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Research Article

Euler-Lagrange Type Cubic Operators and Their Norms on X_λ Space

Abbas Najati[1] and Asghar Rahimi[2]

[1] *Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, P.O. Box 56199-11367 Ardabili, Iran*
[2] *Department of Mathematics, University of Maragheh, P.O. Box 55181-83111, Maragheh, East Azarbayjan, Iran*

Correspondence should be addressed to Abbas Najati, a.najati@uma.ac.ir

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We will introduce linear operators and obtain their exact norms defined on the function spaces X_λ and Z_λ^5. These operators are constructed from the Euler-Lagrange type cubic functional equations and their Pexider versions.

1. Introduction

Let X and Y be complex normed spaces. For a fixed nonnegative real number λ, we denote by X_λ the linear space of all functions $f : X \to Y$ (with pointwise operations) for which there exists a constant $M_f \geq 0$ with

$$\|f(x)\| \leq M_f e^{\lambda\|x\|} \tag{1.1}$$

for all $x \in X$. It is easy to show that the space X_λ with the norm

$$\|f\| := \sup_{x\in X}\{e^{-\lambda\|x\|}\|f(x)\|\} \tag{1.2}$$

is a normed space. Let us denote by X_λ^n the linear space of all functions $\phi : \underbrace{X \times \cdots \times X}_{n \text{ times}} \to Y$ (with pointwise operations) for which there exists a constant $M_\phi \geq 0$ with

$$\|\phi(x_1, \ldots, x_n)\| \leq M_\phi e^{\lambda\sum_{i=1}^{n}\|x_i\|} \tag{1.3}$$

for all $x_1, \dots, x_n \in X$. It is not difficult to show that the space X_λ^n with the norm

$$\|\phi\| := \sup_{x_1,\dots,x_n \in X} \left\{ \|\phi(x_1, \dots, x_n)\| e^{-\lambda \sum_{i=1}^{n} \|x_i\|} \right\} \tag{1.4}$$

is a normed space.

We denote by Z_λ^m the normed space $\bigoplus_{i=1}^m X_\lambda = \{(f_1, \dots, f_m) : f_1, \dots, f_m \in X_\lambda\}$ (with pointwise operations) together with the norm

$$\|(f_1, \dots, f_m)\| := \max\{\|f_1\|, \dots, \|f_m\|\}. \tag{1.5}$$

The norms of the Pexiderized Cauchy, quadratic, and Jensen operators on the function space X_λ have been investigated by Czerwik and Dlutek [1, 2]. In [3], Moslehian et al. have extended the results of [2] to the Pexiderized generalized Jensen and Pexiderized generalized quadratic operators on the function space X_λ and provided more general results regarding their norms.

In [4], Jung investigated the norm of the cubic operator on the function space Z_λ^5.

A function $f : X \to Y$ is called a cubic function if and only if f is a solution function of the cubic functional equation

$$f(x+y) + f(x-y) = 2f\left(\frac{1}{2}x + y\right) + 2f\left(\frac{1}{2}x - y\right) + 12f\left(\frac{1}{2}x\right). \tag{1.6}$$

Jun and Kim [5] proved that when both X and Y are real vector spaces, a function $f : X \to Y$ satisfies (1.6) if and only if there exists a function $B : X \times X \times X \to Y$ such that $f(x) = B(x, x, x)$ for all $x \in X$, and B is symmetric for each fixed one variable and is additive for fixed two variables.

In [6], the authors introduced the following Euler-Lagrange-type cubic functional equation, which is equivalent to (1.6),

$$f(x+y) + f(x-y) = af\left(\frac{1}{a}x + y\right) + af\left(\frac{1}{a}x - y\right) + 2a(a^2 - 1)f\left(\frac{1}{a}x\right) \tag{1.7}$$

for fixed integers a with $a \neq 0, \pm 1$. Moreover, Jun and Kim [7] introduced the following Euler-Lagrange-type cubic functional equation

$$f\left(\frac{1}{a}x + \frac{1}{b}y\right) + f\left(\frac{1}{b}x + \frac{1}{a}y\right) = (a+b)(a-b)^2\left[f\left(\frac{1}{ab}x\right) + f\left(\frac{1}{ab}y\right)\right] + ab(a+b)f\left(\frac{1}{ab}x + \frac{1}{ab}y\right) \tag{1.8}$$

for fixed integers a, b with $a, b \neq 0$, $a \pm b \neq 0$, and they proved the following theorem.

