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Research Article

Stability of a Quadratic Functional Equation in the Spaces of Generalized Functions

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Making use of the pullbacks, we reformulate the following quadratic functional equation: $f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x)$ in the spaces of generalized functions. Also, using the fundamental solution of the heat equation, we obtain the general solution and prove the Hyers-Ulam stability of this equation in the spaces of generalized functions such as tempered distributions and Fourier hyperfunctions.

1. Introduction

Functional equations can be solved by reducing them to differential equations. In this case, we need to assume differentiability up to a certain order of the unknown functions, which is not required in direct methods. From this point of view, there have been several works dealing with functional equations based on distribution theory. In the space of distributions, one can differentiate freely the underlying unknown functions. This can avoid the question of regularity. Actually using distributional operators, it was shown that some functional equations in distributions reduce to the classical ones when the solutions are locally integrable functions [1–4].

Another approach to distributional analogue for functional equations is via the use of the regularization of distributions [5, 6]. More exactly, this method gives essentially the same formulation as in [1–4], but it can be applied to the Hyers-Ulam stability [7–10] for functional equations in distributions [11–14].

In accordance with the notions in [11–14], we reformulate the following quadratic functional equation:

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x) \tag{1.1}$$

in the spaces of generalized functions. Also, we obtain the general solution and prove the Hyers-Ulam stability of (1.1) in the spaces of generalized functions such as $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions and $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions.

The functional equation (1.1) was first solved by Kannappan [15]. In fact, he proved that a function on a real vector space is a solution of (1.1) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x,x) + A(x)$. In addition, Jung [16] investigated Hyers-Ulam stability of (1.1) on restricted domains, and applied the result to the study of an interesting asymptotic behavior of the quadratic functions.

As a matter of fact, we reformulate (1.1) and related inequality in the spaces of generalized functions as follows. For $u \in \mathcal{S}'(\mathbb{R}^n)$ or $u \in \mathcal{F}'(\mathbb{R}^n)$,

$$u \circ A + u \circ P_1 + u \circ P_2 + u \circ P_3 = u \circ B_1 + u \circ B_2 + u \circ B_3, \tag{1.2}$$

$$\left\| u \circ A + u \circ P_1 + u \circ P_2 + u \circ P_3 - u \circ B_1 - u \circ B_2 - u \circ B_3 \right\| \le \epsilon, \tag{1.3}$$

where A, B_1, B_2, B_3, P_1, P_2, and P_3 are the functions defined by

$$\begin{gathered} A(x,y,z) = x + y + z, \\ P_1(x,y,z) = x, \qquad P_2(x,y,z) = y, \qquad P_3(x,y,z) = z, \\ B_1(x,y,z) = x + y, \qquad B_2(x,y,z) = y + z, \qquad B_3(x,y,z) = z + x. \end{gathered} \tag{1.4}$$

Here, $\circ$ denotes the pullbacks of generalized functions, and $\|v\| \le \epsilon$ in (1.3) means that $|\langle v, \varphi\rangle| \le \epsilon \|\varphi\|_{L^1}$ for all test functions φ.

As a consequence, we prove that every solution u of inequality (1.3) can be written uniquely in the form

$$u(x) = u(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j + \sum_{1 \le i \le n} b_i x_i + \mu, \tag{1.5}$$

where μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le 13/3$.

2. Preliminaries

We first introduce briefly spaces of some generalized functions such as tempered distributions and Fourier hyperfunctions. Here, we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 2.1 (see [17, 18]). One denotes by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. A linear functional u on $\mathcal{S}(\mathbb{R}^n)$ is said to be *tempered distribution* if there are a constant $C \geq 0$ and a nonnegative integer N such that

$$\left|\langle u, \varphi\rangle\right| \leq C \sum_{|\alpha|,|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \left|x^\alpha \partial^\beta \varphi\right| \tag{2.2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Imposing the growth condition on $\|\cdot\|_{\alpha,\beta}$ in (2.1), a new space of test functions has emerged as follows.

