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Research Article

Boundedness of Parametrized Littlewood-Paley Operators with Nondoubling Measures

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Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition that there exists a positive constant C such that $\mu(B(x,r)) \leq Cr^n$ for all $x \in \mathbb{R}^d$, $r > 0$ and some fixed $n \in (0,d]$. In this paper, the authors prove that for suitable indexes ρ and λ, the parametrized g_λ^* function $\mathcal{M}_\lambda^{*,\rho}$ is bounded on $L^p(\mu)$ for $p \in [2,\infty)$ with the assumption that the kernel of the operator $\mathcal{M}_\lambda^{*,\rho}$ satisfies some Hörmander-type condition, and is bounded from $L^1(\mu)$ into weak $L^1(\mu)$ with the assumption that the kernel satisfies certain slightly stronger Hörmander-type condition. As a corollary, $\mathcal{M}_\lambda^{*,\rho}$ with the kernel satisfying the above stronger Hörmander-type condition is bounded on $L^p(\mu)$ for $p \in (1,2)$. Moreover, the authors prove that for suitable indexes ρ and λ, $\mathcal{M}_\lambda^{*,\rho}$ is bounded from $L^\infty(\mu)$ into $\mathrm{RBLO}(\mu)$ (the space of regular bounded lower oscillation functions) if the kernel satisfies the Hörmander-type condition, and from the Hardy space $H^1(\mu)$ into $L^1(\mu)$ if the kernel satisfies the above stronger Hörmander-type condition. The corresponding properties for the parametrized area integral $\mathcal{M}_S^\rho$ are also established in this paper.

1. Introduction

Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition that for all $x \in \mathbb{R}^d$ and all $r > 0$:

$$\mu(B(x,r)) \leq C_0 r^n, \tag{1.1}$$

where C_0 and n are positive constants and $n \in (0,d]$, and $B(x,r)$ is the open ball centered at x and having radius r. Such a measure μ may be nondoubling. We recall that a measure μ is said to be doubling, if there is a positive constant C such that for any $x \in \mathrm{supp}(\mu)$ and $r > 0$, $\mu(B(x,2r)) \leq C\mu(B(x,r))$. It is well known that the doubling condition on underlying measures is a key assumption in the classical theory of harmonic analysis. However, in recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid if the underlying measure is a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies (1.1) (see [1–8]). The motivation for developing the analysis with nondoubling measures and some examples of nondoubling measures can be found in [9]. We only point out that the analysis with nondoubling measures played a striking role in solving the long-standing open Painlevé's problem by Tolsa in [10].

Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$. Assume that there exists a positive constant C such that for any $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$|K(x,y)| \leq C|x-y|^{-(n-1)}, \tag{1.2}$$

and for any $x, y, y' \in \mathbb{R}^d$,

$$\int_{|x-y|\geq 2|y-y'|} \left[\left|K(x,y)-K(x,y')\right| + \left|K(y,x)-K(y',x)\right|\right] \frac{1}{|x-y|} d\mu(x) \leq C. \tag{1.3}$$

The parametrized Marcinkiewicz integral $\mathcal{M}^\rho(f)$ associated to the above kernel K and the measure μ as in (1.1) is defined by

$$\mathcal{M}^\rho(f)(x) \equiv \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|x-y|\leq t} \frac{K(x,y)}{|x-y|^{1-\rho}} f(y)d\mu(y)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^d, \tag{1.4}$$

where $\rho \in (0,\infty)$. The parametrized area integral $\mathcal{M}_S^\rho$ and g_λ^* function $\mathcal{M}_\lambda^{*,\rho}$ are defined, respectively, by

$$\mathcal{M}_S^\rho(f)(x) \equiv \left(\int_0^\infty \int_{|y-x|<t} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{K(y,z)}{|y-z|^{1-\rho}} f(z)d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2}, \quad x \in \mathbb{R}^d, \tag{1.5}$$

$$\mathcal{M}_\lambda^{*,\rho}(f)(x) \equiv \left\{\iint_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{K(y,z)}{|y-z|^{1-\rho}} f(z)d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+1}}\right\}^{1/2}, \quad x \in \mathbb{R}^d, \tag{1.6}$$

where $\mathbb{R}_+^{d+1} = \{(y,t) : y \in \mathbb{R}^d, t > 0\}$, $\rho \in (0,\infty)$, and $\lambda \in (1,\infty)$. It is easy to verify that if μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$, and

$$K(x,y) = \frac{\Omega(x-y)}{|x-y|^{d-1}} \tag{1.7}$$

with Ω homogeneous of degree zero and $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ for some $\alpha \in (0,1]$, then K satisfies (1.2) and (1.3). Under these conditions, $\mathcal{M}^\rho$ in (1.4) is just the parametrized Marcinkiewicz integral introduced by Hörmander in [11], and $\mathcal{M}_S^\rho$ and $\mathcal{M}_\lambda^{*,\rho}$ as in (1.5) and (1.6), respectively, are the parametrized area integral and the parametrized g_λ^* function considered by Sakamoto and Yabuta in [12]. We point out that the study of the Littlewood-Paley operators is motivated by their important roles in harmonic analysis and PDE [13, 14]. Since the Littlewood-Paley operators of high dimension were first introduced by Stein in [15], a lot of papers focus on these operators, among them we refer to [16–21] and their references.

When $\rho = 1$, the operator $\mathcal{M}^{\rho}$ as in (1.4) is just the Marcinkiewicz integral with nondoubling measures in [22], where the boundedness of such operator in Lebesgue spaces and Hardy spaces was established under the assumption that $\mathcal{M}^{\rho}$ is bounded on $L^2(\mu)$. Throughout this paper, we always assume that the parametrized Marcinkiewicz integral with nondoubling measures $\mathcal{M}^{\rho}$ as in (1.4) is bounded on $L^2(\mu)$. By a similar argument in [22], it is easy to obtain the boundedness of the parametrized Marcinkiewicz integral $\mathcal{M}^{\rho}$ with $\rho \in (0,\infty)$ from $L^1(\mu)$ into weak $L^1(\mu)$, from the Hardy space $H^1(\mu)$ into $L^1(\mu)$, and from $L^{\infty}(\mu)$ into $\mathrm{RBLO}(\mu)$ (the space of regular bounded lower oscillation functions; see Definition 2.5 below). As a corollary, it is easy to see that $\mathcal{M}^{\rho}$ is bounded on $L^p(\mu)$ with $p \in (1,\infty)$.

The main purpose of this paper is to establish some similar results for the parametrized area integral $\mathcal{M}_S^{\rho}$ and the parametrized g_λ^* function $\mathcal{M}_\lambda^{*,\rho}$ as in (1.5) and (1.6), respectively.

This paper is organized as follows. In the rest of Section 1, we will make some conventions and recall some necessary notation. In Section 2, we will establish the boundedness of $\mathcal{M}_\lambda^{*,\rho}$ as in (1.6) in Lebesgue spaces $L^p(\mu)$ for any $p \in (1,\infty)$. And we will also consider the endpoint estimates for the cases $p = 1$ and $p = \infty$. In Section 3, we will prove that $\mathcal{M}_\lambda^{*,\rho}$ as in (1.6) is bounded from $H^1(\mu)$ into $L^1(\mu)$. And in the last section, the corresponding results for the parametrized area function $\mathcal{M}_S^{\rho}$ as in (1.5) are established.

For a cube $Q \subset \mathbb{R}^d$ we mean a closed cube whose sides parallel to the coordinate axes and we denote its side length by $l(Q)$ and its center by x_Q. Let $\alpha > 1$ and $\beta > \alpha^n$. We say that a cube Q is an (α,β)-doubling cube if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ denotes the cube with the same center as Q and $l(\alpha Q) = \alpha l(Q)$. For definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K_{Q,R} \equiv 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{\left[l(2^k Q)\right]^n}, \tag{1.8}$$

where $N_{Q,R}$ is the smallest positive integer k such that $l(2^k Q) \ge l(R)$ (see [23]).

In what follows, C denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote simply by $A \lesssim B$ if there exists a positive constant C such that $A \le CB$; and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1,\infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2. Boundedness of $\mathcal{M}_\lambda^{*,\rho}$ in Lebesgue spaces

This section is devoted to the behavior of the parametrized g_λ^* function $\mathcal{M}_\lambda^{*,\rho}$ in Lebesgue spaces.

Theorem 2.1. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (1.3)*, and let $\mathcal{M}_\lambda^{*,\rho}$ be as in* (1.6) *with $\rho \in (0,\infty)$ and $\lambda \in (1,\infty)$. Then for any $p \in [2,\infty)$, $\mathcal{M}_\lambda^{*,\rho}$ is bounded on $L^p(\mu)$.*

To obtain the boundedness of $\mathcal{M}_\lambda^{*,\rho}$ in $L^p(\mu)$ with $p \in [1,2)$, we introduce the following condition on the kernel K, that is, for some fixed $\sigma > 2$,

$$\sup_{\substack{r>0, y, y' \in \mathbb{R}^d \\ |y-y'| \le r}} \sum_{l=1}^{\infty} l^{\sigma} \int_{2^l r < |x-y| \le 2^{l+1} r} \left[\left| K(x,y) - K(x,y') \right| + \left| K(y,x) - K(y',x) \right| \right] \frac{1}{|x-y|} d\mu(x) \le C, \tag{2.1}$$

which is slightly stronger than (1.3).

Theorem 2.2. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (2.1)*, and let $\mathcal{M}_\lambda^{*,\rho}$ be as in* (1.6) *with $\rho \in (n/2, \infty)$ and $\lambda \in (2, \infty)$. Then $\mathcal{M}_\lambda^{*,\rho}$ is bounded from $L^1(\mu)$ into weak $L^1(\mu)$, namely, there exists a positive constant C such that for any $\beta > 0$ and any $f \in L^1(\mu)$,*

$$\mu\left(\left\{x \in \mathbb{R}^d : \mathcal{M}_\lambda^{*,\rho}(f)(x) > \beta\right\}\right) \le \frac{C}{\beta} \|f\|_{L^1(\mu)}. \tag{2.2}$$

By the Marcinkiewicz interpolation theorem, and Theorems 2.1 and 2.2, we can immediately obtain the $L^p(\mu)$-boundedness of the operator $\mathcal{M}_\lambda^{*,\rho}$ for $p \in (1,2)$.

Corollary 2.3. *Under the same assumption of Theorem 2.2, $\mathcal{M}_\lambda^{*,\rho}$ is bounded on $L^p(\mu)$ for any $p \in (1,2)$.*

Remark 2.4. We point out that it is still unclear if condition (2.1) in Theorem 2.2 and Corollary 2.3 can be weakened.

Now we turn to discuss the property of the operator $\mathcal{M}_\lambda^{*,\rho}$ in $L^\infty(\mu)$. To this end, we need to recall the definition of the space RBLO(μ) (the space of regular bounded lower oscillation functions).