Theorem 1.1 (see [7, Theorem 2.1]). *Let X and Y be real vector spaces. If a mapping $f : X \to Y$ satisfies the functional equation* (1.6)*, then f satisfies the functional equation* (1.8)*.*

We will introduce linear operators which are constructed from the Euler-Lagrange-type cubic and the Pexiderization of the Euler-Lagrange-type cubic functional equations (1.7) and (1.8).

Definition 1.2. The operators $C_1^P, C_2^P : Z_\lambda^5 \to X_\lambda^2$ are defined by

$$\begin{aligned} C_1^P(f_1,\ldots,f_5)(x,y) &:= f_1(x+y) + f_2(x-y) - m f_3\left(\frac{1}{m}x + y\right) \\ &\quad - m f_4\left(\frac{1}{m}x - y\right) - 2m(m^2-1) f_5\left(\frac{1}{m}x\right), \\ C_2^P(f_1,\ldots,f_5)(x,y) &:= f_1\left(\frac{1}{a}x + \frac{1}{b}y\right) + f_2\left(\frac{1}{b}x + \frac{1}{a}y\right) - (a+b)(a-b)^2\left[f_3\left(\frac{1}{ab}x\right) + f_4\left(\frac{1}{ab}y\right)\right] \\ &\quad - ab(a+b) f_5\left(\frac{1}{ab}x + \frac{1}{ab}y\right), \end{aligned} \tag{1.9}$$

where a, b, and m are fixed integers with $a, b \neq 0$, $a \pm b \neq 0$, and $m \neq 0, \pm 1$.

Definition 1.3. The operators $C_1, C_2 : X_\lambda \to X_\lambda^2$ are defined by

$$\begin{aligned} C_1(f)(x,y) &:= f(x+y) + f(x-y) - m f\left(\frac{1}{m}x + y\right) \\ &\quad - m f\left(\frac{1}{m}x - y\right) - 2m(m^2-1) f\left(\frac{1}{m}x\right), \\ C_2(f)(x,y) &:= f\left(\frac{1}{a}x + \frac{1}{b}y\right) + f\left(\frac{1}{b}x + \frac{1}{a}y\right) \\ &\quad - (a+b)(a-b)^2\left[f\left(\frac{1}{ab}x\right) + f\left(\frac{1}{ab}y\right)\right] - ab(a+b) f\left(\frac{1}{ab}x + \frac{1}{ab}y\right), \end{aligned} \tag{1.10}$$

where a, b, and m are fixed integers with $a, b \neq 0$, $a \pm b \neq 0$, and $m \neq 0, \pm 1$.

In this paper, we will give the exact norms of the operators C_1^P, C_2^P on the function space Z_λ^5, and norms of the operators C_1, C_2 on the function space X_λ. The results extend the results of [4].

2. Main results

Throughout this section, a, b, and m are fixed integers with $a, b \neq 0$, $a \pm b \neq 0$, and $m \neq 0, \pm 1$.

The next theorems give us the exact norms of operators C_1^P, C_2^P, C_1, and C_2.

Theorem 2.1. *The operator $C_1^P : Z_\lambda^5 \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_1^P\| = 2|m|^3 + 2. \tag{2.1}$$

Proof. First, we show that $\|C_1^P\| \leq 2|m|^3 + 2$. Since

$$\max\left\{\|x+y\|, \|x-y\|, \left\|\frac{1}{m}x + y\right\|, \left\|\frac{1}{m}x - y\right\|, \left\|\frac{1}{m}x\right\|\right\} \leq \|x\| + \|y\| \tag{2.2}$$

