n)(n-\beta\Phi_n)}{1-\beta}|b_n| \qquad (4.2) \\ &\quad\le \sum_{n=1}^{\infty}\frac{n^jC(m,n)(n-\alpha\Phi_n)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n^jC(m,n)(n-\alpha\Phi_n)}{1-\alpha}|b_n| \le 1, \end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Therefore $f_j * F_j \in \overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha) \subset \overline{\mathcal{MHS}_s^{(k)}}(j,m,\beta)$. □

We now examine the convex combination of $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$.

Let the functions $f_{j,t}$ be defined, for $t = 1,2,\ldots,\rho$, by

$$f_{j,t}(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}|a_{n,t}|z^n + (-1)^j\sum_{n=1}^{\infty}|b_{n,t}|\overline{z}^n. \tag{4.3}$$

Theorem 4.2. *Let the functions $f_{j,t}$ defined by (4.3) be in the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ for every $t = 1,2,\ldots,\rho$. Then, the functions $\xi_t(z)$ defined by*

$$\xi_t(z) = \sum_{t=1}^{\rho} c_t f_{j_n}(z), \quad (0 \le c_t \le 1), \tag{4.4}$$

are also in the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$, where $\sum_{t=1}^{\rho} c_t = 1$.

Proof. According to the definition of ξ_t, we can write

$$\xi_t(z) = \frac{(-1)^j}{z} + \sum_{n=1}^{\infty}\left(\sum_{t=1}^{\rho} c_t a_{n,t}\right)z^n + (-1)^j\sum_{n=1}^{\infty}\left(\sum_{t=1}^{\rho} c_t b_{n,t}\right)\overline{z}^n. \tag{4.5}$$

Further, since $f_{j,t}(z)$ are in $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ for every $(t = 1,2,\ldots,\rho)$. Then by (2.7), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left[(n+\alpha\Phi_n)\left(\sum_{t=1}^{\rho} c_t|a_{n,t}|\right) + (n-\alpha\Phi_n)\left(\sum_{t=1}^{\rho} c_t|b_{n,t}|\right)\right]n^j C(m,n)\right\} \\
&\quad = \sum_{t=1}^{\rho} c_t\left(\sum_{n=1}^{\infty}[(n+\alpha\Phi_n)|a_{n,t}| + (n-\alpha\Phi_n)|b_{n,t}|]n^j C(m,n)\right) \\
&\quad \le \sum_{t=1}^{\rho} c_t(1-\alpha) \le 1-\alpha.
\end{aligned} \tag{4.6}$$

Hence, the theorem follows. □

Corollary 4.3. *The class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$ is close under convex linear combination.*

Proof. Let the functions $f_{j,t}(z)$ $(t = 1,2)$ defined by (4.3) be in the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Then, the function $\varphi(z)$ defined by

$$\varphi(z) = \mu f_{j,1}(z) + (1-\mu)f_{j,2}(z), \quad (0 \le \mu \le 1), \tag{4.7}$$

is in the class $\overline{\mathcal{MHS}_s^{(k)}}(j,m,\alpha)$. Also, by taking $\rho = 2$, $\xi_1 = \mu$, and $\xi_2 = (1-\mu)$ in Theorem 4.2, we have the corollary. □

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