Definition 2.5. Let $\eta \in (1, \infty)$. A μ-locally integrable function f on $\mathbb{R}^d$ is said to be in the space RBLO(μ) if there exists a positive constant C such that for any (η, η^{d+1})-doubling cube Q,

$$m_Q(f) - \operatorname*{ess\,inf}_{x \in Q} f(x) \le C, \tag{2.3}$$

and for any two (η, η^{d+1})-doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \le C K_{Q,R}. \tag{2.4}$$

The minimal constant C as above is defined to be the norm of f in the space RBLO(μ) and denoted by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Remark 2.6. The space RBLO(μ) was introduced by Jiang in [24], where the (η, η^{d+1})-doubling cube was replaced by $(4\sqrt{d}, (4\sqrt{d})^{n+1})$-doubling cube. It was pointed out in [25] that it is convenient in applications to replace $(4\sqrt{d}, (4\sqrt{d})^{n+1})$-doubling cubes by (η, η^{d+1})-doubling cubes with $\eta \in (1, \infty)$ in the definition of RBLO(μ). Moreover, it was proved in [25] that the definition is independent of the choices of the constant $\eta \in (1, \infty)$. The space RBLO(μ) is a subspace of RBMO(μ) which was introduced by Tolsa in [23].

Theorem 2.7. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (1.3)*, and let $\mathcal{M}_\lambda^{*,\rho}$ be as in* (1.6) *with $\rho \in (0,\infty)$ and $\lambda \in (1,\infty)$. Then for any $f \in L^\infty(\mu)$, $\mathcal{M}_\lambda^{*,\rho}(f)$ is either infinite everywhere or finite almost everywhere. More precisely, if $\mathcal{M}_\lambda^{*,\rho}(f)$ is finite at some point $x_0 \in \mathbb{R}^d$, then $\mathcal{M}_\lambda^{*,\rho}(f)$ is finite almost everywhere and*

$$\left\|\mathcal{M}_\lambda^{*,\rho}(f)\right\|_{\mathrm{RBLO}(\mu)} \le C\|f\|_{L^\infty(\mu)}, \tag{2.5}$$

where the positive constant C is independent of f.

We point out that Theorem 2.7 is also new even when μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

In the rest part of Section 2, we will prove Theorems 2.1, 2.2, and 2.7, respectively. To prove Theorem 2.1, we first recall some basic facts and establish a technical lemma. For $\eta > 1$, let $M_{(\eta)}$ be the noncentered maximal operator defined by

$$M_{(\eta)}f(x) \equiv \sup_{\substack{Q \ni x \\ Q\,\text{cube}}} \frac{1}{\mu(\eta Q)}\int_Q |f(y)|\,d\mu(y), \quad x \in \mathbb{R}^d. \tag{2.6}$$

It is well known that $M_{(\eta)}$ is bounded on $L^p(\mu)$ provided that $p \in (1,\infty)$ (see [23]). The following lemma which is of independent interest plays an important role in our proofs.

Lemma 2.8. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (1.3)*, and $\eta \in (1,\infty)$. Let $\mathcal{M}^\rho$ be as in* (1.4) *and $\mathcal{M}_\lambda^{*,\rho}$ be as in* (1.6) *with $\rho \in (0,\infty)$ and $\lambda \in (1,\infty)$. Then for any nonnegative function ϕ, there exists a positive constant C such that for all $f \in L^p(\mu)$ with $p \in (1,\infty)$,*

$$\int_{\mathbb{R}^d} \left[\mathcal{M}_\lambda^{*,\rho}(f)(x)\right]^2 \phi(x)\,d\mu(x) \le C\int_{\mathbb{R}^d} \left[\mathcal{M}^\rho(f)(x)\right]^2 M_{(\eta)}(\phi)(x)\,d\mu(x). \tag{2.7}$$

Proof. Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left[\mathcal{M}_\lambda^{*,\rho}(f)(x)\right]^2 \phi(x)\,d\mu(x) \\
&\quad= \int_{\mathbb{R}^d}\iint_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\rho}\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} f(z)\,d\mu(z)\right|^2 \frac{d\mu(y)\,dt}{t^{n+1}}\phi(x)\,d\mu(x) \\
&\quad\le \int_{\mathbb{R}^d}\int_0^\infty \left|\frac{1}{t^\rho}\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} f(z)\,d\mu(z)\right|^2 \frac{dt}{t} \sup_{t>0}\left[\int_{\mathbb{R}^d} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{\phi(x)}{t^n}\,d\mu(x)\right] d\mu(y) \\
&\quad= \int_{\mathbb{R}^d} \left[\mathcal{M}^\rho(f)(y)\right]^2 \sup_{t>0}\left[\int_{\mathbb{R}^d} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{\phi(x)}{t^n}\,d\mu(x)\right] d\mu(y).
\end{aligned} \tag{2.8}$$

Thus, to prove Lemma 2.8, it suffices to verify that for any $y \in \mathbb{R}^d$,

$$\sup_{t>0}\int_{\mathbb{R}^d} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{\phi(x)}{t^n}\,d\mu(x) \lesssim M_{(\eta)}(\phi)(y). \tag{2.9}$$

For any fixed $y \in \mathbb{R}^d$ and $t > 0$, write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{\phi(x)}{t^n} d\mu(x) \\
&\quad = \int_{|x-y|\le t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{\phi(x)}{t^n} d\mu(x) + \int_{|x-y|> t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{\phi(x)}{t^n} d\mu(x) \\
&\quad \equiv \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned} \tag{2.10}$$

Let Q_y be the cube with center at y and side length $l(Q_y) = 2t$. Obviously, $\{x : |x-y| < t\} \subset Q_y$, which leads to

$$\mathrm{E}_1 \le \int_{|x-y|\le t} \frac{\phi(x)}{t^n} d\mu(x) \lesssim \frac{1}{\mu(\eta Q_y)} \int_{Q_y} \phi(x) d\mu(x) \lesssim M_{(\eta)}\phi(y). \tag{2.11}$$

As for E_2, a straightforward computation proves that

$$\begin{aligned}
\mathrm{E}_2 &\le \sum_{k=0}^{\infty} \int_{2^k t<|x-y|\le 2^{k+1}t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{\phi(x)}{t^n} d\mu(x) \\
&\lesssim \sum_{k=0}^{\infty} \left(\frac{1}{2^k} \right)^{n\lambda} \left(2^{k+1}t\right)^n \frac{1}{\left(2^{k+1}t\right)^n} \int_{|x-y|\le 2^{k+1}t} \phi(x) d\mu(x) \\
&\lesssim M_{(\eta)}(\phi)(y).
\end{aligned} \tag{2.12}$$

Combining the estimates for E_1 and E_2 yields (2.9), which completes the proof of Lemma 2.8. □

Proof of Theorem 2.1. For the case of $p = 2$, choosing $\phi(x) = 1$ in Lemma 2.8, then we easily obtain that

$$\int_{\mathbb{R}^d} \left[\mathcal{M}_{\lambda}^{*,\rho}(f)(x)\right]^2 d\mu(x) \lesssim \int_{\mathbb{R}^d} \left[\mathcal{M}^{\rho}(f)(x)\right]^2 d\mu(x), \tag{2.13}$$

which, along with the boundedness of $\mathcal{M}^{\rho}$ in $L^2(\mu)$, immediately yields that Theorem 2.1 holds in this case.

For the case of $p \in (2, \infty)$, let q be the index conjugate to $p/2$. Then from Lemma 2.8 and the Hölder inequality, it follows that

$$\begin{aligned}
\left\|\mathcal{M}_{\lambda}^{*,\rho}(f)\right\|_{L^p(\mu)}^2 &= \sup_{\phi\ge 0,\, \|\phi\|_{L^q(\mu)}\le 1} \int_{\mathbb{R}^d} \left[\mathcal{M}_{\lambda}^{*,\rho}(f)(x)\right]^2 \phi(x) d\mu(x) \\
&\lesssim \sup_{\phi\ge 0,\, \|\phi\|_{L^q(\mu)}\le 1} \int_{\mathbb{R}^d} \left[\mathcal{M}^{\rho}(f)(x)\right]^2 M_{(\eta)}\phi(x) d\mu(x) \\
&\lesssim \left\|\mathcal{M}^{\rho}(f)\right\|_{L^p(\mu)}^2 \sup_{\phi\ge 0,\, \|\phi\|_{L^q(\mu)}\le 1} \left\|M_{(\eta)}\phi\right\|_{L^q(\mu)} \\
&\lesssim \|f\|_{L^p(\mu)}^2 \sup_{\phi\ge 0,\, \|\phi(x)\|_{L^q(\mu)}\le 1} \|\phi\|_{L^q(\mu)} \\
&\lesssim \|f\|_{L^p(\mu)}^2,
\end{aligned} \tag{2.14}$$

which completes the proof of Theorem 2.1. □

To prove Theorem 2.2, we need the following Calderón-Zygmund decomposition with nondoubling measures (see [23] or [26]).

Lemma 2.9. *Let $p \in [1, \infty)$. For any $f \in L^p(\mu)$ and $\lambda > 0$ ($\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$), one has the following.*

(a) *There exists a family of almost disjoint cubes $\{Q_j\}_j$ (i.e., $\sum_j \chi_{Q_j} \le C$) such that*

$$\frac{1}{\mu(2Q_j)}\int_{Q_j}|f(x)|^p d\mu(x) > \frac{\lambda^p}{2^{d+1}},$$
$$\frac{1}{\mu(2\eta Q_j)}\int_{\eta Q_j}|f(x)|^p d\mu(x) \le \frac{\lambda^p}{2^{d+1}} \quad \forall \eta > 2, \tag{2.15}$$
$$|f(x)| \le \lambda \quad \mu\text{-a.e. on } \mathbb{R}^d \setminus \cup_j Q_j.$$

(b) *For each j, let R_j be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_j$, $k \in \mathbb{N}$, and let $\omega_j = \chi_{Q_j}/\sum_k \chi_{Q_k}$. Then, there exists a family of functions φ_j with $\operatorname{supp}(\varphi_j) \subset R_j$ satisfying*

$$\int_{\mathbb{R}^d}\varphi_j(x)d\mu(x) = \int_{Q_j} f(x)\omega_j(x)d\mu(x), \qquad \sum_j|\varphi_j(x)| \le B\lambda \tag{2.16}$$

(where B is some constant), and when $p = 1$,

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(R_j) \le C\int_{Q_j}|f(x)|d\mu(x); \tag{2.17}$$

when $p \in (1, \infty)$,

$$\left[\int_{R_j}|\varphi_j(x)|^p d\mu(x)\right]^{1/p}[\mu(R_j)]^{1/p'} \le \frac{C}{\lambda^{p-1}}\int_{Q_j}|f(x)|^p d\mu(x). \tag{2.18}$$

Remark 2.10. From the proof of the Calderón-Zygmund decomposition with nondoubling measures (see [23] or [26]), it is easy to see that if we replace R_j with R_j', the smallest $(6\sqrt{d}, (6\sqrt{d})^{n+1})$-doubling cube of the form $(6\sqrt{d})^k Q_j$ ($k \in \mathbb{N}$), the above conclusions (a) and (b) still hold. Here and hereafter, when we mention R_j in Lemma 2.9 we always mean R_j'.