Definition 2.2 (see [19]). One denotes by $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.3}$$

for some positive constants A, B depending only on φ. One says that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some $A, B > 0$, and denotes by $\mathcal{F}'(\mathbb{R}^n)$ the strong dual of $\mathcal{F}(\mathbb{R}^n)$ and calls its elements *Fourier hyperfunctions* .

It can be verified that the seminorms (2.3) are equivalent to

$$\|\varphi\|_{h,k} = \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|} \alpha!} < \infty \tag{2.4}$$

for some constants $h, k > 0$. It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{F}'(\mathbb{R}^n). \tag{2.5}$$

From the above inclusions, it suffices to say that one considers (1.2) and (1.3) in the space $\mathcal{F}'(\mathbb{R}^n)$.

In order to obtain the general solution and prove the Hyers-Ulam stability of (1.1) in the space $\mathcal{F}'(\mathbb{R}^n)$, one employs the n-dimensional heat kernel, that is, the fundamental solution of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}_x^n \times \mathbb{R}_t^+$ given by

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\dfrac{|x|^2}{4t}\right), & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{2.6}$$

In view of (2.1), one sees that $E_t(\cdot)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ for each $t > 0$. Thus, its *Gauss transform*

$$\widetilde{u}(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y) \rangle, \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.7}$$

is well defined for each $u \in \mathcal{F}'(\mathbb{R}^n)$. In relation to the Gauss transform, it is well known that the *semigroup property* of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{2.8}$$

holds for convolution. Moreover, the following result holds [20].

Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then, its Gauss transform $\widetilde{u}(x,t)$ is a C^∞-solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widetilde{u}(x,t) = 0 \tag{2.9}$$

satisfying what follows.

(i) There exist positive constants C, M, and N such that

$$\left|\widetilde{u}(x,t)\right| \le C t^{-M} (1 + |x|)^N \text{ in } \mathbb{R}^n \times (0, \delta). \tag{2.10}$$

(ii) $\widetilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \widetilde{u}(x,t) \varphi(x) dx. \tag{2.11}$$

Conversely, every C^∞-solution $U(x,t)$ of the heat equation satisfying the growth condition (2.10) can be uniquely expressed as $U(x,t) = \widetilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$.

Analogously, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results [21]. In this case, the estimate (2.10) is replaced by what follows.

For every $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$\left|\widetilde{u}(x,t)\right| \le C_\epsilon \exp\left(\epsilon\left(|x| + \frac{1}{t}\right)\right) \text{ in } \mathbb{R}^n \times (0, \delta). \tag{2.12}$$

3. General solution and stability in $\mathcal{F}'(\mathbb{R}^n)$

We will now consider the general solution and the Hyers-Ulam stability of (1.1) in the space $\mathcal{F}'(\mathbb{R}^n)$. Convolving the tensor product $E_t(\xi)E_s(\eta)E_r(\zeta)$ of n-dimensional heat kernels in both sides of (1.2), we have

$$\begin{aligned}
\big[(u\circ A)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \langle u\circ A, E_t(x-\xi)E_s(y-\eta)E_r(z-\zeta)\rangle \\
&= \left\langle u_\xi, \iint E_t(x-\xi+\eta+\zeta)E_s(y-\eta)E_r(z-\zeta)\,d\eta\, d\zeta\right\rangle \\
&= \left\langle u_\xi, \iint E_t(x+y+z-\xi-\eta-\zeta)E_s(\eta)E_r(\zeta)\,d\eta\, d\zeta\right\rangle \\
&= \langle u_\xi, (E_t*E_s*E_r)(x+y+z-\xi)\rangle \\
&= \langle u_\xi, (E_{t+s+r})(x+y+z-\xi)\rangle \\
&= \tilde{u}(x+y+z, t+s+r),
\end{aligned} \tag{3.1}$$

