

Research Article

On Multivariate Grüss Inequalities

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The main purpose of the present paper is to establish some new Grüss integral inequalities in n independent variables. Our results in special cases yield some of the recent results on Pachpatte's, Mitrinović's, and Ostrowski's inequalities, and provide new estimates on such types of inequalities.

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1. Introduction

The well-known Grüss integral inequality [1] can be stated as follows (see [2, page 296]):

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leq \frac{1}{4}(P-p)(Q-q), \quad (1.1)$$

provided that f and g are two integrable functions on $[a, b]$ such that $p \leq f(x) \leq P$, $q \leq g(x) \leq Q$, for all $x \in [a, b]$, where p, P, q, Q are real constants.

Many generalizations, extensions, and variants of this inequality (1.1) have appeared in the literature, see [1–8] and the references given therein. The main purpose of the present paper is to establish several multivariate Grüss integral inequalities. Our results provide a new estimates on such type of inequalities.

2. Main results

In what follows, \mathbb{R} denotes the set of real numbers, \mathbb{R}^n the n -dimensional Euclidean space. Let $D = \{(x_1, \dots, x_n) : a_i \leq x_i \leq b_i (i = 1, \dots, n)\}$. For a function $u(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote the

first-order partial derivatives by $(\partial u(x)/\partial x_i)$ ($i = 1, \dots, n$) and $\int_D u(x)dx$ the n -fold integral $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \dots, x_n) dx_1 \cdots dx_n$.

For continuous functions $p(x), q(x) : D \rightarrow \mathbb{R}$ which are differentiable on D and $w(x) : D \rightarrow [0, \infty)$ an integrable function such that $\int_D w(x)dx > 0$, we use the notation

$$G[w, p, q]_n := \int_D w(x)p(x)q(x)dx - \frac{(\int_D w(x)p(x)dx)(\int_D w(x)q(x)dx)}{\int_D w(x)dx} \quad (2.1)$$

to simplify the details of presentation. Furthermore, if $\sum_{i=1}^n (\partial h/\partial x_i) \cdot (x_i - y_i) \neq 0$, for any $x, y \in D$, we use the abbreviations

$$G[\Sigma_c, w, g, h]_n$$

$$:= \frac{\int_D (\int_D (\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i) / \sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)) w(y) dy) g(x) h(x) w(x) dx}{\int_D w(y) dy} - \frac{\int_D (\int_D (\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i) / \sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)) w(y) h(y) dy) g(x) w(x) dx}{\int_D w(y) dy},$$

$$G[\Sigma_d, w, f, h]_n$$

$$:= \frac{\int_D (\int_D (\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i) / \sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)) w(y) dy) f(x) h(x) w(x) dx}{\int_D w(y) dy} - \frac{\int_D (\int_D (\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i) / \sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)) w(y) h(y) dy) f(x) w(x) dx}{\int_D w(y) dy}. \quad (2.2)$$

It is clear that if

$$\frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} = 1, \quad (2.3)$$

then $G[\Sigma_c, w, g, h]_n = G[w, g, h]_n$ and $G[\Sigma_d, w, f, h]_n = G[w, f, h]_n$.

Our main results are established in the following theorems.

Theorem 2.1. Let $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous functions on D . If f, g are differentiable on the interior of D and $w(x) : D \rightarrow [0, \infty)$ an integrable function such that $\int_D w(x)dx > 0$. If $\sum_{i=1}^n (\partial h/\partial x_i) \cdot (x_i - y_i) \neq 0$, for every $x \in D$, then

$$|G[w, f, g]_n| \leq \frac{1}{2} \{ |G[\Sigma_c, w, g, h]_n| + |G[\Sigma_d, w, f, h]_n| \}. \quad (2.4)$$