Proof of Theorem 2.2. Let $f \in L^1(\mu)$ and $\beta > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ (note that if $0 < \beta \le 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$, the estimate (2.2) obviously holds). Applying Lemma 2.9 to f at the level β, we obtain $f(x) \equiv g(x) + b(x)$ with

$$g(x) \equiv f(x)\chi_{\mathbb{R}^d\setminus\bigcup_j Q_j}(x) + \sum_j \varphi_j(x), \qquad b(x) \equiv \sum_j[\omega_j(x)f(x) - \varphi_j(x)] = \sum_j b_j(x), \tag{2.19}$$

where ω_j, φ_j, Q_j, and R_j are the same as in Lemma 2.9. It is easy to see that $\|g\|_{L^\infty(\mu)} \lesssim \beta$ and $\|g\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}$. By the boundedness of $\mathcal{M}_\lambda^{*,\rho}$ in $L^2(\mu)$, we easily obtain that

$$\mu(\{x \in \mathbb{R}^d : \mathcal{M}_\lambda^{*,\rho}(g)(x) > \beta\}) \le \beta^{-2}\|\mathcal{M}_\lambda^{*,\rho}(g)\|_{L^2(\mu)}^2 \lesssim \beta^{-1}\|f\|_{L^1(\mu)}. \tag{2.20}$$

From (a) of Lemma 2.9, it follows that

$$\mu(\cup_j 2Q_j) \lesssim \beta^{-1}\sum_j \int_{Q_j} |f(x)|d\mu(x) \lesssim \beta^{-1}\int_{\mathbb{R}^d} |f(x)|d\mu(x), \tag{2.21}$$

and therefore, the proof of Theorem 2.2 can be deduced to proving that

$$\mu(\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : \mathcal{M}_\lambda^{*,\rho}(b)(x) > \beta\}) \lesssim \beta^{-1}\int_{\mathbb{R}^d} |f(x)|d\mu(x). \tag{2.22}$$

For each fixed j, let $R_j^* = 6\sqrt{d}R_j$. Notice that

$$\begin{aligned} &\mu(\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : \mathcal{M}_\lambda^{*,\rho}(b)(x) > \beta\}) \\ &\quad \le \beta^{-1}\left\{\sum_j \int_{\mathbb{R}^d\setminus R_j^*} \mathcal{M}_\lambda^{*,\rho}(b_j)(x)d\mu(x) + \sum_j \int_{R_j^*\setminus 2Q_j} \mathcal{M}_\lambda^{*,\rho}(b_j)(x)d\mu(x)\right\}. \end{aligned} \tag{2.23}$$

Thus, it suffices to prove that for each fixed j,

$$\int_{\mathbb{R}^d\setminus R_j^*} \mathcal{M}_\lambda^{*,\rho}(b_j)(x)d\mu(x) \lesssim \int_{Q_j} |f(x)|d\mu(x), \tag{2.24}$$

$$\int_{R_j^*\setminus 2Q_j} \mathcal{M}_\lambda^{*,\rho}(b_j)(x)d\mu(x) \lesssim \int_{Q_j} |f(x)|d\mu(x). \tag{2.25}$$

To verify (2.24), for each fixed j, let $B_j = B(x_{Q_j}, 2\sqrt{d}l(R_j))$, and write

$$\begin{aligned} &\int_{\mathbb{R}^d\setminus R_j^*} \mathcal{M}_\lambda^{*,\rho}(b_j)(x)d\mu(x) \\ &\quad \le \int_{\mathbb{R}^d\setminus R_j^*} \left[\iint_{|y-x|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|\le t} \frac{K(y,z)b_j(z)}{|y-z|^{1-\rho}}d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2} d\mu(x) \\ &\qquad + \int_{\mathbb{R}^d\setminus R_j^*} \left[\iint_{\substack{|y-x|\ge t\\ y\in B_j}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|\le t} \frac{K(y,z)b_j(z)}{|y-z|^{1-\rho}}d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2} d\mu(x) \\ &\qquad + \int_{\mathbb{R}^d\setminus R_j^*} \left[\iint_{\substack{|y-x|\ge t\\ y\in \mathbb{R}^d\setminus B_j}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|\le t} \frac{K(y,z)b_j(z)}{|y-z|^{1-\rho}}d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2} d\mu(x) \\ &\quad \equiv \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3. \end{aligned} \tag{2.26}$$

For each fixed j, further decompose

$$\begin{aligned} \mathrm{F}_1 &\le \int_{\mathbb{R}^d\setminus R_j^*} \left[\iint_{\substack{|y-x|<t\\ y\in 4R_j}} \left|\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} b_j(z)d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2} d\mu(x) \\ &\quad + \int_{\mathbb{R}^d\setminus R_j^*} \left[\iint_{\substack{|y-x|<t\\ y\in \mathbb{R}^d\setminus 4R_j}} \left|\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} b_j(z)d\mu(z)\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2} d\mu(x) \\ &\equiv \mathrm{H}_1 + \mathrm{H}_2. \end{aligned} \tag{2.27}$$

It is easy to see that for any $x \in \mathbb{R}^d \setminus R_j^*$, $y \in 4R_j$ with $|y-x| < t$ and $z \in R_j$, $|x - x_{Q_j}| - 2\sqrt{d}l(R_j) \le |x-y| < t$ and $|y-z| < 4\sqrt{d}l(R_j)$. This fact along the Minkowski inequality and (1.2) leads to

$$\begin{aligned}
\mathrm{H}_1 &\le \int_{\mathbb{R}^d \setminus R_j^*} \int_{R_j} |b_j(z)| \left[\iint_{\substack{|y-x|<t \\ |y-z|\le t \\ y \in 4R_j}} \frac{|K(y,z)|^2}{|y-z|^{2-2\rho}} \frac{d\mu(y)dt}{t^{n+2\rho+1}} \right]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{\mathbb{R}^d \setminus R_j^*} \int_{R_j} |b_j(z)| \left[\int_{|y-z|<4\sqrt{d}l(R_j)} \frac{1}{|y-z|^{2n-2\rho}} \right. \\
&\qquad\qquad \left. \times \left(\int_{|x-x_{Q_j}|-2\sqrt{d}l(R_j)}^{\infty} \frac{dt}{t^{n+2\rho+1}} \right) d\mu(y) \right]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{R_j} |b_j(z)| \left[\int_{|y-z|<4\sqrt{d}l(R_j)} \frac{1}{|y-z|^{2n-2\rho}} d\mu(y) \right]^{1/2} d\mu(z) \int_{\mathbb{R}^d \setminus R_j^*} \frac{1}{|x-x_{Q_j}|^{(n+2\rho)/2}} d\mu(x) \\
&\lesssim \|b_j\|_{L^1(\mu)}.
\end{aligned} \tag{2.28}$$

As for H_2, first write

$$\begin{aligned}
\mathrm{H}_2 &\le \int_{\mathbb{R}^d \setminus R_j^*} \left[\iint_{\substack{|y-x|<t,\, y \in \mathbb{R}^d \setminus 4R_j \\ t \le |y-x_{Q_j}|+2\sqrt{d}l(R_j)}} \left| \int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} b_j(z)d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}} \right]^{1/2} d\mu(x) \\
&\quad + \int_{\mathbb{R}^d \setminus R_j^*} \left[\iint_{\substack{|y-x|<t,\, y \in \mathbb{R}^d \setminus 4R_j \\ t > |y-x_{Q_j}|+2\sqrt{d}l(R_j)}} \left| \int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} b_j(z)d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}} \right]^{1/2} d\mu(x) \\
&\equiv \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned} \tag{2.29}$$

Notice that for any $z \in R_j$, $x \in \mathbb{R}^d \setminus R_j^*$ and $y \in \mathbb{R}^d \setminus 4R_j$, $|y-z| \sim |y - x_{Q_j}|$, and $|x - x_{Q_j}| < 5\sqrt{d}|y - x_{Q_j}|$. Thus, by (1.2) and the Minkowski inequality, we obtain that

$$\begin{aligned}
\mathrm{J}_1 &\lesssim \int_{\mathbb{R}^d \setminus R_j^*} \int_{R_j} |b_j(z)| \left[\int_{\mathbb{R}^d \setminus 4R_j} \frac{1}{|y-z|^{2n-2\rho}} \left(\int_{|y-z|}^{|y-x_{Q_j}|+2\sqrt{d}l(R_j)} \frac{dt}{t^{n+2\rho+1}} \right) d\mu(y) \right]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{\mathbb{R}^d \setminus R_j^*} \int_{R_j} |b_j(z)| \left[\int_{\mathbb{R}^d \setminus 4R_j} \frac{1}{|y-x_{Q_j}|^{n+1/2}} \frac{l(R_j)}{|y-x_{Q_j}|^{2n+1/2}} d\mu(y) \right]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{R_j} |b_j(z)| \left[\int_{\mathbb{R}^d \setminus 4R_j} \frac{l(R_j)^{1/2}}{|y-x_{Q_j}|^{n+1/2}} d\mu(y) \right]^{1/2} d\mu(z) \int_{\mathbb{R}^d \setminus R_j^*} \frac{l(R_j)^{1/4}}{|x-x_{Q_j}|^{n+1/4}} d\mu(x) \\
&\lesssim \|b_j\|_{L^1(\mu)}.
\end{aligned} \tag{2.30}$$

On the other hand, it is easy to verify that for any $y \in \mathbb{R}^d \setminus 4R_j$ and $t > |y - x_{Q_j}| + 2\sqrt{d}l(R_j)$, $R_j \subset \{z : |y - z| \leq t\}$ and $|x - x_{Q_j}| < 2t$. Choose $0 < \epsilon < \min\{1/2, (\lambda - 2)n/2, \rho - n/2, \sigma/2 - 1\}$ (we always take ϵ to satisfy this restriction in our proof). The vanishing moment of b_j on R_j and the Minkowski inequality give us that

$$\begin{aligned}
\mathrm{J}_2 &= \int_{\mathbb{R}^d \setminus R_j^*} \left\{ \iint_{\substack{|y-x|<t, y \in \mathbb{R}^d \setminus 4R_j \\ t > |y - x_{Q_j}| + 2\sqrt{d}l(R_j)}} \left| \int_{|y-z| \leq t} \left[\frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y, x_{Q_j})}{\left|y - x_{Q_j}\right|^{1-\rho}} \right] b_j(z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}} \right\}^{1/2} d\mu(x) \\
&\leq \int_{\mathbb{R}^d \setminus R_j^*} \int_{R_j} \left|b_j(z)\right| \left\{ \int_{\mathbb{R}^d \setminus 4R_j} \left| \frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y, x_{Q_j})}{\left|y - x_{Q_j}\right|^{1-\rho}} \right|^2 \right. \\
&\qquad \left. \times \left[\int_{|y - x_{Q_j}| + 2\sqrt{d}l(R_j)}^{\infty} \frac{\left[\log\left(t/l(R_j)\right)\right]^{2+2\epsilon} dt}{t^{2\rho-n+1} t^{2n} \left[\log\left(t/l(R_j)\right)\right]^{2+2\epsilon}} \right] d\mu(y) \right\}^{1/2} d\mu(z) d\mu(x) \\
&\lesssim \int_{\mathbb{R}^d \setminus R_j^*} \frac{1}{|x - x_{Q_j}|^n \left[\log\left(\left|x - x_{Q_j}\right| / l(R_j)\right)\right]^{1+\epsilon}} \int_{R_j} \left|b_j(z)\right| \\
&\qquad \times \left\{ \int_{\mathbb{R}^d \setminus 4R_j} \left| \frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y, x_{Q_j})}{\left|y - x_{Q_j}\right|^{1-\rho}} \right|^2 \right. \\
&\qquad \left. \times \left[\int_{|y - x_{Q_j}| + 2\sqrt{d}l(R_j)}^{\infty} \frac{\left[\log\left(t/l(R_j)\right)\right]^{2+2\epsilon}}{t^{2\rho-n+1}} dt \right] d\mu(y) \right\}^{1/2} d\mu(z) d\mu(x).
\end{aligned} \tag{2.31}$$