for all $x, y \in X$, we get

$$\begin{aligned}
\|C_1^P(f_1,\dots,f_5)\| &= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \left\| f_1(x+y) + f_2(x-y) - m f_3\left(\frac{1}{m}x + y\right) \right. \\
&\qquad \left. - m f_4\left(\frac{1}{m}x - y\right) - 2m(m^2-1) f_5\left(\frac{1}{m}x\right) \right\| \\
&\le \sup_{x,y\in X} e^{-\lambda\|x+y\|}\|f_1(x+y)\| + \sup_{x,y\in X} e^{-\lambda\|x-y\|}\|f_2(x-y)\| \\
&\quad + |m| \sup_{x,y\in X} e^{-\lambda\|(1/m)x+y\|} \left\| f_3\left(\frac{1}{m}x + y\right)\right\| \\
&\quad + |m| \sup_{x,y\in X} e^{-\lambda\|(1/m)x-y\|} \left\| f_4\left(\frac{1}{m}x - y\right)\right\| \\
&\quad + 2|m|(m^2-1)\sup_{x\in X} e^{-\lambda\|(1/m)x\|} \left\| f_5\left(\frac{1}{m}x\right)\right\| \\
&= \|f_1\| + \|f_2\| + |m|\|f_3\| + |m|\|f_4\| + 2|m|(m^2-1)\|f_5\| \\
&\le (2|m|^3 + 2)\max\{\|f_1\|, \|f_2\|, \|f_3\|, \|f_4\|, \|f_5\|\} \\
&= (2|m|^3 + 2)\|(f_1, f_2, f_3, f_4, f_5)\|
\end{aligned} \tag{2.3}$$

for each $(f_1,\dots,f_5) \in Z_\lambda^5$. This implies that

$$\|C_1^P\| \le 2|m|^3 + 2. \tag{2.4}$$

Now, let $v \in Y$ be such that $\|v\| = 1$ and let $\{\xi_n\}_n$ be a sequence of positive real numbers decreasing to 0. We define

$$f_n(x) = \begin{cases} e^{2\lambda\xi_n} v, & \text{if } \|x\| = 2\xi_n \text{ or } \|x\| = 0, \\ -\dfrac{|m|}{m} e^{2\lambda\xi_n} v, & \text{if } \|mx\| = |m+1|\xi_n,\ \|mx\| = |m-1|\xi_n \text{ or } \|mx\| = \xi_n, \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

for all $x \in X$. Hence we have

$$e^{-\lambda\|x\|}\|f_n(x)\| = \begin{cases} e^{2\lambda\xi_n}, & \text{if } \|x\| = 0, \\ 1, & \text{if } \|x\| = 2\xi_n, \\ e^{(2-|(m+1)/m|)\lambda\xi_n}, & \text{if } \|mx\| = |m+1|\xi_n, \\ e^{(2-|(m-1)/m|)\lambda\xi_n}, & \text{if } \|mx\| = |m-1|\xi_n, \\ e^{(2-1/|m|)\lambda\xi_n}, & \text{if } \|mx\| = \xi_n, \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n, with

$$\|f_n\| = e^{2\lambda\xi_n}. \tag{2.7}$$

Let $u \in X$ be such that $\|u\| = 1$ and take $x_0, y_0 \in X$ as $x_0 = y_0 = \xi_n u$. Then it follows from the definition of f_n that

$$\begin{aligned}\|C_1^P(f_n,\ldots,f_n)\| = \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \Big\| f_n(x+y) + f_n(x-y) - m f_n\Big(\frac{1}{m}x+y\Big) \\ - m f_n\Big(\frac{1}{m}x-y\Big) - 2m(m^2-1) f_n\Big(\frac{1}{m}x\Big)\Big\| \\ \geq e^{-2\lambda\xi_n}\|e^{2\lambda\xi_n}v + e^{2\lambda\xi_n}v + |m|e^{2\lambda\xi_n}v + |m|e^{2\lambda\xi_n}v + 2|m|(m^2-1)e^{2\lambda\xi_n}v\| \\ = 2|m|^3 + 2.\end{aligned} \tag{2.8}$$

If on the contrary $\|C_1^P\| < 2|m|^3 + 2$, then there exists a $\delta > 0$ such that

$$\|C_1^P(f_n,\ldots,f_n)\| \leq (2|m|^3 + 2 - \delta)\|(f_n,\ldots,f_n)\| \tag{2.9}$$

for all positive integers n. So it follows from (2.7), (2.8), and (2.9) that

$$2|m|^3 + 2 \leq \|C_1^P f_n,\ldots,f_n)\| \leq (2|m|^3 + 2 - \delta)e^{2\lambda\xi_n} \tag{2.10}$$

for all positive integers n. Since $\lim_{n\to\infty} e^{2\lambda\xi_n} = 1$, the right-hand side of (2.10) tends to $2|m|^3 + 2 - \delta$ as $n\to\infty$, whence $2|m|^3 + 2 \leq 2|m|^3 + 2 - \delta$, which is a contradiction. Hence we have $\|C_1^P\| = 2|m|^3 + 2$. □

Theorem 2.1 of [4] is a result of Theorem 2.1 for $m = 2$.

