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Research Article

Oscillation of Higher-Order Neutral-Type Periodic Differential Equations with Distributed Arguments

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We derive oscillation criteria for general-type neutral differential equations $[x(t)+\alpha x(t-\tau)+\beta x(t+\tau)]^{(n)} = \delta\int_a^b x(t-s)\mathrm{d}_s q_1(t,s) + \delta\int_c^d x(t+s)\mathrm{d}_s q_2(t,s) = 0$, $t \geq t_0$, where $t_0 \geq 0$, $\delta = \pm 1$, $\tau > 0$, $b > a \geq 0$, $d > c \geq 0$, α and β are real numbers, the functions $q_1(t,s) : [t_0,\infty)\times[a,b] \to \mathbb{R}$ and $q_2(t,s) : [t_0,\infty)\times[c,d] \to \mathbb{R}$ are nondecreasing in s for each fixed t, and τ is periodic and continuous with respect to t for each fixed s. In certain special cases, the results obtained generalize and improve some existing ones in the literature.

1. Introduction

In this paper, we study the oscillatory behavior of neutral equations of the form

$$\begin{aligned}&\left[x(t)+\alpha,x(t-\tau)+\beta,x(t+\tau)\right]^{(n)}\\&\quad = \delta\int_a^b x(t-s)\mathrm{d}_s q_1(t,s) + \delta\int_c^d x(t+s)\mathrm{d}_s q_2(t,s) = 0\end{aligned} \tag{1.1}$$

for $t \geq t_0$, where $t_0 \geq 0$ is a fixed real number and $\delta = \pm 1$.

We assume throughout the paper that the following conditions hold.

(H1) τ, a, b, c, d, α, β are real numbers such that $\tau > 0$, $b > a \geq 0$, and $d > c \geq 0$.

(H2) $q_1 : [t_0,\infty)\times[a,b] \to \mathbb{R}$ and $q_2 : [t_0,\infty)\times[c,d] \to \mathbb{R}$ are nondecreasing in s for each fixed t, and τ periodic and continuous with respect to t for each fixed s, respectively.

(H3) For some $T_0 \geq t_0$,

$$\mathrm{d}_s q_i(t,s) \geq 0, \quad q_i(t,s) \neq 0 \ \forall (t,s) \in [T_0,\infty)\times[a,b]. \tag{1.2}$$

By a proper solution of (1.1) we mean a real-valued continuous function $x(t)$ which is locally absolutely continuous on $[t_0,\infty)$ along with its derivatives up to the order $n-1$ inclusively, satisfies (1.1) almost everywhere, and $\sup\{|x(s)| : s \geq t\} > 0$ for $t \in [t_0,\infty)$. As usual such a solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative.

Neutral-type equations of the form (1.1), in many particular cases, appear in mathematical modeling problems such as in networks containing lossless transmission lines and also in some variational problems [1]. Therefore, the oscillatory behavior of solutions of such equations in various special cases has been both theoretical and practical interest over the past few decades, receiving considerable attention of many authors (see [1–28] and the references therein).

In this article, we aim to establish some oscillation criteria for solutions of (1.1) which generalize and improve certain known results obtained for less general-type neutral differential equations. The main results of this paper are the comparison theorems contained in the next section where we relate the oscillation of solutions of (1.1) to nonexistence of eventually positive solutions of some nonneutral differential inequalities. These comparison theorems can be used to obtain more concrete oscillation criteria for solutions of (1.1). The last section is therefore devoted to such results, where we provide some oscillation criteria which in some sense extend to (1.1) the ones given by Agarwal and Grace in [3].

We will rely on the following well-known lemma of Kiguradze.

Lemma 1.1. *Let u be real-valued function which is locally absolutely continuous on $[t_*,\infty)$ along with its derivatives up to the order $n-1$ inclusively. If $u(t) > 0$, $u^{(n)}(t) \leq 0$ for $t \geq t_*$, and $u^{(n)}(t) \neq 0$ in any neighborhood of ∞, then there exist $t_1 \geq t_*$ and $l \in \{0,\dots,n-1\}$ such that $l+n$ is odd and for $t \geq t_1$,*

$$\begin{aligned} u^{(i)}(t) &> 0 \quad \text{for } i = 0,\dots,l; \\ (-1)^{i+l}u^{(i)}(t) &> 0 \quad \text{for } i = l+1,\dots,n-1. \end{aligned} \tag{1.3}$$

Definition 1.2. A real-valued function u which is locally absolutely continuous on $[t_0,\infty)$ along with its derivatives up to the order $n-1$ inclusively is said to be of degree 0 if $(-1)^i u^{(i)}(t) > 0$ for $i = 0,1,\dots,n$ and of degree n if $u^{(i)}(t) > 0$ for $i = 0,1,\dots,n$.