It follows from [27, Lemma 2.2] that for any $y \in \mathbb{R}^d \setminus 4R_j$,

$$\int_{|y - x_{Q_j}| + 2\sqrt{d}l(R_j)}^{\infty} \frac{\left[\log\left(t/l(R_j)\right)\right]^{2+2\epsilon}}{t^{2\rho-n+1}} dt \lesssim \frac{\left[\log\left(\left|y - x_{Q_j}\right| / l(R_j) + 2\sqrt{d}\right)\right]^{2+2\epsilon}}{\left[\left|y - x_{Q_j}\right| + 2\sqrt{d}l(R_j)\right]^{2\rho-n}}, \tag{2.32}$$

which, together with (2.1), leads to

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \int_{\mathbb{R}^d \setminus R_j^*} \frac{1}{\left|x - x_{Q_j}\right|^n \left[\log\left(\left|x - x_{Q_j}\right| / l(R_j)\right)\right]^{1+\epsilon}} \\
&\qquad \times \int_{R_j} \left|b_j(z)\right| \left[\int_{\mathbb{R}^d \setminus 4R_j} \left| \frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y, x_{Q_j})}{\left|y - x_{Q_j}\right|^{1-\rho}} \right|^2 \right. \\
&\qquad \left. \times \frac{\left[\log\left(\left|y - x_{Q_j}\right| / l(R_j) + 2\sqrt{d}\right)\right]^{2+2\epsilon}}{\left[\left|y - x_{Q_j}\right| + 2\sqrt{d}l(R_j)\right]^{2\rho-n}} d\mu(y) \right]^{1/2} d\mu(z) d\mu(x)
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^d\setminus R_j^*} \frac{1}{\left(\left|x-x_{Q_j}\right|\right)^n \left[\log\left(\left|x-x_{Q_j}\right|/l(R_j)\right)\right]^{1+\epsilon}}$$
$$\times \int_{R_j} \left|b_j(z)\right| \Bigg\{ \sum_{k=1}^{\infty} \frac{(k+1)^{2+2\epsilon}}{\left[2^k l(R_j)\right]^{2\rho-n}}$$
$$\times \Bigg[\int_{2^k l(R_j)\le|y-x_{Q_j}|<2^{k+1}l(R_j)} \frac{\left|K(y,z)-K(y,x_{Q_j})\right|^2}{|y-z|^{2-2\rho}}$$
$$+\left|K(y,x_{Q_j})\right|^2 \left| \frac{1}{|y-z|^{1-\rho}} - \frac{1}{\left|y-x_{Q_j}\right|^{1-\rho}} \right|^2 d\mu(y) \Bigg] \Bigg\}^{1/2} d\mu(z)\,d\mu(x)$$
$$\lesssim \int_{\mathbb{R}^d\setminus R_j^*} \frac{1}{\left(\left|x-x_{Q_j}\right|\right)^n \left[\log\left(\left|x-x_{Q_j}\right|/l(R_j)\right)\right]^{1+\epsilon}}$$
$$\times \int_{R_j} \left|b_j(z)\right| \Bigg\{ \sum_{k=1}^{\infty} \frac{(k+1)^{2+2\epsilon}}{\left[2^k l(R_j)\right]^{2\rho-n}}$$
$$\times \Bigg[\int_{2^k l(R_j)\le|y-x_{Q_j}|<2^{k+1}l(R_j)} \frac{1}{\left[2^k l(R_j)\right]^{n-2\rho}} \frac{\left|K(y,z)-K(y,x_{Q_j})\right|}{|y-z|}$$
$$+\frac{l(R_j)^2}{\left|y-x_{Q_j}\right|^{2n-2\rho+2}} d\mu(y) \Bigg] \Bigg\}^{1/2} d\mu(z)\,d\mu(x)$$
$$\lesssim \int_{\mathbb{R}^d\setminus R_j^*} \frac{1}{\left|x-x_{Q_j}\right|^n \left[\log\left(\left|x-x_{Q_j}\right|/l(R_j)\right)\right]^{1+\epsilon}} \int_{R_j} \left|b_j(z)\right| \left[1+\sum_{k=1}^{\infty}\frac{(k+1)^{2+2\epsilon}}{2^{2k}}\right]^{1/2} d\mu(z)\,d\mu(x)$$
$$\lesssim \|b_j\|_{L^1(\mu)}. \tag{2.33}$$

Combining the estimates for H_1, J_1, and J_2 yields

$$F_1 \lesssim \|b_j\|_{L^1(\mu)} \lesssim \int_{Q_j} |f(x)|\,d\mu(x). \tag{2.34}$$

To estimate F_2, first notice that for any $y\in B_j$, $x\in\mathbb{R}^d\setminus R_j^*$, and $z\in R_j$, $|y-x|\ge|x-x_{Q_j}|/2$, $|y-z|\le 4\sqrt{d}\,l(R_j)$, and $|x-y|\sim|x-x_{Q_j}|$. Thus, by the Minkowski inequality and (1.2), we easily obtain that

$$F_2 \le \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} \left|b_j(z)\right| \left[\iint_{\substack{|y-x|\ge t\\ |y-z|\le t\\ y\in B_j}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon} \frac{\left|K(y,z)\right|^2}{|y-z|^{2-2\rho}} \frac{d\mu(y)\,dt}{t^{n+2\rho+1}} \right]^{1/2} d\mu(z)\,d\mu(x)$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j}\left|b_j(z)\right|\left[\int_{|y-z|\le 4\sqrt{d}l(R_j)}\frac{1}{\left|x-x_{Q_j}\right|^{2n+2\epsilon}|y-z|^{n-\epsilon}}\right.\\
&\qquad\qquad\left.\times\left(\int_0^{|y-x|}t^{\epsilon-1}dt\right)d\mu(y)\right]^{1/2}d\mu(z)d\mu(x)\\
&\lesssim \int_{R_j}\left|b_j(z)\right|\left[\int_{|y-z|\le 4\sqrt{d}l(R_j)}\frac{1}{|y-z|^{n-\epsilon}}d\mu(y)\right]^{1/2}d\mu(z)\int_{\mathbb{R}^d\setminus R_j^*}\frac{1}{\left|x-x_{Q_j}\right|^{n+\epsilon/2}}d\mu(x)\\
&\lesssim \|b_j\|_{L^1(\mu)}\\
&\lesssim \int_{Q_j}\left|f(x)\right|d\mu(x).
\end{aligned}\tag{2.35}$$

It remains to estimate F_3. By (1.2), we can write

$$\begin{aligned}
\mathrm{F}_3 \lesssim &\int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j}\left|b_j(z)\right|\left[\iint_{\substack{|y-z|\le t\le|y-x|,\, y\in\mathbb{R}^d\setminus B_j\\ t\le|y-x_{Q_j}|+C_\epsilon l(R_j)\\ |x-x_{Q_j}|\le 2|y-x_{Q_j}|}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2n-2\rho}}\frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}d\mu(z)d\mu(x)\\
&+\int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j}\left|b_j(z)\right|\left[\iint_{\substack{|y-z|\le t\le|y-x|,\, y\in\mathbb{R}^d\setminus B_j\\ t\le|y-x_{Q_j}|+C_\epsilon l(R_j)\\ |x-x_{Q_j}|>2|y-x_{Q_j}|}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2n-2\rho}}\frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}d\mu(z)d\mu(x)\\
&+\int_{\mathbb{R}^d\setminus R_j^*}\left[\iint_{\substack{t\le|y-x|,\, y\in\mathbb{R}^d\setminus B_j\\ |y-x_{Q_j}|+C_\epsilon l(R_j)<t}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|\le t}\frac{K(y,z)}{|y-z|^{1-\rho}}b_j(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}d\mu(x)\\
\equiv\ &\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3,
\end{aligned}\tag{2.36}$$

where $C_\epsilon = 8\sqrt{d}e^{(2+2\epsilon)/\epsilon}$. Note that for any $y\in\mathbb{R}^d\setminus B_j$ and $z\in R_j$ with $|y-z|\le t\le|y-x|$, then $|y-z|\sim|y-x_{Q_j}|$ and $|y-x_{Q_j}|\le t+\sqrt{d}l(R_j)$. Consequently,

$$\begin{aligned}
\mathrm{L}_1 \le &\int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j}\left|b_j(z)\right|\left[\int_{\substack{y\in\mathbb{R}^d\setminus B_j\\ |x-x_{Q_j}|\le 2|y-x_{Q_j}|}}\left(\int_{|y-x_{Q_j}|-\sqrt{d}l(R_j)}^{|y-x_{Q_j}|+C_\epsilon l(R_j)}\frac{dt}{t^{n+2\rho+1}}\right)\right.\\
&\qquad\qquad\left.\times\frac{1}{|y-z|^{2n-2\rho}}d\mu(y)\right]^{1/2}d\mu(z)d\mu(x)\\
&\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j}\left|b_j(z)\right|\left[\int_{\substack{y\in\mathbb{R}^d\setminus B_j\\ |x-x_{Q_j}|\le 2|y-x_{Q_j}|}}\frac{l(R_j)}{\left|y-x_{Q_j}\right|^{3n+1}}d\mu(y)\right]^{1/2}d\mu(z)d\mu(x)\\
&\lesssim \|b_j\|_{L^1(\mu)}.
\end{aligned}\tag{2.37}$$