Corollary 2.2. *The operator $C_1 : X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_1\| = 2|m|^3 + 2. \tag{2.11}$$

Proof. The result follows from the proof of Theorem 2.1. □

Theorem 2.3. *The operator $C_2^P : Z_\lambda^5 \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_2^P\| = 2|a+b|(a-b)^2 + |ab(a+b)| + 2. \tag{2.12}$$

Proof. Since

$$\max\left\{\left\|\frac{1}{a}x + \frac{1}{b}y\right\|, \left\|\frac{1}{b}x + \frac{1}{a}y\right\|, \left\|\frac{1}{ab}x\right\|, \left\|\frac{1}{ab}y\right\|, \left\|\frac{1}{ab}x + \frac{1}{ab}y\right\|\right\} \leq \|x\| + \|y\| \tag{2.13}$$

for all $x, y \in X$, we get

$$\begin{aligned}
\|C_2^P(f_1,\ldots,f_5)\| &= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\left\| f_1\left(\frac{1}{a}x+\frac{1}{b}y\right)+f_2\left(\frac{1}{b}x+\frac{1}{a}y\right)\right.\\
&\qquad -(a+b)(a-b)^2\left[f_3\left(\frac{1}{ab}x\right)+f_4\left(\frac{1}{ab}y\right)\right]\\
&\qquad \left. -ab(a+b)f_5\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\right\|\\
&\le \sup_{x,y\in X} e^{-\lambda\|(1/a)x+(1/b)y\|}\left\|f_1\left(\frac{1}{a}x+\frac{1}{b}y\right)\right\|\\
&\quad + \sup_{x,y\in X} e^{-\lambda\|(1/b)x+(1/a)y\|}\left\|f_2\left(\frac{1}{b}x+\frac{1}{a}y\right)\right\|\\
&\quad + |a+b|(a-b)^2\sup_{x\in X} e^{-\lambda(\|(1/ab)x\|)}\left\|f_3\left(\frac{1}{ab}x\right)\right\|\\
&\quad + |a+b|(a-b)^2\sup_{y\in X} e^{-\lambda(\|(1/ab)y\|)}\left\|f_4\left(\frac{1}{ab}y\right)\right\|\\
&\quad + |ab(a+b)|\sup_{x,y\in X} e^{-\lambda\|(1/ab)x+(1/ab)y\|}\left\|f_5\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\right\|\\
&\le \|f_1\|+\|f_2\|+|a+b|(a-b)^2(\|f_3\|+\|f_4\|)+|ab(a+b)|\|f_5\|\\
&\le \left(2|a+b|(a-b)^2+|ab(a+b)|+2\right)\max\{\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|,\|f_5\|\}\\
&= \left(2|a+b|(a-b)^2+|ab(a+b)|+2\right)\|(f_1,f_2,f_3,f_4,f_5)\|
\end{aligned} \tag{2.14}$$

for each $(f_1,\ldots,f_5) \in Z_\lambda^5$. This implies that

$$\|C_2^P\| \le 2|a+b|(a-b)^2+|ab(a+b)|+2. \tag{2.15}$$

Let η be a real number such that

$$\eta \notin \left\{0,1,\frac{1-a}{b},\frac{1-b}{a},\frac{a-1}{1-b},\frac{b-1}{1-a},\frac{a}{1-b},\frac{b}{1-a}\right\}. \tag{2.16}$$

Now, let $u \in X, v \in Y$ be such that $\|u\| = \|v\| = 1$ and let $\{\xi_n\}_n$ be a sequence of positive real numbers decreasing to 0. We define

$$f_n(x) = \begin{cases} e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \left(\dfrac{1}{a}+\dfrac{\eta}{b}\right)\xi_n u, \text{ or } x = \left(\dfrac{1}{b}+\dfrac{\eta}{a}\right)\xi_n u,\\ -\dfrac{|a+b|}{a+b}e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \dfrac{1}{ab}\xi_n u, \text{ or } x = \dfrac{\eta}{ab}\xi_n u,\\ -\dfrac{|ab(a+b)|}{ab(a+b)}e^{\lambda(1+|\eta|)\xi_n}v, & \text{if } x = \dfrac{1+\eta}{ab}\xi_n u,\\ 0, & \text{otherwise} \end{cases} \tag{2.17}$$