2. Comparison theorems

We will make reference to nonexistence of eventually positive solutions of nonneutral-type differential inequalities of the form

$$w^{(n)}(t) + \frac{1}{\lambda}\int_a^b w(t+h-s)\,\mathrm{d}_s q_1(t,s) + \frac{1}{\lambda}\int_c^d w(t+h+s)\,\mathrm{d}_s q_2(t,s) \leq 0, \tag{E_h^λ}$$

$$w^{(n)}(t) - \frac{1}{\mu}\int_a^b w(t+k-s)\,\mathrm{d}_s q_1(t,s) - \frac{1}{\mu}\int_c^d w(t+k+s)\,\mathrm{d}_s q_2(t,s) \geq 0, \tag{E_k^μ}$$

where h, k, λ, μ are real numbers with $\lambda > 0$ and $\mu > 0$.

We may begin with the following comparison theorem.

THEOREM 2.1. *Let $\delta = 1$, $\alpha \geq 0$, $\beta < 0$, and $1+\alpha+\beta > 0$. Suppose that*

(a) *equation (E_k^μ) with $\mu = 1+\alpha+\beta$ and $k = 0$ has no eventually positive solution of degree n;*

(b) *equation (E_h^λ) with $\lambda = -\beta$ and $h = -\tau$ has no eventually positive solution of degree 0 whenever n is odd;*

(c) *equation (E_k^μ) with $\mu = 1+\alpha$ and $k = \tau$ has no eventually positive solution of degree 0 whenever n is even.*

Then every solution $x(t)$ of (1.1) is oscillatory.

Proof. Suppose that there exists an eventually positive solution $x(t)$ of (1.1). Letting

$$y(t) = x(t) + \alpha x(t-\tau) + \beta x(t+\tau), \tag{2.1}$$

we see that

$$y^{(n)}(t) = \int_a^b x(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d x(t+s)\mathrm{d}_s q_2(t,s) \tag{2.2}$$

is eventually nonnegative by (H3), and therefore the derivatives $y^{(i)}(t)$, $i = 0,1,\dots,n-1$, are eventually of fixed sign. It suffices to show that $y(t)$ cannot be of fixed sign.

Case 1. Let $y(t) < 0$ eventually. We easily see that $y(t) \geq \beta x(t+\tau)$ and hence eventually,

$$x(t) \geq \frac{1}{\beta} y(t-\tau). \tag{2.3}$$

It follows from (2.2), (2.3), and (H3) that eventually,

$$y^{(n)}(t) - \int_a^b \frac{y(t-\tau-s)}{\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{y(t-\tau+s)}{\beta}\mathrm{d}_s q_2(t,s) \geq 0. \tag{2.4}$$

There are two cases: (i) $y'(t) < 0$ and (ii) $y'(t) > 0$ eventually.

If (i) holds, then as $y(t) < 0$ eventually there exists a positive constant k such that $y(t) \leq -k$ eventually. Let $T \geq t_0$ be sufficiently large. Then we see from (2.4) that

$$y^{(n-1)}(t) - y^{(n-1)}(T) \geq -\frac{k}{\beta}\int_T^t Q_1(s)ds, \quad Q_1(t) = \int_a^b \mathrm{d}_s q_1(t,s), \tag{2.5}$$

from which by noting that the function Q_1 is positive and periodic (hence bounded), we get $y^{(n-1)}(t) \to \infty$ as $t \to \infty$. Since $y^{(n)}(t) \geq 0$ eventually, it follows that $y(t)$ is eventually positive, a contradiction.

Suppose that (ii) holds. In view of Lemma 1.1, we see that n must be odd. Setting $y = -v$ in (2.4) we have

$$v^{(n)}(t) - \int_a^b \frac{v(t-\tau-s)}{\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{v(t-\tau+s)}{\beta}\mathrm{d}_s q_2(t,s) \leq 0. \tag{2.6}$$

Applying Lemma 1.1, we easily see that $(-1)^i v^{(i)}(t) > 0$ eventually for $i = 0,1,\dots,n-1$, which contradicts our assumption (b). Therefore $y(t)$ cannot be eventually negative.