A trivial computation involving the fact that $|x-y| > |x-x_{Q_j}|/2$ for any $x \in \mathbb{R}^d \setminus R_j^*$ and $y \in \mathbb{R}^d \setminus B_j$ satisfying $|x-x_{Q_j}| > 2|y-x_{Q_j}|$ proves that

$$\begin{aligned}
\mathrm{L}_2 &\leq \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} |b_j(z)| \Bigg[\int_{\substack{y\in\mathbb{R}^d\setminus B_j\\ |x-x_{Q_j}|>2|y-x_{Q_j}|}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon} \frac{1}{|y-z|^{2n-2\rho}} \\
&\qquad \times \left(\int_{|y-x_{Q_j}|-\sqrt{d}l(R_j)}^{|y-x_{Q_j}|+C_\epsilon l(R_j)} \frac{dt}{t^{n+2\rho+1}}\right) d\mu(y)\Bigg]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} |b_j(z)| \Bigg[\int_{\substack{y\in\mathbb{R}^d\setminus B_j\\ |x-x_{Q_j}|>2|y-x_{Q_j}|}} \left(\int_{|y-x_{Q_j}|-\sqrt{d}l(R_j)}^{|y-x_{Q_j}|+C_\epsilon l(R_j)} \frac{1}{t^{2\rho-n-2\epsilon+1}} dt\right) \\
&\qquad \times \frac{1}{|x-y|^{2n+2\epsilon}|y-z|^{2n-2\rho}} d\mu(y)\Bigg]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} |b_j(z)| \left[\int_{\mathbb{R}^d\setminus B_j} \frac{1}{|y-z|^{n-2\epsilon+1}} \frac{l(R_j)}{|x-x_{Q_j}|^{2n+2\epsilon}} d\mu(y)\right]^{1/2} d\mu(z)d\mu(x) \\
&\lesssim \|b_j\|_{L^1(\mu)}.
\end{aligned} \tag{2.38}$$

Finally, let us estimate L_3. It is easy to see that for any $y \in \mathbb{R}^d \setminus B_j$ and $t > |y-x_{Q_j}| + C_\epsilon l(R_j)$, $R_j \subset \{z : |y-z| \leq t\}$ and $t + |x-y| \geq |x-x_{Q_j}| + C_\epsilon l(R_j)$. Thus, from the vanishing moment of b_j on R_j, ϵit follows that

$$\begin{aligned}
\mathrm{L}_3 &\leq \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} |b_j(z)| \Bigg[\iint_{\substack{y\in\mathbb{R}^d\setminus B_j\\ t>|y-x_{Q_j}|+C_\epsilon l(R_j)\\ |y-z|\leq t\leq|y-x|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\qquad \times \left|\frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y,x_{Q_j})}{|y-x_{Q_j}|^{1-\rho}}\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\Bigg]^{1/2} d\mu(z)d\mu(x) \\
&= \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} |b_j(z)| \Bigg[\iint_{\substack{y\in\mathbb{R}^d\setminus B_j\\ t>|y-x_{Q_j}|+C_\epsilon l(R_j)\\ |y-z|\leq t\leq|y-x|}} \frac{t^{\lambda n}}{(t+|x-y|)^{2n}\left[\log\left((t+|x-y|)/l(R_j)\right)\right]^{2+2\epsilon}} \\
&\qquad \times \frac{\left[\log\left((t+|x-y|)/l(R_j)\right)\right]^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}} \\
&\qquad \times \left|\frac{K(y,z)}{|y-z|^{1-\rho}} - \frac{K(y,x_{Q_j})}{|y-x_{Q_j}|^{1-\rho}}\right|^2 \frac{d\mu(y)dt}{t^{n+2\rho+1}}\Bigg]^{1/2} d\mu(z)d\mu(x)
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} \frac{|b_j(z)|}{\left(\left|x-x_{Q_j}\right|+C_\epsilon l(R_j)\right)^n\left[\log\left(\left(\left|x-x_{Q_j}\right|+C_\epsilon l(R_j)\right)/l(R_j)\right)\right]^{1+\epsilon}}$$
$$\times\left[\int_{\substack{y\in\mathbb{R}^d\setminus B_j\\ |y-x|\geq|y-x_{Q_j}|+C_\epsilon l(R_j)}}\left|\frac{K(y,z)}{|y-z|^{1-\rho}}-\frac{K(y,x_{Q_j})}{\left|y-x_{Q_j}\right|^{1-\rho}}\right|^2\right.$$
$$\times\int_{|y-x_{Q_j}|+C_\epsilon l(R_j)}^{|y-x|}\frac{t^{\lambda n}\left[\log\left((t+|x-y|)/l(R_j)\right)\right]^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}}$$
$$\left.\times\frac{1}{t^{n+2\rho+1}}dt\,d\mu(y)\right]^{1/2}d\mu(z)d\mu(x)$$
$$\lesssim \int_{\mathbb{R}^d\setminus R_j^*}\int_{R_j} \frac{\left|b_j(z)\right|}{\left[\left|x-x_{Q_j}\right|+C_\epsilon l(R_j)\right]^n\left[\log\left(\left(\left|x-x_{Q_j}\right|+C_\epsilon l(R_j)\right)/l(R_j)\right)\right]^{1+\epsilon}}$$
$$\times\left[\int_{\mathbb{R}^d\setminus B_j}\left|\frac{K(y,z)}{|y-z|^{1-\rho}}-\frac{K(y,x_{Q_j})}{\left|y-x_{Q_j}\right|^{1-\rho}}\right|^2\right.$$
$$\left.\times\frac{\left[\log\left(\left(\left|y-x_{Q_j}\right|+C_\epsilon l(R_j)\right)/l(R_j)\right)\right]^{2+2\epsilon}}{\left[\left|y-x_{Q_j}\right|+C_\epsilon l(R_j)\right]^{2\rho-n}}d\mu(y)\right]^{1/2}d\mu(z)d\mu(x), \tag{2.39}$$

where in the penultimate inequality, we have used the following inequality

$$\int_{|y-x_{Q_j}|+C_\epsilon l(R_j)}^{|y-x|}\frac{\left[\log\left((t+|x-y|)/l(R_j)\right)\right]^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}t^{n+2\rho+1-\lambda n}}dt\lesssim\frac{\left[\log\left(\left(\left|y-x_{Q_j}\right|+C_\epsilon l(R_j)\right)/l(R_j)\right)\right]^{2+2\epsilon}}{\left(\left|y-x_{Q_j}\right|+C_\epsilon l(R_j)\right)^{2\rho-n}}, \tag{2.40}$$

which can be proved by the same way as in [28, page 357]. Thus, by an argument similar to the estimate of (2.33), we obtain that

$$\mathrm{L}_3\lesssim\left\|b_j\right\|_{L^1(\mu)}. \tag{2.41}$$

Combining the estimates for L_1, L_2, and L_3 yields that

$$\mathrm{F}_3\lesssim\left\|b_j\right\|_{L^1(\mu)}\lesssim\int_{Q_j}\left|f(x)\right|d\mu(x), \tag{2.42}$$

which along with the estimates for F_1 and F_2 leads to (2.24).

Now we turn to prove the estimate (2.25). Observe that if supp $(h)\subset I$ for some cube I, then by (1.2), we have that for any $s>1$ and any $x\in\mathbb{R}^d\setminus sI$,

$$\begin{aligned}\mathcal{M}_\lambda^{*,\rho}(h)(x)&\leq\int_I\left|h(z)\right|\left[\iint_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{|y-z|^{2n-2\rho}}\frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}d\mu(z)\\&\leq\int_I\left|h(z)\right|\left[\mathrm{M}_1(z)+\mathrm{M}_2(z)+\mathrm{M}_3(z)\right]d\mu(z),\end{aligned} \tag{2.43}$$

where

$$\begin{aligned}
\mathrm{M}_1(z) &\equiv \left[\iint_{\substack{|y-z|\le t\\ 2|y-z|>|x-z|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{1}{|y-z|^{2n-2\rho}} \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}, \\
\mathrm{M}_2(z) &\equiv \left[\iint_{\substack{|y-z|\le t,|y-x|<t\\ 2|y-z|\le|x-z|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{1}{|y-z|^{2n-2\rho}} \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}, \qquad (2.44)\\
\mathrm{M}_3(z) &\equiv \left[\iint_{\substack{|y-z|\le t,|y-x|\ge t\\ 2|y-z|\le|x-z|}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \frac{1}{|y-z|^{2n-2\rho}} \frac{d\mu(y)dt}{t^{n+2\rho+1}}\right]^{1/2}.
\end{aligned}$$

Some trivial computation leads to that for any $x \in \mathbb{R}^d \setminus sI$ and $z \in I$,

$$\begin{aligned}
\mathrm{M}_1(z) &\le \left[\int_{2|y-z|>|x-z|} \frac{1}{|y-z|^{2n}} \left(\int_{|x-z|/2}^{\infty} \frac{dt}{t^{n+1}}\right) d\mu(y)\right]^{1/2} \\
&\lesssim \left[\frac{1}{|x-z|^n} \int_{2|y-z|>|x-z|} \frac{1}{|y-z|^{2n}} d\mu(y)\right]^{1/2} \qquad (2.45)\\
&\lesssim \frac{1}{|x-x_I|^n}.
\end{aligned}$$

As for $\mathrm{M}_2(z)$, notice that for any $x, y, z \in \mathbb{R}^d$ satisfying $|y-x| < t$ and $2|y-z| \le |x-z|$, $|x-z|/2 < t$. From this fact and $\rho \in (n/2, \infty)$, it follows that for any $x \in \mathbb{R}^d \setminus sI$ and $z \in I$,

$$\mathrm{M}_2(z) \le \left[\int_{2|y-z|\le|x-z|} \frac{1}{|y-z|^{2n-2\rho}} \left(\int_{(1/2)|x-z|}^{\infty} \frac{dt}{t^{n+2\rho+1}}\right) d\mu(y)\right]^{1/2} \lesssim \frac{1}{|x-x_I|^n}. \qquad (2.46)$$

To estimate $\mathrm{M}_3(z)$, we first have that for any $x, y, z \in \mathbb{R}^d$ satisfying $2|y-z| \le |x-z|$, $2|y-x| \ge |x-z|$, and $|y-x| \le 3|x-z|/2$. Consequently, for any $x \in \mathbb{R}^d \setminus sI$ and $z \in I$,

$$\begin{aligned}
\mathrm{M}_3(z) &\le \left[\int_{2|y-z|\le|x-z|} \frac{1}{|y-z|^{n-\epsilon}} \frac{1}{|x-z|^{2n+2\epsilon}} \left(\int_0^{|y-x|} \frac{dt}{t^{1-\epsilon}}\right) d\mu(y)\right]^{1/2} \\
&= \left[\int_{2|y-z|\le|x-z|} \frac{|y-x|^{\epsilon}}{|x-z|^{2n+2\epsilon}} \frac{1}{|y-z|^{n-\epsilon}} d\mu(y)\right]^{1/2} \qquad (2.47)\\
&\lesssim \frac{1}{|x-x_I|^n}.
\end{aligned}$$

Combining the estimates for $\mathrm{M}_1(z)$, $\mathrm{M}_2(z)$, and $\mathrm{M}_3(z)$, we obtain that for any $x \in \mathbb{R}^d \setminus sI$,

$$\mathcal{M}_{\lambda}^{*,\rho}(h)(x) \lesssim \frac{1}{|x-x_I|^n} \int_I |h(z)| d\mu(z). \qquad (2.48)$$