and similarly we obtain

$$\begin{aligned}
\big[(u\circ P_1)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(x,t), \\
\big[(u\circ P_2)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(y,s), \\
\big[(u\circ P_3)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(z,r), \\
\big[(u\circ B_1)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(x+y,t+s), \\
\big[(u\circ B_2)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(y+z,s+r), \\
\big[(u\circ B_3)*\big(E_t(\xi)E_s(\eta)E_r(\zeta)\big)\big](x,y,z) &= \tilde{u}(z+x,r+t),
\end{aligned} \tag{3.2}$$

where $\tilde{u}$ is the Gauss transform of u. Thus, (1.2) is converted into the classical functional equation

$$\tilde{u}(x+y+z,t+s+r)+\tilde{u}(x,t)+\tilde{u}(y,s)+\tilde{u}(z,r) = \tilde{u}(x+y,t+s)+\tilde{u}(y+z,s+r)+\tilde{u}(z+x,r+t) \tag{3.3}$$

for all $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$. For that reason, we first prove the following lemma which is essential to prove the main result.

Lemma 3.1. *Suppose that a function $f:\mathbb{R}^n\times(0,\infty)\to\mathbb{C}$ satisfies*

$$f(x+y+z,t+s+r)+f(x,t)+f(y,s)+f(z,r) = f(x+y,t+s)+f(y+z,s+r)+f(z+x,r+t) \tag{3.4}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. Also, assume that $f(x,t)$ is continuous and 2-times differentiable with respect to x and t, respectively. Then, there exist constants $a_{ij}, b_i, c_i, d, e \in \mathbb{C}$ such that

$$f(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_i x_j + \sum_{1\le i\le n} b_i x_i + t\sum_{1\le i\le n} c_i x_i + dt^2 + et \tag{3.5}$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t > 0$.

Proof. In view of (3.4), $f(x,0^+) := \lim_{t\to 0^+} f(x,t)$ exists for each $x \in \mathbb{R}^n$. Letting $t = s = r \to 0^+$ in (3.4), we see that $f(x,0^+)$ satisfies (1.1). By the result as that in [15], there exist a symmetric biadditive function B and an additive function A such that

$$f(x,0^+) = B(x,x) + A(x) \tag{3.6}$$

for all $x \in \mathbb{R}^n$. From the hypothesis that $f(x,t)$ is continuous with respect to x, we have

$$f(x,0^+) = \sum_{1\le i\le j\le n} a_{ij}x_i x_j + \sum_{1\le i\le n} b_i x_i \tag{3.7}$$

for some $a_{ij}, b_i \in \mathbb{C}$. We now define a function h as $h(x,t) := f(x,t) - f(x,0^+) - f(0,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$. Putting $x = y = z = 0$ and $t = s = r \to 0^+$ in (3.4), we have $f(0,0^+) = 0$. From the definition of h and $f(0,0^+) = 0$, we see that h satisfies $h(0,t) = 0, h(x,0^+) = 0$, and

$$h(x+y+z, t+s+r) + h(x,t) + h(y,s) + h(z,r) = h(x+y, t+s) + h(y+z, s+r) + h(z+x, r+t) \tag{3.8}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. Putting $y = z = 0$ in (3.8), we get

$$h(x, t+s+r) + h(x,t) = h(x, t+s) + h(x, r+t). \tag{3.9}$$

Now letting $t \to 0^+$ in (3.9) yields

$$h(x, s+r) = h(x,s) + h(x,r). \tag{3.10}$$

Given the continuity, $h(x,t)$ can be written as

$$h(x,t) = h(x,1)t \tag{3.11}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Setting $x = 0$, $t = 1$, and $s = r \to 0^+$ in (3.8), we obtain

$$h(y+z, 1) = h(y,1) + h(z,1) \tag{3.12}$$

for all $y, z \in \mathbb{R}^n$. This shows that $h(x,1)$ is additive. Thus, $h(x,t)$ can be written in the form

$$h(x,t) = t\sum_{1\le i\le n} c_i x_i \tag{3.13}$$

for some $c_i \in \mathbb{C}$. Now we are going to find the general solution of $f(0,t)$. Putting $x = y = z = 0$ in (3.4), we obtain

$$f(0,t+s+r) + f(0,t) + f(0,s) + f(0,r) = f(0,t+s) + f(0,s+r) + f(0,r+t). \tag{3.14}$$