Case 2. Let $y(t) > 0$ eventually. Because of the linearity and the periodicity conditions, $x(t-\tau)$, $x(t+\tau)$, and hence $y(t)$ is also a solution (1.1). Likewise,

$$w(t) = y(t) + \alpha y(t-\tau) + \beta y(t+\tau) \tag{2.7}$$

is a solution of (1.1). Thus, we may write that eventually,

$$w^{(n)}(t) = \int_a^b y(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d y(t+s)\mathrm{d}_s q_2(t,s); \tag{2.8}$$

$$\left[w(t) + \alpha w(t-\tau) + \beta w(t+\tau)\right]^{(n)} = \int_a^b w(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d w(t+s)\mathrm{d}_s q_2(t,s) = 0. \tag{2.9}$$

Using the procedure in Case 1, one can see that $w(t)$ cannot be eventually negative. So $w(t)$ is eventually positive. Clearly, $y'(t)$ is either eventually positive or eventually negative.

If $y'(t) > 0$ eventually, then from (2.8) we get

$$\begin{aligned} w^{(n)}(t-\tau) &= \int_a^b y(t-\tau-s)\mathrm{d}_s q_1(t,s) + \int_c^d y(t-\tau+s)\mathrm{d}_s q_2(t,s) \\ &\le \int_a^b y(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d y(t+s)\mathrm{d}_s q_2(t,s) \\ &= w^{(n)}(t). \end{aligned} \tag{2.10}$$

Since y is bounded from below, integration of (2.8) from a sufficiently large T to t and letting $t \to \infty$ result in $w^{(n-1)}(t) \to \infty$ and hence $w^{(i)}(t) > 0$ eventually for each $i = 0, 1, \dots, n$. Using (2.10), we obtain from (2.9) that

$$w^{(n)}(t) - \int_a^b \frac{w(t-s)}{1+\alpha+\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t+s)}{1+\alpha+\beta}\mathrm{d}_s q_2(t,s) \ge 0. \tag{2.11}$$

Since (2.11) contradicts (a), $y'(t)$ cannot be eventually positive.

If $y'(t) < 0$ eventually, then one can similarly obtain

$$w^{(n)}(t-\tau) \ge w^{(n)}(t). \tag{2.12}$$

Since n is even in this case, $y'(t)$ is eventually increasing. It follows from

$$w'(t) = y'(t) + \alpha y'(t-\tau) + \beta y'(t+\tau) \le (1+\alpha+\beta)y'(t+\tau) \tag{2.13}$$

that w' is eventually negative as well. In fact, by Lemma 1.1, we see that $(-1)^i w^{(i)}(t) > 0$ eventually for $i = 0, 1, \dots, n-1$. Now, using (2.12) we get

$$w^{(n)}(t) - \int_a^b \frac{w(t+\tau-s)}{1+\alpha}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t+\tau+s)}{1+\alpha}\mathrm{d}_s q_2(t,s) \ge 0. \tag{2.14}$$

Having an eventually positive solution w of degree 0 of inequality (2.14) contradicts (c). The proof is complete. □

The proof of the next theorem is similar, and hence we omit it.

THEOREM 2.2. *Let $\delta = 1$, $\alpha < 0$, $\beta \geq 0$, and $1+\alpha+\beta > 0$. Suppose that*

(a) *equation (E_k^μ) with $\mu = 1+\beta$ and $k = -\tau$ has no eventually positive solution of degree n;*

(b) *equation (E_h^λ) with $\lambda = -\alpha$ and $h = \tau$ has no eventually positive solution of degree 0 whenever n is odd;*

(c) *equation (E_k^μ) with $\mu = 1+\alpha+\beta$ and $k = \tau$ has no eventually positive solution of degree 0 whenever n is even.*

Then every solution $x(t)$ of (1.1) is oscillatory.

THEOREM 2.3. *Let $\delta = 1$, $\alpha \geq 0$, and $\beta \geq 0$. Suppose that*

(a) *equation (E_k^μ) with $\mu = 1+\alpha+\beta$ and $k = -\tau$ has no eventually positive solution of degree n;*

(b) *equation (E_k^μ) with $\mu = 1+\alpha+\beta$ and $k = \tau$ has no eventually positive solution of degree 0 whenever n is even.*

Then every solution $x(t)$ of (1.1) is oscillatory.