On the other hand, it follows from [26, Lemma 2.3] (see also [23, Lemma 2.1]) that

$$\int_{R_j^* \setminus 2Q_j} \frac{1}{|x-x_{Q_j}|^n} d\mu(x) \lesssim 1. \qquad (2.49)$$

This fact together with (2.48) tells us that

$$\int_{R_j^*\setminus 2Q_j} \mathcal{M}_\lambda^{*,\rho}(\omega_j f)(x)d\mu(x) \lesssim \int_{R_j^*\setminus 2Q_j} \frac{1}{|x-x_{Q_j}|^n} d\mu(x) \int_{Q_j} |f(y)|d\mu(y) \lesssim \int_{Q_j} |f(y)|d\mu(y). \tag{2.50}$$

The last estimate and the following trivial estimate that

$$\begin{aligned}\int_{R_j^*} \mathcal{M}_\lambda^{*,\rho}(\varphi_j)(x)d\mu(x) &\leq \left[\int_{R_j^*} |\mathcal{M}_\lambda^{*,\rho}(\varphi_j)(x)|^2 d\mu(x)\right]^{1/2} \mu(R_j^*)^{1/2} \\ &\lesssim \left[\int_{R_j^*} |\varphi_j(x)|^2 d\mu(x)\right]^{1/2} \mu(R_j)^{1/2} \\ &\lesssim \int_{Q_j} |f(x)|d\mu(x),\end{aligned} \tag{2.51}$$

which is obtained by the Hölder inequality and the $L^2(\mu)$-boundedness of $\mathcal{M}_\lambda^{*,\rho}$, imply the inequality (2.25). This finishes the proof of Theorem 2.2. □

Proof of Theorem 2.7. Recalling that the definition of RBLO(μ) is independent of the choices of the constant $\eta \in (1,\infty)$, we choose $\eta = 16\sqrt{d}$ in our proof. Hence, to prove Theorem 2.7, it is enough to prove for any $f \in L^\infty(\mu)$, if $\mathcal{M}_\lambda^{*,\rho}(f)(x_0) < \infty$ for some point $x_0 \in \mathbb{R}^d$, then for any $(16\sqrt{d}, (16\sqrt{d})^{d+1})$-doubling cube $Q \ni x_0$,

$$m_Q[\mathcal{M}_\lambda^{*,\rho}(f)] - \operatorname*{ess\,inf}_{x\in Q} \mathcal{M}_\lambda^{*,\rho}(f)(x) \lesssim \|f\|_{L^\infty(\mu)}, \tag{2.52}$$

and for any two $(16\sqrt{d}, (16\sqrt{d})^{d+1})$-doubling cubes $R \supset Q$,

$$m_Q[\mathcal{M}_\lambda^{*,\rho}(f)] - m_R[\mathcal{M}_\lambda^{*,\rho}(f)] \lesssim K_{Q,R}\|f\|_{L^\infty(\mu)}. \tag{2.53}$$

We first verify (2.52). For each fixed cube Q, let B be the smallest ball which contains Q and has the same center as Q. Denote by r the radius of B. Decompose f as

$$f(x) = f(x)\chi_{8B}(x) + f(x)\chi_{\mathbb{R}^d\setminus 8B}(x) \equiv f_1(x) + f_2(x), \tag{2.54}$$

and write

$$\begin{aligned}\mathcal{M}_\lambda^{*,\rho}(f_2)(x) \leq{}& \left[\int_0^r\int_{\mathbb{R}^d}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|\leq t}\frac{K(y,z)}{|y-z|^{1-\rho}}f_2(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2} \\ &+\left[\int_r^\infty\int_{\mathbb{R}^d}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|\leq t}\frac{K(y,z)}{|y-z|^{1-\rho}}f_2(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2} \\ \equiv{}& \mathcal{M}_{\lambda,0}^{*,\rho}(f_2)(x) + \mathcal{M}_{\lambda,\infty}^{*,\rho}(f_2)(x).\end{aligned} \tag{2.55}$$

Thus,

$$\begin{aligned} m_Q[\mathcal{M}_\lambda^{*,\rho}(f)] - \operatorname*{ess\,inf}_{x\in Q} \mathcal{M}_\lambda^{*,\rho}(f)(x) &\lesssim m_Q[\mathcal{M}_\lambda^{*,\rho}(f_1)] + m_Q[\mathcal{M}_{\lambda,0}^{*,\rho}(f_2)] + \sup_{x'\in Q}\left|\mathcal{M}_{\lambda,\infty}^{*,\rho}(f)(x') - \mathcal{M}_{\lambda,\infty}^{*,\rho}(f_2)(x')\right| \\ &\quad + \frac{1}{\mu(Q)}\int \sup_{x'\in Q}\left|\mathcal{M}_{\lambda,\infty}^{*,\rho}(f_2)(x) - \mathcal{M}_{\lambda,\infty}^{*,\rho}(f_2)(x')\right| d\mu(x). \end{aligned} \tag{2.56}$$

By the Hölder inequality and $L^2(\mu)$-boundedness of $M_\lambda^{*,\rho}$, we obtain that

$$m_Q[\mathcal{M}_\lambda^{*,\rho}(f_1)] \le \frac{1}{[\mu(Q)]^{1/2}}\left[\int_{\mathbb{R}^d}[\mathcal{M}_\lambda^{*,\rho}(f_1)(x)]^2 d\mu(x)\right]^{1/2} \lesssim \|f\|_{L^\infty(\mu)}. \tag{2.57}$$

From (1.2) and the fact that for any $x\in Q\subset B$, $z\in\mathbb{R}^d\setminus 8B$, $y\in\mathbb{R}^d$ satisfying $|x-y|<r$, and $t\le r$, $\{z\in\mathbb{R}^d : z\in(\mathbb{R}^d\setminus 8B)\}\cap\{z\in\mathbb{R}^d : |y-z|\le t\}=\varnothing$, it follows that

$$\begin{aligned} \mathcal{M}_{\lambda,0}^{*,\rho}(f_2)(x) &\lesssim \left[\int_0^r\int_{|x-y|\ge r}\frac{1}{|x-y|^{\lambda n}}\left|\int_{|y-z|\le t}\frac{1}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n-\lambda n+2\rho+1}}\right]^{1/2}\|f\|_{L^\infty(\mu)} \\ &\lesssim \left[\int_{|x-y|\ge r}\frac{1}{|x-y|^{\lambda n}}d\mu(y)\int_0^r\frac{1}{t^{n-\lambda n+1}}dt\right]^{1/2}\|f\|_{L^\infty(\mu)} \\ &\lesssim \|f\|_{L^\infty(\mu)}, \end{aligned} \tag{2.58}$$

which gives us that

$$m_Q[\mathcal{M}_{\lambda,0}^{*,\rho}(f_2)] \lesssim \|f\|_{L^\infty(\mu)}. \tag{2.59}$$

Obviously, for any $x'\in Q$, $z\in 8B$, and $y\in\mathbb{R}^d$ with $|x'-y|>16r$, $|x'-y|\sim|y-z|$. Some computation involving this fact and (1.2) yields that

$$\begin{aligned} &\sup_{x'\in Q}\left|\mathcal{M}_{\lambda,\infty}^{*,\rho}(f_2)(x') - \mathcal{M}_{\lambda,\infty}^{*,\rho}(f)(x')\right| \\ &\quad\le \sup_{x'\in Q}\left(\int_r^\infty\int_{\mathbb{R}^d}\left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{K(y,z)f_1(z)}{|y-z|^{1-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n+1}}\right)^{1/2} \\ &\quad\lesssim \sup_{x'\in Q}\left(\int_r^\infty\int_{|x'-y|\le 16r}\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{|f_1(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n+1}}\right)^{1/2} \\ &\qquad+\sup_{x'\in Q}\left(\int_r^\infty\int_{\substack{|x'-y|>16r\\|x'-y|<t}}\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{|f_1(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n+1}}\right)^{1/2} \\ &\qquad+\sup_{x'\in Q}\left(\int_r^\infty\int_{\substack{|x'-y|>16r\\|x'-y|\ge t}}\frac{1}{|x'-y|^{\lambda n}}\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{|f_1(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n-\lambda n+1}}\right)^{1/2} \\ &\quad\lesssim \|f\|_{L^\infty(\mu)} + \sup_{x'\in Q}\left(\int_r^\infty\int_{|x'-y|<t}\left|\int_{8B}\frac{|f_1(z)|}{r^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)\,dt}{t^{n+2\rho+1}}\right)^{1/2} \\ &\qquad+\sup_{x'\in Q}\left(\int_{|x'-y|>16r}\frac{1}{|x-y|^{\lambda n+2n}}\int_r^{|x'-y|}\left|\int_{8B}|f_1(z)|d\mu(z)\right|^2\frac{dt\,d\mu(y)}{t^{n-\lambda n+1}}\right)^{1/2} \\ &\quad\lesssim \|f\|_{L^\infty(\mu)}. \end{aligned} \tag{2.60}$$

Thus, the proof of the estimate (2.52) can be reduced to proving that for any $x, x' \in Q$,

$$\left|\mathcal{M}^{*,\rho}_{\lambda,\infty}(f_2)(x) - \mathcal{M}^{*,\rho}_{\lambda,\infty}(f_2)(x')\right| \lesssim \|f\|_{L^\infty(\mu)}. \tag{2.61}$$

For any $x, x' \in Q$, write

$$\begin{aligned}
&\left|\mathcal{M}^{*,\rho}_{\lambda,\infty}(f_2)(x) - \mathcal{M}^{*,\rho}_{\lambda,\infty}(f_2)(x')\right| \\
&\quad\le \left(\int_r^\infty \int_{|x-y|>8r} \left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n} - \left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\right| \right. \\
&\qquad\qquad \left. \times \left|\frac{1}{t^\rho}\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} f_2(z)\,d\mu(z)\right|^2 \frac{d\mu(y)\,dt}{t^{n+1}}\right)^{1/2} \\
&\qquad + \left(\int_r^\infty \int_{|x-y|\le 8r} \left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n} - \left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\right| \right. \\
&\qquad\qquad \left. \times \left|\frac{1}{t^\rho}\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{1-\rho}} f_2(z)\,d\mu(z)\right|^2 \frac{d\mu(y)\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad\equiv \mathrm{U}_1 + \mathrm{U}_2.
\end{aligned} \tag{2.62}$$

It follows from the mean value theorem that for any $x, x' \in Q \subset B$ and $y \in \mathbb{R}^d$ with $|x-y| > 8r$,

$$\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n} - \left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\right| \lesssim \frac{|x-x'|}{t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n+1}, \tag{2.63}$$

which, along with (1.2), tells us that

$$\begin{aligned}
\mathrm{U}_1 &\lesssim \left(\int_r^\infty \int_{|x-y|>8r} \frac{|x-x'|}{(t+|x-y|)^{\lambda n+1}} \left|\int_{|y-z|\le t} \frac{1}{|y-z|^{n-\rho}}\,d\mu(z)\right|^2 \frac{d\mu(y)\,dt}{t^{n+2\rho+1-n\lambda}}\right)^{1/2} \|f\|_{L^\infty(\mu)} \\
&\lesssim \left(\int_{|x-y|>8r} \frac{r}{|x-y|^{\lambda n+1}} \int_r^{|x-y|} \frac{1}{t^{n-\lambda n+1}}\,dt\,d\mu(y) + \int_r^\infty \int_{|x-y|\le t} \frac{r}{t^{n+2}}\,d\mu(y)\,dt\right)^{1/2} \|f\|_{L^\infty(\mu)} \\
&\lesssim \|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{2.64}$$

As for U_2, first note that for any $x, y \in \mathbb{R}^d$ satisfying $|y-x| \le 8r$ and $t > r$, $t + |y-x| \le 9t$, and then

$$\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n} - \left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\right| \lesssim \frac{|x-x'|}{t}. \tag{2.65}$$

Therefore,

$$\begin{aligned}
\mathrm{U}_2 &\lesssim \left(\int_r^\infty \int_{|x-y|\le 8r} |x-x'| \left|\frac{1}{t^\rho}\int_{|y-z|\le t} \frac{1}{|y-z|^{n-\rho}}\,d\mu(z)\right|^2 \frac{d\mu(y)\,dt}{t^{n+2}}\right)^{1/2} \|f\|_{L^\infty(\mu)} \\
&\lesssim \left(\int_r^\infty \frac{r^{n+1}}{t^{n+2}}\,dt\right)^{1/2} \|f\|_{L^\infty(\mu)} \\
&\lesssim \|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{2.66}$$

The estimates for U_1 and U_2 yield (2.61).