Proof. Let $x, y \in D$ with $x \neq y$. From the n -dimensional version of the Cauchy's mean value theorem (see [9]), we have

$$f(x) - f(y) = \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} (h(x) - h(y)),$$

$$g(x) - g(y) = \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} (h(x) - h(y)), \quad (2.5)$$

where $c = (y_1 + \alpha(x_1 - y_1), \dots, y_n + \alpha(x_n - y_n))$ and $d = (y_1 + \beta(x_1 - y_1), \dots, y_n + \beta(x_n - y_n))$ ($0 < \alpha < 1, 0 < \beta < 1$). Multiplying both sides of (2.5) by $g(x)$ and $f(x)$, respectively, and adding, we get

$$\begin{aligned} 2f(x)g(x) - g(x)f(y) - f(x)g(y) &= \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} (g(x)h(x) - g(x)h(y)) \\ &\quad + \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} (f(x)h(x) - f(x)h(y)). \end{aligned} \quad (2.6)$$

Multiplying both sides of (2.6) by $w(y)$ and integrating the resulting identity with respect to y over D , we have

$$\begin{aligned} &2 \left(\int_D w(y) dy \right) f(x)g(x) - g(x) \int_D w(y)f(y) dy - f(x) \int_D w(y)g(y) dy \\ &= \left(\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \right) g(x)h(x) \\ &\quad - g(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y) dy \\ &\quad + \left(\int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \right) f(x)h(x) \\ &\quad - f(x) \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y) dy. \end{aligned} \quad (2.7)$$

Next, multiplying both sides of (2.7) by $w(x)$ and integrating the resulting identity with respect to x over D , we have

$$\begin{aligned} &2 \left(\int_D w(y) dy \right) \int_D w(x)f(x)g(x) dx - \left(\int_D w(x)g(x) dx \right) \left(\int_D w(y)f(y) dy \right) \\ &\quad - \left(\int_D w(x)f(x) dx \right) \left(\int_D w(y)g(y) dy \right) \\ &= \int_D \left(\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \right) g(x)h(x)w(x) dx \\ &\quad - \int_D \left(\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y) dy \right) g(x)w(x) dx \\ &\quad + \int_D \left(\int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \right) f(x)h(x)w(x) dx \\ &\quad - \int_D \left(\int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y) dy \right) f(x)w(x) dx. \end{aligned} \quad (2.8)$$

From (2.8), it is easy to observe that

$$|G[w, f, g]_n| \leq \frac{1}{2} \{ |G[\Sigma_c, w, g, h]_n| + |G[\Sigma_d, w, f, h]_n| \}. \quad (2.9)$$

The proof is complete. \square

Remark 2.2. When $n = 1$, we have $D = [a_1, b_1]$ and

$$\frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{f'(c)}{h'(c)}, \quad \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} = \frac{g'(d)}{h'(d)}, \quad (2.10)$$

where $c = y_1 + \alpha(x_1 - y_1)$, $0 < \alpha < 1$, and $d = y_1 + \beta(x_1 - y_1)$, $0 < \beta < 1$. In this case, (2.4) reduces to the following inequality which was given by Pachpatte in [8]:

$$|G[w, f, g]| \leq \frac{1}{2} \left\{ \left\| \frac{f'}{h'} \right\|_{\infty} |G[w, g, h]| + \left\| \frac{g'}{h'} \right\|_{\infty} |G[w, f, h]| \right\}, \quad (2.11)$$

where $f(x), g(x), h(x) : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b) , $w : [a, b] \rightarrow [0, \infty)$ is an integrable function with $\int_a^b w(x) dx > 0$, $\|\cdot\|_{\infty}$ is the sup norm, and

$$G[w, p, q] := \int_a^b w(x)p(x)q(x) dx - \frac{(\int_a^b w(x)p(x) dx)(\int_a^b w(x)q(x) dx)}{\int_a^b w(x) dx}. \quad (2.12)$$

Remark 2.3. If

$$\frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} = 1, \quad (2.13)$$

we have $G[\Sigma_c, w, g, h]_n = G[w, f, g]_n$ and $G[\Sigma_d, w, f, h]_n = G[w, f, h]_n$. In this case, (2.4) reduces to the following interesting inequality:

$$|G[w, f, g]_n| \leq \frac{1}{2} \{ |G[w, g, h]_n| + |G[w, f, h]_n| \}. \quad (2.14)$$