Proof. Suppose that there exists an eventually positive solution $x(t)$ of (1.1). Let

$$\begin{aligned} y(t) &= x(t)+\alpha x(t-\tau)+\beta x(t+\tau), \\ w(t) &= y(t)+\alpha y(t-\tau)+\beta y(t+\tau). \end{aligned} \tag{2.15}$$

Clearly,

$$y^{(n)}(t) = \int_a^b x(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d x(t+s)\mathrm{d}_s q_2(t,s) \tag{2.16}$$

is eventually nonnegative and therefore $y^{(i)}(t)$, $i = 0,1,\ldots,n-1$, are eventually of fixed sign. Further, $y(t)$ is eventually positive. There are two possibilities to consider, namely, $y'(t) > 0$ eventually or $y'(t) < 0$ eventually.

Case 1. Let $y'(t) > 0$ eventually. In this case, it is easily seen that $w^{(i)}(t) > 0$ eventually for $i = 0,1,\ldots,n$. From

$$w^{(n)} = \int_a^b y(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d y(t+s)\mathrm{d}_s q_2(t,s), \tag{2.17}$$

we obtain that eventually,

$$w^{(n)}(t-\tau) \leq w^{(n)}(t) \leq w^{(n)}(t+\tau). \tag{2.18}$$

Using this inequality and the fact that $w(t)$ is a solution of (1.1), we have

$$w^{(n)}(t) - \int_a^b \frac{w(t-\tau-s)}{1+\alpha+\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t-\tau+c)}{1+\alpha+\beta}\mathrm{d}_s q_2(t,s) \geq 0. \tag{2.19}$$

We easily obtain from (2.19) a contradiction to our assumption (a).

Case 2. Let $y'(t) < 0$ eventually. Then we have $w'(t) < 0$ eventually. By Lemma 1.1, n is odd and $(-1)^i w^{(i)}(t) > 0$ eventually for $i = 0,1,2,\ldots,n-1$. Following the steps in the previous case, we arrive at

$$w^{(n)}(t-\tau) \geq w^{(n)}(t) \geq w^{(n)}(t+\tau), \tag{2.20}$$

and hence

$$w^{(n)}(t) - \int_a^b \frac{w(t-\tau-s)}{1+\alpha+\beta} \mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t-\tau+c)}{1+\alpha+\beta} \mathrm{d}_s q_2(t,s) \geq 0. \tag{2.21}$$

Since (2.21) contradicts (b), this case is not possible either. Thus, the proof is complete. □

Theorem 2.4. *Let $\delta = 1$, $\alpha \leq 0$, $\beta \leq 0$, and $\alpha+\beta < 0$. Suppose that*

(a) *equation (E_k^μ) with $\mu = 1$ and $k = 0$ has no eventually positive solution of degree n;*

(b) *equation (E_h^λ) with $\lambda = -\alpha+\beta$ and $h = \tau$ has no eventually positive solution of degree 0 whenever n is odd;*

(c) *equation (E_k^μ) with $\mu = 1$ and $k = 0$ has no eventually positive solution of degree 0 whenever n is even.*

Then every solution $x(t)$ of (1.1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Define

$$\begin{aligned} y(t) &= x(t) + \alpha x(t-\tau) + \beta x(t+\tau), \\ v(t) &= y(t) + \alpha y(t-\tau) + \beta y(t+\tau). \end{aligned} \tag{2.22}$$

Clearly, $y(t)$ and $v(t)$ are solutions of (1.1). Moreover,

$$y^{(n)}(t) = \int_a^b x(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d x(t+s)\mathrm{d}_s q_2(t,s), \tag{2.23}$$

$$v^{(n)}(t) = \int_a^b y(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d y(t+s)\mathrm{d}_s q_2(t,s). \tag{2.24}$$

From (2.23) and (H3), we see that $y^{(i)}(t)$, $i = 0,1,\ldots,n-1$, are eventually of fixed sign. We will consider the two possibilities $y(t) < 0$ eventually and $y(t) > 0$ eventually.

Case 1. Let $y(t) < 0$ eventually. In this case, we have $v(t) \geq y(t)$ and $v^{(n)}(t) \leq 0$ eventually. There are two possibilities: (i) $y'(t) < 0$ or (ii) $y'(t) > 0$ eventually.