Differentiating (3.14) with respect to t, we have

$$f'(0,t+s+r) + f'(0,t) = f'(0,t+s) + f'(0,r+t) \tag{3.15}$$

for all $t, s, r > 0$. Similarly, differentiation of (3.15) with respect to s yields

$$f''(0,t+s+r) = f''(0,t+s) \tag{3.16}$$

which shows that $f''(0,t)$ is a constant function. By virtue of $f(0,0^+) = 0$, $f(0,t)$ can be written as

$$f(0,t) = dt^2 + et \tag{3.17}$$

for some $d, e \in \mathbb{C}$. Combining (3.7), (3.13), and (3.17), $f(x,t)$ can be written in the form

$$f(x,t) = f(x,0^+) + h(x,t) + f(0,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i + t\sum_{1\le i\le n} c_ix_i + dt^2 + et \tag{3.18}$$

for some $a_{ij}, b_i, c_i, d, e \in \mathbb{C}$. This completes the proof. □

As an immediate consequence of Lemma 3.1, we establish the general solution of (1.1) in the space $\mathcal{F}'(\mathbb{R}^n)$.

Theorem 3.2. *Every solution u in $\mathcal{F}'(\mathbb{R}^n)$ of*

$$u \circ A + u \circ P_1 + u \circ P_2 + u \circ P_3 = u \circ B_1 + u \circ B_2 + u \circ B_3 \tag{3.19}$$

has the form

$$u(x) = u(x_1, \ldots, x_n) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i \tag{3.20}$$

for some $a_{ij}, b_i \in \mathbb{C}$.

Proof. As we see above, if we convolve the tensor product $E_t(\xi)E_s(\eta)E_r(\zeta)$ of n-dimensional heat kernels in both sides of (3.19), then (3.19) is converted into the classical functional equation

$$\tilde{u}(x+y+z,t+s+r) + \tilde{u}(x,t) + \tilde{u}(y,s) + \tilde{u}(z,r) = \tilde{u}(x+y,t+s) + \tilde{u}(y+z,s+r) + \tilde{u}(z+x,r+t) \tag{3.21}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$, where $\widetilde{u}$ is the Gauss transform of u. According to Lemma 3.1, $\widetilde{u}(x,t)$ is of the form

$$\widetilde{u}(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i + t\sum_{1\le i\le n} c_ix_i + dt^2 + et \tag{3.22}$$

for some constants $a_{ij}, b_i, c_i, d, e \in \mathbb{C}$. Now letting $t \to 0^+$, we have

$$u = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i \tag{3.23}$$

which completes the proof. □

We now in a position to state and prove the main result of this paper.

Theorem 3.3. *Suppose that u in $\mathcal{F}'(\mathbb{R}^n)$ satisfies the inequality*

$$\left\| u\circ A + u\circ P_1 + u\circ P_2 + u\circ P_3 - u\circ B_1 - u\circ B_2 - u\circ B_3 \right\| \le \varepsilon. \tag{3.24}$$

Then, there exists a function T defined by

$$T(x) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i, \quad a_{ij}, b_i \in \mathbb{C}, \tag{3.25}$$

such that

$$\left\| u - T(x) \right\| \le \frac{13}{3}\varepsilon. \tag{3.26}$$

Proof. Convolving the tensor product $E_t(\xi)E_s(\eta)E_r(\zeta)$ of n-dimensional heat kernels in both sides of (3.24), we have the classical functional inequality