Remark 2.4. If $h(x) = \sum_{i=1}^n x_i$, then (2.5) reduces to the following results, respectively,

$$f(x) - f(y) = \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i), \quad g(x) - g(y) = \sum_{i=1}^n \frac{\partial g(d)}{\partial x_i} (x_i - y_i). \quad (2.15)$$

Furthermore, letting $w(y) = 1$, (2.7) reduces to

$$\begin{aligned} & \left| f(x)g(x) - \frac{1}{2M}g(x) \int_D f(y)dy - \frac{1}{2M}f(x) \int_D g(y)dy \right| \\ & \leq \frac{1}{2M} \sum_{i=1}^n \left(|g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} \right) E_i(x), \end{aligned} \quad (2.16)$$

where $M = \text{mes}D := \prod_{i=1}^n (b_i - a_i)$, and $E_i(x) := \int_D |x_i - y_i| dy$. This is precisely a new inequality established by Pachpatte in [6]. If, in addition, $g(x) \equiv 1$, then inequality (2.16) reduces to the inequality established by Mitrinović in [2], which is in turn a generalization of the well-known Ostrowski inequality.

Theorem 2.5. *Let f, g, h be as in Theorem 2.1. Then,*

$$\begin{aligned}
|G[w, f, g]_n| &\leq \frac{1}{\left(\int_D w(y) dy\right)^2} \\
&\times \left| \int_D \left(w(x) h^2(x) \int_D \frac{\sum_{i=1}^n (\partial f(c) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c) / \partial x_i)(x_i - y_i)} w(y) dy \right. \right. \\
&\quad \cdot \left. \int_D \frac{\sum_{i=1}^n (\partial g(d) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d) / \partial x_i)(x_i - y_i)} w(y) dy \right) dx \\
&\quad + \int_D \left(w(x) \int_D \frac{\sum_{i=1}^n (\partial f(c) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c) / \partial x_i)(x_i - y_i)} w(y) h(y) dy \right. \\
&\quad \cdot \left. \int_D \frac{\sum_{i=1}^n (\partial g(d) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d) / \partial x_i)(x_i - y_i)} w(y) h(y) dy \right) dx \\
&\quad - 2 \int_D \left(w(x) h(x) \int_D \frac{\sum_{i=1}^n (\partial f(c) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c) / \partial x_i)(x_i - y_i)} w(y) dy \right. \\
&\quad \cdot \left. \int_D \frac{\sum_{i=1}^n (\partial g(d) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d) / \partial x_i)(x_i - y_i)} w(y) h(y) dy \right) dx \Big|. \tag{2.17}
\end{aligned}$$

Proof. Multiplying both sides of (2.5) by $w(y)$ and integrate the resulting identities with respect to y on D , we get, respectively,

$$\begin{aligned}
&\left(\int_D w(y) dy \right) f(x) - \int_D w(y) f(y) dy \\
&= h(x) \int_D \frac{\sum_{i=1}^n (\partial f(c) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c) / \partial x_i)(x_i - y_i)} w(y) dy - \int_D \frac{\sum_{i=1}^n (\partial f(c) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c) / \partial x_i)(x_i - y_i)} w(y) h(y) dy, \\
&\left(\int_D w(y) dy \right) g(x) - \int_D w(y) g(y) dy \\
&= h(x) \int_D \frac{\sum_{i=1}^n (\partial g(d) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d) / \partial x_i)(x_i - y_i)} w(y) dy - \int_D \frac{\sum_{i=1}^n (\partial g(d) / \partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d) / \partial x_i)(x_i - y_i)} w(y) h(y) dy. \tag{2.18}
\end{aligned}$$

Multiplying the left sides and right sides of (2.18), we get

$$\begin{aligned}
& \left(\int_D w(y) dy \right)^2 f(x)g(x) - \left(\int_D w(y) dy \right) f(x) \left(\int_D w(y)g(y) dy \right) \\
& - \left(\int_D w(y) dy \right) g(x) \left(\int_D w(y)f(y) dy \right) + \left(\int_D w(y)f(y) dy \right) \left(\int_D w(y)g(y) dy \right) \\
& = h^2(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \\
& + \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \\
& - h(x) \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \\
& - h(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) h(y) dy.
\end{aligned} \tag{2.19}$$

