

## Research Article

# John-Nirenberg Type Inequalities for the Morrey-Campanato Spaces

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We give John-Nirenberg type inequalities for the Morrey-Campanato spaces on  $\mathbb{R}^n$ .

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Given a function  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  and a cube  $Q$  on  $\mathbb{R}^n$ , let  $f_Q$  denote the average of  $f$  on  $Q$ ,  $f_Q = (1/|Q|) \int_Q f(x) dx$ . We say that  $f$  has bounded mean oscillation if there is a constant  $C$  such that for any cube  $Q$ ,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq C. \quad (1)$$

The space of functions with this property is denoted by BMO. For  $f \in \text{BMO}$ , define the norm on BMO by

$$\|f\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx. \quad (2)$$

John and Nirenberg [1] obtained the following well-known John-Nirenberg inequality for BMO.

**Theorem 1.** *Let  $f \in \text{BMO}$  and  $\|f\|_{\text{BMO}} \neq 0$ . Then there exist positive constants  $C_1$  and  $C_2$ , depending only on the dimension, such that for all cube  $Q$  and any  $\lambda > 0$ ,*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1 e^{-C_2 \lambda / \|f\|_{\text{BMO}}} |Q|. \quad (3)$$

Suppose  $f$  is a locally integrable function on  $\mathbb{R}^n$ ,  $Q$  is a cube, and  $s$  is a nonnegative integer; let  $P_Q f(x)$  be the unique polynomial of degree at most  $s$  such that

$$\int_Q (f(x) - P_Q(f)(x))x^\alpha dx = 0 \quad (4)$$

for all  $0 \leq |\alpha| \leq s$ . Moreover, for any  $x \in Q$ ,

$$|P_Q(f)(x)| \leq \frac{A}{|Q|} \int_Q |f(x)| dx, \quad (5)$$

where the constant  $A$  is independent of  $f$  and  $Q$ . Clearly,  $A \geq 1$ .

For  $\beta \geq 0$ ,  $s \geq 0$ ,  $1 \leq q < \infty$ , we will say that a locally integrable function  $f(x)$  belongs to the Morrey-Campanato spaces  $L(\beta, q, s)$  if

$$\|f\|_{L(\beta, q, s)} = \sup_Q |Q|^{-\beta} \left\{ \frac{1}{|Q|} \int_Q |f(x) - P_Q(f)(x)|^q dx \right\}^{1/q} < \infty, \quad (6)$$

where  $Q$  is a cube. Then if  $f - g$  is a polynomial of degree at most  $s$ ,  $g$  also satisfies (6) and  $\|f\|_{L(\beta, q, s)} = \|g\|_{L(\beta, q, s)}$ . If this is the case we say that  $f$  and  $g$  are equivalent, the quotient space divided by such equivalence classes will be denoted by  $L(\beta, q, s)$ , and (6) defines its norm.

These spaces played an important role in the study of partial differential equations and they were studied extensively. Reader is referred, in particular, to [2–4]. Recently, Deng et al. [5] and Duong and Yan [6] gave several new characterizations for the Morrey-Campanato spaces.

As noted in [2], for  $\beta = 0$  and  $1 \leq q \leq \infty$ , these spaces are variants of the BMO space. For  $\beta > 0$  and  $s \geq [n\beta]$ , the spaces  $L(\beta, q, s)$  are variants of the homogeneous Lipschitz spaces  $\dot{\Lambda}_{n\beta}(\mathbb{R}^n)$  which are duals of certain Hardy spaces. See also [1].

In [7], we proved a John-Nirenberg-type inequality for homogeneous Lipschitz spaces  $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ ,  $0 < \alpha < 1$ . In this note, we will show that a similar inequality is also true for the Morrey-Campanato spaces  $L(\beta, q, s)$  on  $\mathbb{R}^n$ , where  $\beta$  is nonnegative,  $1 \leq q \leq \infty$ , and the integer  $s \geq 0$ . Our main result can be stated as follows.