for all $x \in X$. Hence we have

$$e^{-\lambda\|x\|}\|f_n(x)\| = \begin{cases} e^{(1+|\eta|-|1/a+\eta/b|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{a}+\frac{\eta}{b}\right)\xi_n u, \\ e^{(1+|\eta|-|1/b+\eta/a|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{b}+\frac{\eta}{a}\right)\xi_n u, \\ e^{(1+|\eta|-|1/ab|)\lambda\xi_n}, & \text{if } x = \frac{1}{ab}\xi_n u, \\ e^{(1+|\eta|-|\eta/ab|)\lambda\xi_n}, & \text{if } x = \frac{\eta}{ab}\xi_n u, \\ e^{(1+|\eta|-|(1+\eta)/ab|)\lambda\xi_n}, & \text{if } x = \frac{1+\eta}{ab}\xi_n u, \\ 0, & \text{otherwise} \end{cases} \tag{2.18}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n, with

$$\begin{aligned} \|f_n\| = \max\{&e^{(1+|\eta|-|1/a+\eta/b|)\lambda\xi_n}, e^{(1+|\eta|-|1/b+\eta/a|)\lambda\xi_n}, \\ &e^{(1+|\eta|-|1/ab|)\lambda\xi_n}, e^{(1+|\eta|-|\eta/ab|)\lambda\xi_n}, e^{(1+|\eta|-|(1+\eta)/ab|)\lambda\xi_n}\}. \end{aligned} \tag{2.19}$$

Let $x_0, y_0 \in X$ be such that $x_0 = \xi_n u$ and $y_0 = \eta\xi_n u$. Then it follows from the definition of f_n that

$$\begin{aligned} \|C_2^P(f_n,\ldots,f_n)\| = \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \bigg\| &f_n\left(\frac{1}{a}x+\frac{1}{b}y\right) + f_n\left(\frac{1}{b}x+\frac{1}{a}y\right) \\ &- (a+b)(a-b)^2\left[f_n\left(\frac{1}{ab}x\right)+f_n\left(\frac{1}{ab}y\right)\right] \\ &- ab(a+b)f_n\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\bigg\| \\ \geq e^{-\lambda(1+|\eta|)\xi_n}\|&e^{\lambda(1+|\eta|)\xi_n} + e^{\lambda(1+|\eta|)\xi_n} + 2|a+b|(a-b)^2e^{\lambda(1+|\eta|)\xi_n} + |ab(a+b)|e^{\lambda(1+|\eta|)\xi_n}\| \\ = 2|a+b|&(a-b)^2 + |ab(a+b)| + 2, \end{aligned} \tag{2.20}$$

so that

$$\|C_2^P(f_n,\ldots,f_n)\| \geq 2|a+b|(a-b)^2 + |ab(a+b)| + 2. \tag{2.21}$$

If on the contrary $\|C_2^P\| < 2|a+b|(a-b)^2 + |ab(a+b)| + 2$, then there exists a $\delta > 0$ such that

$$\|C_2^P(f_n,\ldots,f_n)\| \leq (2|a+b|(a-b)^2 + |ab(a+b)| + 2 - \delta)\|(f_n,\ldots,f_n)\| \tag{2.22}$$

for all positive integers n. So it follows from (2.21) and (2.22) that

$$2|a+b|(a-b)^2+|ab(a+b)|+2 \leq \|C_2^P(f_n,\ldots,f_n)\| \leq (2|a+b|(a-b)^2+|ab(a+b)|+2-\delta)\|f_n\| \tag{2.23}$$

for all positive integers n. Since $\lim_{n\to\infty}\xi_n = 0$, it follows from (2.19) that $\lim_{n\to\infty}\|f_n\| = 1$, so the right-hand side of (2.23) tends to $2|a+b|(a-b)^2+|ab(a+b)|+2-\delta$ as $n\to\infty$, whence

$$2|a+b|(a-b)^2+|ab(a+b)|+2 \leq 2|a+b|(a-b)^2+|ab(a+b)|+2-\delta, \tag{2.24}$$

which is a contradiction. Hence we have $\|C_2^P\| = 2|a+b|(a-b)^2+|ab(a+b)|+2$. □

Corollary 2.4. *The operator $C_2 : X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|C_2\| = 2|a+b|(a-b)^2+|ab(a+b)|+2. \tag{2.25}$$

Proof. The result follows from the proof of Theorem 2.3. □

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