Now we prove that $\mathcal{M}_{\lambda}^{*,\rho}(f)$ satisfies (2.53). Let $Q \subset R$ be any two $(16\sqrt{d}, (16\sqrt{d})^{d+1})$-doubling cubes. Set $N \equiv N_{Q,R} + 1$. For any $x \in Q$ and any $y \in R$, write

$$\begin{aligned}\mathcal{M}_{\lambda}^{*,\rho}(f)(x) \leq{}& \mathcal{M}_{\lambda,0}^{*,\rho}(f_1)(x) + \mathcal{M}_{\lambda,0}^{*,\rho}(f_2)(x) + \mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{4Q})(x)\\ &+ \mathcal{M}_{\lambda,\infty}^{*,\rho}\left(\sum_{k=2}^{N_{Q,R}} f\chi_{2^{k+1}Q\setminus 2^k Q}\right)(x) + \mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{\mathbb{R}^d\setminus 2^N Q})(y)\\ &+ \left[\mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{\mathbb{R}^d\setminus 2^N Q})(x) - \mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{\mathbb{R}^d\setminus 2^N Q})(y)\right].\end{aligned} \tag{2.67}$$

By (1.2), we obtain that for any $x \in Q$,

$$\begin{aligned}&\mathcal{M}_{\lambda,\infty}^{*,\rho}\left(\sum_{k=2}^{N_{Q,R}} f\chi_{2^{k+1}Q\setminus 2^k Q}\right)(x)\\ &\quad\leq \sum_{k=2}^{N_{Q,R}}\left[\int_r^\infty\int_{2^{k-1}Q}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{\substack{|y-z|\leq t\\ z\in 2^{k+1}Q\setminus 2^k Q}}\frac{|f(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2}\\ &\qquad+\left[\sum_{k=2}^{N_{Q,R}}\int_r^\infty\int_{2^{k+2}Q\setminus 2^{k-1}Q}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{\substack{|y-z|\leq t\\ z\in 2^{k+1}Q\setminus 2^k Q}}\frac{|f(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2}\\ &\qquad+\sum_{k=2}^{N_{Q,R}}\left[\int_r^\infty\int_{\mathbb{R}^d\setminus 2^{k+2}Q}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{\substack{|y-z|\leq t\\ z\in 2^{k+1}Q\setminus 2^k Q}}\frac{|f(z)|}{|y-z|^{n-\rho}}d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2}\\ &\quad\equiv \mathrm{V}_1+\mathrm{V}_2+\mathrm{V}_3.\end{aligned} \tag{2.68}$$

The Minkowski inequality involving the fact that for any $y \in 2^{k-1}Q$ and $z \in 2^{k+1}Q \setminus 2^k Q$, $|y-z| \sim |z-x_Q|$ and $t \geq |y-z| \geq 2^{k-2}l(Q)$ gives us that

$$\begin{aligned}\mathrm{V}_1 &\lesssim \|f\|_{L^\infty(\mu)}\sum_{k=2}^{N_{Q,R}}\int_{2^{k+1}Q\setminus 2^k Q}\frac{1}{|z-x_Q|^n}\left(\int_{2^{k-2}l(Q)}^\infty\int_{2^{k-1}Q}\frac{1}{t^{n+1}}d\mu(y)dt\right)^{1/2}d\mu(z)\\ &\lesssim \|f\|_{L^\infty(\mu)}\sum_{k=2}^{N_{Q,R}}\int_{2^{k+1}Q\setminus 2^k Q}\frac{1}{|z-x_Q|^n}d\mu(z)\\ &\lesssim K_{Q,R}\|f\|_{L^\infty(\mu)}.\end{aligned} \tag{2.69}$$

It is easy to verify that for any $y \in 2^{k+2}Q \setminus 2^{k-1}Q$ and $x \in Q$, $|y-x| \sim |y-x_Q|$, which leads to

$$\begin{aligned}\mathrm{V}_2 \lesssim{}& \left[\sum_{k=2}^{N_{Q,R}}\int_{2^{k+2}Q\setminus 2^{k-1}Q}\int_{|y-x|}^\infty\frac{1}{t^{n+1}}dt\,d\mu(y)\right]^{1/2}\|f\|_{L^\infty(\mu)}\\ &+\left[\sum_{k=2}^{N_{Q,R}}\int_{2^{k+2}Q\setminus 2^{k-1}Q}\frac{1}{|x-y|^{\lambda n}}\int_0^{|x-y|}\frac{1}{t^{n-\lambda n+1}}dt\,d\mu(y)\right]^{1/2}\|f\|_{L^\infty(\mu)}\\ \lesssim{}& \left[\sum_{k=2}^{N_{Q,R}}\int_{2^{k+2}Q\setminus 2^{k-1}Q}\frac{1}{|y-x_Q|^n}d\mu(y)\right]^{1/2}\|f\|_{L^\infty(\mu)}\\ \lesssim{}& K_{Q,R}\|f\|_{L^\infty(\mu)}.\end{aligned} \tag{2.70}$$

To estimate V_3, we first have that for any $x \in Q$, $z \in 2^{k+1}Q \setminus 2^kQ$, and $y \in \mathbb{R}^d \setminus 2^{k+2}Q$, $|y - x_Q| \sim |y - x| \sim |y - z|$ and $|z - x_Q| \lesssim 2^k l(Q)$. This fact and the Minkowski inequality state that

$$\begin{aligned}
\mathrm{V}_3 &\lesssim \|f\|_{L^\infty(\mu)} \sum_{k=2}^{N_{Q,R}} \int_{2^{k+1}Q\setminus 2^kQ} \left\{ \left[\iint_{\mathbb{R}^d\setminus 2^{k+2}Q} \frac{1}{|y - x_Q|^{2n}} \int_{|y-x|}^{\infty} \frac{1}{t^{n+1}} dt\, d\mu(y) \right]^{1/2} \right. \\
&\qquad \left. + \left[\iint_{\mathbb{R}^d\setminus 2^{k+2}Q} \frac{1}{|y - x_Q|^{2n+\lambda n}} \int_0^{|y-x|} \frac{1}{t^{n-\lambda n+1}} dt\, d\mu(y) \right]^{1/2} \right\} d\mu(z) \\
&\lesssim \|f\|_{L^\infty(\mu)} \sum_{k=2}^{N_{Q,R}} \int_{2^{k+1}Q\setminus 2^kQ} \frac{1}{|z - x_Q|^n} d\mu(z) \\
&\lesssim K_{Q,R} \|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{2.71}$$

Combining the estimates for V_1, V_2, and V_3 yields that

$$\mathcal{M}_{\lambda,\infty}^{*,\rho}\left(\sum_{k=2}^{N_{Q,R}} f \chi_{2^{k+1}Q\setminus 2^kQ} \right)(x) \lesssim K_{Q,R} \|f\|_{L^\infty(\mu)}. \tag{2.72}$$

An argument similar to the estimate of (2.60) shows that for any $y \in R$,

$$\mathcal{M}_{\lambda,\infty}^{*,\rho}\left(f \chi_{\mathbb{R}^d\setminus 2^NQ}\right)(y) \le \mathcal{M}_{\lambda}^{*,\rho}(f)(y) + C\|f\|_{L^\infty(\mu)}. \tag{2.73}$$

By some estimate similar to that for (2.61), we easily obtain that for any $x, y \in R$,

$$\left|\mathcal{M}_{\lambda,\infty}^{*,\rho}\left(f \chi_{\mathbb{R}^d\setminus 2^NQ}\right)(x) - \mathcal{M}_{\lambda,\infty}^{*,\rho}\left(f \chi_{\mathbb{R}^d\setminus 2^NQ}\right)(y)\right| \lesssim \|f\|_{L^\infty(\mu)}. \tag{2.74}$$

Therefore, for any $x \in Q$ and $y \in R$,

$$\mathcal{M}_{\lambda}^{*,\rho}(f)(x) - \mathcal{M}_{\lambda}^{*,\rho}(f)(y) \lesssim \mathcal{M}_{\lambda,0}^{*,\rho}(f_1)(x) + \mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{4Q})(x) + K_{Q,R}\|f\|_{L^\infty(\mu)}. \tag{2.75}$$

Taking mean value over Q for x, and over R for y, then yields

$$\begin{aligned}
m_Q\left[\mathcal{M}_{\lambda}^{*,\rho}(f)\right] - m_R\left[\mathcal{M}_{\lambda}^{*,\rho}(f)\right] &\lesssim m_Q\left[\mathcal{M}_{\lambda,0}^{*,\rho}(f_1)\right] + m_Q\left[\mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{4Q})\right] + K_{Q,R}\|f\|_{L^\infty(\mu)} \\
&\lesssim K_{Q,R}\|f\|_{L^\infty(\mu)},
\end{aligned} \tag{2.76}$$

where we used the fact that $m_Q[\mathcal{M}_{\lambda,0}^{*,\rho}(f_1)] \lesssim \|f\|_{L^\infty(\mu)}$ and $m_Q[\mathcal{M}_{\lambda,\infty}^{*,\rho}(f\chi_{4Q})] \lesssim \|f\|_{L^\infty(\mu)}$, which can be proved by a way similar to that for the estimate (2.57). This finishes the proof of Theorem 2.7. □

Remark 2.11. From the proofs of Theorems 2.2 and 2.7, we can see that if we replace the assumption that $\mathcal{M}^\rho$ as (1.4) is bounded on $L^2(\mu)$ by the one that $\mathcal{M}_{\lambda}^{*,\rho}$ is bounded on $L^2(\mu)$, then Theorems 2.2 and 2.7 still hold. Therefore, applying the interpolation theorem (see [23, Theorem 7.1]) between the endpoint estimates that $\mathcal{M}_{\lambda}^{*,\rho}$ is bounded from $L^\infty(\mu)$ into $\mathrm{RBLO}(\mu)$, which is a subspace of $\mathrm{RBMO}(\mu)$, and the boundedness of $\mathcal{M}_{\lambda}^{*,\rho}$ in $L^2(\mu)$, we can obtain that $\mathcal{M}_\lambda^{*,\rho}$ as in (1.6) is bounded on $L^p(\mu)$ for $p \in [2,\infty)$ with the kernel satisfies (1.2) and (1.3). On the other hand, it follows from the Marcinkiewicz interpolation theorem that $\mathcal{M}_\lambda^{*,\rho}$ as in (1.6) is also bounded on $L^p(\mu)$ for $p \in (1,2)$ with the kernel satisfying (1.2) and (2.1).