**Theorem 2** (John-Nirenberg-type inequality). *Given  $\beta \geq 0$  and  $s \geq 0$ , let  $f \in L(\beta, 1, s)$  and  $\|f\|_{L(\beta, 1, s)} \neq 0$ . Then there exist positive constants  $C_1$  and  $C_2$ , depending only on the dimension, such that for all cube  $Q$  and any  $\lambda > 0$ ,*

$$|\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda\}| \leq C_1 e^{-C_2 \lambda / \|f\|_{L(\beta, 1, s)}} |Q|. \quad (7)$$

*Proof.* Let  $Q$  be a fixed cube and let  $\lambda_0$  be some positive real number which will be determined later. Applying the Calderon-Zygmund decomposition to the function  $|Q|^{-\beta} |f(x) - P_Q(f)(x)|$  at height  $\lambda_0$  to obtain a family of subcubes  $\{Q_j\}$  of  $Q$  with disjoint interiors such that

$$|Q|^{-\beta} |f(x) - P_Q(f)(x)| \leq \lambda_0 \quad \text{a.e. } Q \setminus \bigcup_{j=1}^{\infty} Q_j, \quad (8)$$

$$\lambda_0 < \frac{1}{|Q_j|} \int_{Q_j} |Q|^{-\beta} |f(x) - P_Q(f)(x)| dx \leq 2^n \lambda_0 \quad \text{for any } j, \quad (9)$$

$$\sum_{j=1}^{\infty} |Q_j| \leq \frac{1}{\lambda_0} \int_Q |Q|^{-\beta} |f(x) - P_Q(f)(x)| dx. \quad (10)$$

By (5), for any  $x \in Q_j$ , we get

$$\begin{aligned} |P_Q(f)(x) - P_{Q_j}(f)(x)| &= |P_{Q_j}(P_Q(f) - P_{Q_j}(f))(x)| \\ &\leq \frac{A}{|Q_j|} \int_{Q_j} |P_Q(f)(y) - P_{Q_j}(f)(y)| dy. \end{aligned} \quad (11)$$

Thus for any  $x \in Q_j$ , by (9) we have

$$\begin{aligned} |Q|^{-\beta} |P_Q(f)(x) - P_{Q_j}(f)(x)| &\leq \frac{A}{|Q_j|} \int_{Q_j} |Q|^{-\beta} |P_Q(f)(y) - P_{Q_j}(f)(y)| dy \\ &\leq \frac{A}{|Q_j|} \int_{Q_j} |Q|^{-\beta} |f(y) - P_Q(f)(y)| dy + \frac{A}{|Q_j|} \int_{Q_j} |Q_j|^{-\beta} |f(y) - P_{Q_j}(f)(y)| dy \\ &\leq A2^n \lambda_0 + A \|f\|_{L(\beta,1,s)}. \end{aligned} \quad (12)$$

Denote  $b = A2^n \lambda_0 + A \|f\|_{L(\beta,1,s)} > \lambda_0$ . For any  $x \in Q_j$ , we have

$$\begin{aligned} |Q|^{-\beta} |f(x) - P_Q(f)(x)| &\leq |Q|^{-\beta} |P_Q(f)(x) - P_{Q_j}(f)(x)| + |Q_j|^{-\beta} |f(x) - P_{Q_j}(f)(x)| \\ &\leq b + |Q_j|^{-\beta} |f(x) - P_{Q_j}(f)(x)|. \end{aligned} \quad (13)$$

Then for any  $\lambda > 0$ , we have

$$\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda + b\} \subset \{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda_0\} \subset \bigcup_{j=1}^{\infty} Q_j. \quad (14)$$

By (13) and (14),

$$\begin{aligned} \{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda + b\} &\subset \bigcup_{j=1}^{\infty} \{x \in Q_j : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda + b\} \\ &\subset \bigcup_{j=1}^{\infty} \{x \in Q_j : |Q_j|^{-\beta} |f(x) - P_{Q_j}(f)(x)| > \lambda\}. \end{aligned} \quad (15)$$

For any  $\lambda > 0$ , we set

$$F_f(\lambda) = \sup_Q \frac{1}{|Q|} |\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda\}|. \quad (16)$$

Clearly,  $F_f(\lambda)$  is a decreasing function on  $[0, \infty)$  and  $F_f(0) \leq 1$ . Using (10), we have

$$\begin{aligned} \frac{1}{|Q|} |\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda + b\}| &\leq F_f(\lambda) \frac{1}{|Q|} \sum_{j=1}^{\infty} |Q_j| \\ &\leq F_f(\lambda) \frac{1}{\lambda_0 |Q|} \int_Q |Q|^{-\beta} |f(x) - P_Q(f)(x)| dx. \end{aligned} \quad (17)$$