$$\left|\widetilde{u}(x+y+z,t+s+r)+\widetilde{u}(x,t)+\widetilde{u}(y,s)+\widetilde{u}(z,r)-\widetilde{u}(x+y,t+s) - \widetilde{u}(y+z,s+r) - \widetilde{u}(z+x,r+t)\right| \le \epsilon \tag{3.27}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$, where $\widetilde{u}$ is the Gauss transform of u. Define a function $f_e : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ by $f_e(x,t) := (1/2)(\widetilde{u}(x,t) + \widetilde{u}(-x,t)) - \widetilde{u}(0,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$. Then, $f_e(-x,t) = f_e(x,t)$, $f_e(0,t) = 0$, and

$$\left|f_e(x+y+z,t+s+r)+f_e(x,t)+f_e(y,s)+f_e(z,r)-f_e(x+y,t+s)-f_e(y+z,s+r)-f_e(z+x,r+t)\right|\le 2\epsilon \tag{3.28}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. Replacing z by $-y$ in (3.28), we have

$$\left|f_e(x,t+s+r) + f_e(x,t) + f_e(y,s) + f_e(y,r) - f_e(x+y,t+s) - f_e(x-y,r+t)\right| \le 2\epsilon. \tag{3.29}$$

Putting $y = z = 0$ in (3.28) yields

$$\left|f_e(x,t+s+r)+f_e(x,t)-f_e(x,t+s)-f_e(x,r+t)\right| \le 2\epsilon. \tag{3.30}$$

Taking (3.29) into (3.30), we obtain

$$\left|f_e(x+y,t+s)+f_e(x-y,r+t)f_e(x,t+s)-f_e(x,r+t)-f_e(y,s)-f_e(y,r)\right| \le 4\epsilon. \tag{3.31}$$

Letting $t \to 0^+$ and switching r by s, we have

$$\left|f_e(x+y,s)+f_e(x-y,s)-2f_e(x,s)-2f_e(y,s)\right| \le 4\epsilon. \tag{3.32}$$

Substituting y, s by x, t, respectively, and then dividing by 4, we lead to

$$\left|\frac{f_e(2x,t)}{4}-f_e(x,t)\right| \le \epsilon. \tag{3.33}$$

Making use of an induction argument, we obtain

$$\left|4^{-k}f_e(2^k x,t)-f_e(x,t)\right| \le \frac{4}{3}\epsilon \tag{3.34}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $t > 0$. Exchanging x by $2^l x$ in (3.34) and then dividing the result by 4^l, we can see that $\{4^{-k}f_e(2^k x,t)\}$ is a Cauchy sequence which converges uniformly. Let $g(x,t) = \lim_{k\to\infty}4^{-k}f_e(2^k x,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$. It follows from (3.28) and (3.34) that $g(x,t)$ is the unique function satisfying

$$\begin{gathered}g(x+y+z,t+s+r)+g(x,t)+g(y,s)+g(z,r)=g(x+y,t+s)+g(y+z,s+r)+g(z+x,r+t),\\ \left|f_e(x,t)-g(x,t)\right| \le \frac{4}{3}\epsilon\end{gathered} \tag{3.35}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. By virtue of Lemma 3.1, g is of the form

$$g(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i + t\sum_{1\le i\le n} c_ix_i + dt^2 + et \tag{3.36}$$

for some constants $a_{ij}, b_i, c_i, d, e \in \mathbb{C}$. Since $f_e(-x,t) = f_e(x,t)$ and $f_e(0,t) = 0$ for all $x \in \mathbb{R}^n$ and $t > 0$, we have

$$g(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j. \tag{3.37}$$

On the other hand, let $f_o : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be the function defined by $f_o(x,t) := (1/2)(\widetilde{u}(x,t) - \widetilde{u}(-x,t))$ for all $x \in \mathbb{R}^n$ and $t > 0$. Then, $f_o(-x,t) = -f_o(x,t)$, $f_o(0,t) = 0$, and

$$\left|f_o(x+y+z,t+s+r)+f_o(x,t)+f_o(y,s)+f_o(z,r)-f_o(x+y,t+s)-f_o(y+z,s+r)-f_o(z+x,r+t)\right| \le \epsilon \tag{3.38}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. Replacing z by $-y$ in (3.38), we have

$$\left|f_o(x,t+s+r) + f_o(x,t) + f_o(y,s) - f_o(y,r) - f_o(x+y,t+s) - f_o(x-y,r+t)\right| \le \epsilon. \tag{3.39}$$