3. Boundedness of $\mathcal{M}_\lambda^{*,\rho}$ in Hardy spaces

In this section, we will prove that the operator $\mathcal{M}_\lambda^{*,\rho}$ as in (1.6) is bounded from $H^1(\mu)$ into $L^1(\mu)$. To state our result, we first recall the definition of the space $H^1(\mu)$ via the "grand" maximal function characterization of Tolsa (see [29]).

Definition 3.1. Given $f \in L^1_{\text{loc}}(\mu)$, set

$$M_\Phi f(x) \equiv \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f(y)\varphi(y)\,d\mu(y) \right|, \tag{3.1}$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,

(ii) $0 \leq \varphi(y) \leq 1/|y-x|^n$ for all $y \in \mathbb{R}^d$,

(iii) $|\nabla\varphi(y)| \leq 1/|y-x|^{n+1}$ for all $y \in \mathbb{R}^d$.

Definition 3.2. The Hardy space $H^1(\mu)$ is defined to be the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} \equiv \|f\|_{L^1(\mu)} + \left\|M_\Phi f\right\|_{L^1(\mu)}. \tag{3.2}$$

Theorem 3.3. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (2.1)*, and $\mathcal{M}_\lambda^{*,\rho}$ be as in* (1.6) *with $\rho \in (n/2,\infty)$ and $\lambda \in (2,\infty)$. Then, $\mathcal{M}_\lambda^{*,\rho}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$.*

We begin with the proof of Theorem 3.3 with the atomic characterization of $H^1(\mu)$ established by Tolsa in [23].

Definition 3.4. Let $\eta \in (1,\infty)$ and $p \in (1,\infty]$. A function $b \in L^1_{\text{loc}}(\mu)$ is called to be an atomic block if

(i) there exists some cube R such that $\operatorname{supp}(b) \subset R$;

(ii) $\int_{\mathbb{R}^d} b(x)\,d\mu(x) = 0$;

(iii) there are functions a_j with supports in cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b \equiv \sum_j \lambda_j a_j$, and

$$\left\|a_j\right\|_{L^\infty(\mu)} \leq \left[\mu(\eta Q_j)\right]^{1/p-1} \left[K_{Q_j,R}\right]^{-1}. \tag{3.3}$$

Then, we define $|b|_{H^{1,p}_{\text{atb}}(\mu)} \equiv \sum_j |\lambda_j|$.

A function $f \in L^1(\mu)$ is said to belong to the space $H^{1,p}_{\text{atb}}(\mu)$ if there exist atomic blocks b_i such that $f \equiv \sum_{i=1}^\infty b_i$ with $\sum_i |b_i|_{H^{1,p}_{\text{atb}}(\mu)} < \infty$. The $H^{1,p}_{\text{atb}}(\mu)$ norm of f is defined by $\|f\|_{H^{1,p}_{\text{atb}}(\mu)} \equiv \inf \sum_i |b_i|_{H^{1,p}_{\text{atb}}(\mu)}$, where the infimum is taken over all the possible decompositions of f in atomic blocks.

It was proved in [23] that the definition of $H^{1,p}_{\text{atb}}(\mu)$ is independent of the chosen constant $\eta \in (1,\infty)$, and for any $p \in (1,\infty]$, all the atomic Hardy spaces $H^{1,p}_{\text{atb}}(\mu)$ are just the Hardy space $H^1(\mu)$ with equivalent norms.

Proof of Theorem 3.3. By a standard argument, it suffices to verify that for any atomic block b as in Definition 3.4 with $\eta = 2$ and $p = \infty$,

$$\left\|\mathcal{M}_{\lambda}^{*,\rho}(b)\right\|_{L^1(\mu)} \lesssim |b|_{H^{1,\infty}_{\text{atb}}(\mu)}. \tag{3.4}$$

Let all the notation be the same as in Definition 3.4. Write

$$\int_{\mathbb{R}^d} \mathcal{M}_{\lambda}^{*,\rho}(b)(x)d\mu(x) = \int_{\mathbb{R}^d\setminus 6\sqrt{d}R} \mathcal{M}_{\lambda}^{*,\rho}(b)(x)d\mu(x) + \int_{6\sqrt{d}R} \mathcal{M}_{\lambda}^{*,\rho}(b)(x)d\mu(x) \equiv \mathrm{W}_1 + \mathrm{W}_2. \tag{3.5}$$

By (2.24) and Definition 3.4, we have

$$\mathrm{W}_1 \lesssim \|b\|_{L^1(\mu)} \lesssim |b|_{H^{1,\infty}_{\text{atb}}(\mu)}. \tag{3.6}$$

To estimate the term W_2, let $b \equiv \sum_j \lambda_j a_j$ be as in (iii) of Definition 3.4, and further write

$$\mathrm{W}_2 \le \sum_j |\lambda_j| \int_{2Q_j} \mathcal{M}_{\lambda}^{*,\rho}(a_j)(x)d\mu(x) + \sum_j |\lambda_j| \int_{6\sqrt{d}R\setminus 2Q_j} \mathcal{M}_{\lambda}^{*,\rho}(a_j)(x)d\mu(x). \tag{3.7}$$

The $L^2(\mu)$-boundedness of $\mathcal{M}_{\lambda}^{*,\rho}$ via the Hölder inequality states that for each fixed j,

$$\int_{2Q_j} \mathcal{M}_{\lambda}^{*,\rho}(a_j)(x)d\mu(x) \le \left\|\mathcal{M}_{\lambda}^{*,\rho}(a_j)\right\|_{L^2(\mu)} \left[\mu(2Q_j)\right]^{1/2} \lesssim \|a_j\|_{L^\infty(\mu)}\mu(2Q_j) \lesssim 1. \tag{3.8}$$

On the other hand, it follows from (2.48) that

$$\begin{aligned}\int_{6\sqrt{d}R\setminus 2Q_j} \mathcal{M}_{\lambda}^{*,\rho}(a_j)(x)d\mu(x) &\lesssim \int_{6\sqrt{d}R\setminus 2Q_j} \frac{1}{\left|x - x_{Q_j}\right|^n} d\mu(x)\|a_j\|_{L^1(\mu)} \\ &\lesssim K_{Q_j,R}\|a_j\|_{L^\infty(\mu)}\mu(Q_j) \\ &\lesssim 1.\end{aligned} \tag{3.9}$$

Thus,

$$\mathrm{W}_2 \lesssim \sum_j |\lambda_j| = |b|_{H^{1,\infty}_{\text{atb}}(\mu)}, \tag{3.10}$$

which completes the proof of Theorem 3.3. □

4. Boundedness of $\mathcal{M}_S^\rho$ in Lebesgue spaces and Hardy spaces

In this section, we will investigate the boundedness for the operator $\mathcal{M}_S^\rho$ as in (1.5) in Lebesgue spaces and Hardy spaces.

It is easy to verify that for any $\rho \in (0,\infty)$, $\lambda \in (1,\infty)$, and $x \in \mathbb{R}^d$,

$$\mathcal{M}_S^\rho(f)(x) \le \mathcal{M}_\lambda^{*,\rho}(f)(x), \tag{4.1}$$

which, together with Theorems 2.1 and 2.2, gives us the boundedness of the operator $\mathcal{M}_S^\rho$ in $L^p(\mu)$ for $p \in [1,\infty)$ as follows.

Theorem 4.1. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (1.3)*, and $\mathcal{M}_S^\rho$ be as in* (1.5) *with $\rho \in (0,\infty)$. Then, for any $p \in [2,\infty)$, $\mathcal{M}_S^\rho$ is bounded on $L^p(\mu)$.*

Theorem 4.2. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (2.1)*, and $\mathcal{M}_S^\rho$ be as in* (1.5) *with $\rho \in (n/2,\infty)$. Then, $\mathcal{M}_S^\rho$ is bounded from $L^1(\mu)$ to weak $L^1(\mu)$.*

By the Marcinkiewicz interpolation theorem, and Theorems 4.1 and 4.2, we easily obtain the $L^p(\mu)$-boundedness of the operator $\mathcal{M}_S^\rho$ for $p \in (1,2)$.

Corollary 4.3. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (2.1)*, and $\mathcal{M}_S^\rho$ be as in* (1.5) *with $\rho \in (n/2,\infty)$. Then, $\mathcal{M}_S^\rho$ is bounded on $L^p(\mu)$ for any $p \in (1,2)$.*

For the case of $p = \infty$, we also obtain the similar result for the operator $\mathcal{M}_\lambda^{*,\rho}$.

Theorem 4.4. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (1.3)*, and $\mathcal{M}_S^\rho$ be as in* (1.5) *with $\rho \in (0,\infty)$. Then, for any $f \in L^\infty(\mu)$, $\mathcal{M}_S^\rho(f)$ is either infinite everywhere or finite almost everywhere. More precisely, if $\mathcal{M}_S^\rho(f)$ is finite at some point $x_0 \in \mathbb{R}^d$, then $\mathcal{M}_S^\rho(f)$ is finite almost everywhere and*

$$\left\|\mathcal{M}_S^\rho(f)\right\|_{\mathrm{RBLO}(\mu)} \le C\|f\|_{L^\infty(\mu)}, \tag{4.2}$$

where the positive constant C is independent of f.

As for the behavior of the operator $\mathcal{M}_S^\rho$ in Hardy spaces, we have the following conclusion.

Theorem 4.5. *Let K be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$ satisfying* (1.2) *and* (2.1)*, and $\mathcal{M}_S^\rho$ be as in* (1.5) *with $\rho \in (n/2,\infty)$. Then, $\mathcal{M}_S^\rho$ is bounded from $H^1(\mu)$ to $L^1(\mu)$.*

We point out that Theorems 4.4 and 4.5 can not be easily deduced from (4.1), and Theorems 2.7 and 3.3. However, using the same method, we can prove the above results more easily than the corresponding results for $\mathcal{M}_\lambda^{*,\rho}$. Here, we omit the proofs for brevity.

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