So for any  $\lambda \geq 0$ , we get  $F_f(\lambda + b) \leq \lambda_0^{-1} \|f\|_{L(\beta,1,s)} F_f(\lambda)$ . Taking  $\lambda_0 = e \|f\|_{L(\beta,1,s)}$ , then  $b = A(2^n e + 1) \|f\|_{L(\beta,1,s)}$  is also a fixed positive number and for any  $\lambda \geq 0$ ,

$$F_f(\lambda + b) \leq \frac{1}{e} F_f(\lambda). \quad (18)$$

By induction argument for any  $k \geq 1$ , we get

$$F_f((k+1)b) \leq e^{-k} F_f(b). \quad (19)$$

Thus, for  $\lambda \in (kb, (k+1)b]$ , we have

$$F_f(\lambda) \leq F_f(kb) \leq e^{-k} F_f(0) \leq e e^{-\lambda/b}. \quad (20)$$

Notice that this inequality is also true for  $\lambda \in [0, b]$ , due to  $F_f(\lambda) \leq F_f(0) = 1 \leq e e^{-\lambda/b}$ . Thus, for any  $\lambda \geq 0$ , we have

$$\frac{1}{|Q|} |\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda\}| \leq e e^{-\lambda/b}. \quad (21)$$

This concludes the proof of the theorem.  $\square$

**Corollary 1.** Given  $\beta \geq 0$ ,  $s \geq 0$ . For all  $q \in [1, \infty)$ , the spaces  $L(\beta, q, s)$  coincide, and the norms  $\|\cdot\|_{L(\beta,q,s)}$  are equivalent, namely,

$$\sup_Q \left( \frac{1}{|Q|} \int_Q [ |Q|^{-\beta} |f(x) - P_Q(f)(x)| ]^q dx \right)^{1/q} \approx \sup_Q \frac{1}{|Q|} \int_Q |Q|^{-\beta} |f(x) - P_Q(f)(x)| dx. \quad (22)$$

*Proof.* It will suffice to prove that  $\|f\|_{L(\beta,q,s)} \leq C_q \|f\|_{L(\beta,1,s)}$  for any  $1 < q < \infty$ . In fact, by (7),

$$\begin{aligned} \int_Q (|Q|^{-\beta} |f(x) - P_Q(f)(x)|)^q dx &\leq q \int_0^\infty \lambda^{q-1} |\{x \in Q : |Q|^{-\beta} |f(x) - P_Q(f)(x)| > \lambda\}| d\lambda \\ &\leq C_1 q |Q| \int_0^\infty \lambda^{q-1} e^{-C_2 \lambda / \|f\|_{L(\beta,1,s)}} d\lambda \end{aligned} \quad (23)$$

make the change of variables  $\mu = C_2 \lambda / \|f\|_{L(\beta,1,s)}$ , then we get

$$\begin{aligned} \frac{1}{|Q|} \int_Q (|Q|^{-\beta} |f(x) - P_Q(f)(x)|)^q dx &\leq C_1 q \left( \frac{\|f\|_{L(\beta,1,s)}}{C_2} \right)^q \int_0^\infty \mu^{q-1} e^{-\mu} d\mu \\ &= C_1 q C_2^{-q} \Gamma(q) (\|f\|_{L(\beta,1,s)})^q \end{aligned} \quad (24)$$

which yields the desired inequality.  $\square$

As a consequence of the proof of Corollary 1, we get two additional results.

**Corollary 2.** Given  $\beta \geq 0$ ,  $s \geq 0$ ,  $1 \leq q < \infty$ , if  $f \in L(\beta, q, s)$ , then there exists  $\lambda > 0$  such that for any cube  $Q$ ,

$$\frac{1}{|Q|} \int_Q e^{\lambda |Q|^{-\beta} |f(x) - P_Q(f)(x)|} dx < \infty. \quad (25)$$

**Corollary 3.** Given  $\beta \geq 0$ ,  $s \geq 0$ ,  $1 \leq q < \infty$ ,  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ , suppose there exist constants  $C_1$ ,  $C_2$ , and  $K$  such that for any cube  $Q$  and  $\lambda > 0$ ,

$$|\{x \in Q : |Q|^{-\beta}|f(x) - P_Q(f)(x)| > \lambda\}| \leq C_1 e^{-C_2 \lambda / K} |Q|. \quad (26)$$

Then  $f \in L(\beta, q, s)$ .

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