Multiplying both sides of (2.19) by $w(x)$ and integrating the resulting identity with respect to x over D , we get

$$\begin{aligned}
& \left(\int_D w(y) dy \right)^2 \int_D w(x) f(x)g(x) dx - \left(\int_D w(y) dy \right) \left(\int_D w(x) f(x) dx \right) \left(\int_D w(y)g(y) dy \right) \\
& - \left(\int_D w(y) dy \right) \left(\int_D w(x)g(x) dx \right) \left(\int_D w(y)f(y) dy \right) \\
& + \left(\int_D w(x) dx \right) \left(\int_D w(y)f(y) dy \right) \left(\int_D w(y)g(y) dy \right) \\
& = \int_D \left(w(x) h^2(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \right) dx \\
& + \int_D \left(w(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \right) dx \\
& - \int_D \left(w(x) h(x) \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \right) dx \\
& - \int_D \left(w(x) h(x) \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y) dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y) h(y) dy \right) dx.
\end{aligned} \tag{2.20}$$

From (2.20), it is easy to arrive at inequality (2.17). The proof of Theorem 2.5 is completed. \square

Remark 2.6. Taking $n = 1$, we have $D = [a_1, b_1]$ and

$$\frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{f'(c)}{h'(c)}, \quad \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} = \frac{g'(d)}{h'(d)}, \quad (2.21)$$

where $c = y_1 + \alpha(x_1 - y_1)$, $0 < \alpha < 1$, and $d = y_1 + \beta(x_1 - y_1)$, $0 < \beta < 1$. In this case, (2.20) becomes the following inequality which was given by Pachpatte in [8]:

$$|G[w, f, g]| \leq \left| \int_a^b w(x)h^2(x)dx - \frac{(\int_a^b w(x)h(x)dx)^2}{\int_a^b w(x)dx} \right| \left\| \frac{f'}{g'} \right\|_{\infty} \left\| \frac{g'}{h'} \right\|_{\infty}, \quad (2.22)$$

where $f(x), g(x), h(x) : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b) , $w : [a, b] \rightarrow [0, \infty)$ is an integrable function with $\int_a^b w(x)dx > 0$, and

$$G[w, p, q] := \int_a^b w(x)p(x)q(x)dx - \frac{(\int_a^b w(x)p(x)dx)(\int_a^b w(x)q(x)dx)}{\int_a^b w(x)dx}. \quad (2.23)$$

Remark 2.7. If $h(x) = \sum_{i=1}^n x_i$, then (2.5) becomes

$$f(x) - f(y) = \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i}(x_i - y_i), \quad g(x) - g(y) = \sum_{i=1}^n \frac{\partial g(d)}{\partial x_i}(x_i - y_i). \quad (2.24)$$

Multiplying the left and right sides of (2.24), we get

$$f(x)g(x) - f(x)g(y) - g(x)f(y) + f(y)g(y) = \left[\sum_{i=1}^n \frac{\partial f(c)}{\partial x_i}(x_i - y_i) \right] \left[\sum_{i=1}^n \frac{\partial g(d)}{\partial x_i}(x_i - y_i) \right]. \quad (2.25)$$

Integrating both sides of (2.25) with respect to y on D , we have the following inequality which was established by Pachpatte in [6]:

$$\begin{aligned} & \left| f(x)g(x) - f(x) \left(\frac{1}{M} \int_D g(y)dy \right) - g(x) \left(\frac{1}{M} \int_D f(y)dy \right) + \frac{1}{M} \int_D f(y)g(y)dy \right| \\ & \leq \frac{1}{M} \int_D \left\{ \left[\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right] \left[\sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} |x_i - y_i| \right] \right\} dy, \end{aligned} \quad (2.26)$$

where $M = \text{mes}D = \prod_{i=1}^n (b_i - a_i)$.

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