Setting $y = z = 0$ in (3.38) yields

$$\left|f_o(x,t+s+r) + f_o(x,t) - f_o(x,t+s) - f_o(x,r+t)\right| \le \epsilon. \tag{3.40}$$

Adding (3.39) to (3.40), we obtain

$$\left|f_o(x+y,t+s) + f_o(x-y,r+t) - f_o(x,t+s) - f_o(x,r+t) - f_o(y,s) + f_o(y,r)\right| \le 2\epsilon. \tag{3.41}$$

Letting $t \to 0^+$ and replacing r by s, we have

$$\left|f_o(x+y,s) + f_o(x-y,s) - 2f_o(x,s)\right| \le 2\epsilon. \tag{3.42}$$

Substituting y, s by x, t, respectively, and then dividing by 2, we lead to

$$\left|\frac{f_o(2x,t)}{2} - f_o(x,t)\right| \le \epsilon. \tag{3.43}$$

Using the iterative method, we obtain

$$\left|2^{-k} f_o(2^k x,t) - f_o(x,t)\right| \le 2\epsilon \tag{3.44}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $t > 0$. From (3.38) and (3.44), we verify that h is the unique function satisfying

$$\begin{gathered} h(x+y+z,t+s+r)+h(x,t)+h(y,s)+h(z,r)=h(x+y,t+s)+h(y+z,s+r)+h(z+x,r+t), \\ \left|f_o(x,t) - h(x,t)\right| \le 2\epsilon \end{gathered} \tag{3.45}$$

for all $x, y, z \in \mathbb{R}^n$ and $t, s, r > 0$. According to Lemma 3.1, there exist $a_{ij}, b_i, c_i, d, e \in \mathbb{C}$ such that

$$h(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + \sum_{1\le i\le n} b_ix_i + t\sum_{1\le i\le n} c_ix_i + dt^2 + et. \tag{3.46}$$

On account of $f_o(-x,t) = f_o(x,t)$ and $f_o(0,t) = 0$ for all $x \in \mathbb{R}^n$ and $t > 0$, we have

$$h(x,t) = \sum_{1\le i\le n} b_i x_i + t \sum_{1\le i\le n} c_i x_i. \tag{3.47}$$

In turn, since $\tilde{u}(x,t) = f_e(x,t) + f_o(x,t) + \tilde{u}(0,t)$, we figure out

$$\begin{aligned} \left|\tilde{u}(x,t) - g(x,t) - h(x,t)\right| &\le \left|f_e(x,t) - g(x,t)\right| + \left|f_o(x,t) - h(x,t)\right| + \left|\tilde{u}(0,t)\right| \\ &\le \frac{10}{3}\epsilon + \left|\tilde{u}(0,t)\right| \end{aligned} \tag{3.48}$$

In view of (3.27), it is easy to see that $c := \limsup_{t\to 0^+} f(0,t)$ exists. Letting $x = y = z = 0$ and $t = s = r \to 0^+$ in (3.27), we have $|c| \le \epsilon$. Finally, taking $t \to 0^+$ in (3.48), we have

$$\left\| u - \left(\sum_{1\le i\le j\le n} a_{ij} x_i x_j + \sum_{1\le i\le n} b_i x_i \right) \right\| \le \frac{13}{3}\epsilon \tag{3.49}$$

which completes the proof. □

Remark 3.4. The above norm inequality (3.49) implies that $u - T(x)$ belongs to $(L^1)' = L^\infty$. Thus, all the solution u in $\mathcal{F}'(\mathbb{R}^n)$ can be written uniquely in the form

$$u = T(x) + \mu, \tag{3.50}$$

where μ is a bounded measurable function such that $||\mu||_{L^\infty} \le (13/3)\epsilon$.

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