If (i) holds, then we see that for some $k > 0$, $y(t) \leq -k$ eventually. Using this fact in (2.24) and integrating the resulting inequality leads to $v^{(n-1)}(t) \to -\infty$ as $t \to \infty$. This together with $v^{(n)}(t) \leq 0$ eventually results in $v^{(i)}(t) < 0$ eventually for $i = 0,1,\ldots,n-1$. Further, we see from (2.24) that

$$v^{(n)}(t) \leq \int_a^b v(t-s)\mathrm{d}_s q_1(t,s) + \int_c^d v(t+s)\mathrm{d}_s q_2(t,s), \tag{2.25}$$

which, on setting $w=-v$, leads to

$$w^{(n)}(t)-\int_a^b w(t-s)\mathrm{d}_s q_1(t,s)-\int_c^d w(t+s)\mathrm{d}_s q_2(t,s)\geq 0. \tag{2.26}$$

Inequality (2.26) contradicts our assumption (a).

Suppose that (ii) holds. In this case, we have $(-1)^i y^{(i)}(t)<0$ eventually for $i=0,1,\dots,n-1$ with n odd. Since $y(t)$ is bounded, $v(t)$ is bounded as well and hence $(-1)^i v^{(i)}(t)>0$ eventually for $i=0,1,\dots,n-1$. Now using (2.24) we see that eventually,

$$\begin{gathered} v^{(n)}(t-\tau)\leq v^{(n)}(t)\leq v^{(n)}(t+\tau), \\ \left[v(t)+\alpha v(t-\tau)+\beta v(t+\tau)\right]^{(n)}\leq(\alpha+\beta)v^{(n)}(t-\tau). \end{gathered} \tag{2.27}$$

Since v is a solution of (1.1), we have

$$v^{(n)}(t)-\int_a^b \frac{v(t+\tau-s)}{\alpha+\beta}\mathrm{d}_s q_1(t,s)-\int_c^d \frac{v(t+\tau+s)}{\alpha+\beta}\mathrm{d}_s q_2(t,s)\leq 0. \tag{2.28}$$

Since (2.28) contradicts (b), the possibility $y'(t)>0$ eventually is ruled out. Thus, Case 1 fails to hold.

Case 2. Suppose that $y(t)>0$ eventually. Since $y(t)$ is a solution of (1.1), $v(t)$ must be eventually positive as in the previous case. In view of $y(t)>v(t)$ eventually, we see from (2.24) that

$$v^{(n)}(t)\geq\int_a^b v(t-s)\mathrm{d}_s q_1(t,s)+\int_c^d v(t+s)\mathrm{d}_s q_2(t,s). \tag{2.29}$$

If $v'(t)>0$ eventually, then so are $v^{(i)}(t)$ for $i=0,1,\dots,n-1$. In case $v'(t)<0$ eventually, we see that n is even and $(-1)^i v^{(i)}(t)>0$ eventually for $i=0,1,\dots,n-1$ which contradicts (c). The proof is complete. □

The next three theorems which are analog to above ones are concerned with (1.1) when $\delta=-1$. Since the proofs are very much alike, we omit them.

Theorem 2.5. *Let $\delta=-1$, $\alpha\geq 0$, and $\beta<0$. Suppose that*

(a) *equation (E_k^μ) with $\mu=-\beta$ and $k=-\tau$ has no eventually positive solution of degree n;*

(b) *equation (E_h^λ) with $\lambda=1+\alpha$ and $h=\tau$ has no eventually positive solution of degree 0 whenever n is odd;*

(c) *equation (E_k^μ) with $\mu=-\beta$ and $k=-\tau$ has no eventually positive solution of degree 0 whenever n is even.*

Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 2.6. *Let $\delta=-1$, $\alpha<0$, and $\beta\geq 0$. Suppose that*

(a) *equation (E_k^μ) with $\mu=-\alpha$ and $k=\tau$ has no eventually positive solution of degree n;*

(b) *equation (E_h^λ) with $\lambda=1+\beta$ and $h=-\tau$ has no eventually positive solution of degree 0 whenever n is odd;*

(c) *equation* (E_k^μ) *with* $\mu = \alpha$ *and* $k = \tau$ *has no eventually positive solution of degree 0 whenever* n *is even.*

Then every solution $x(t)$ *of (1.1) is oscillatory.*

Theorem 2.7. *Let* $\delta = -1$, $\alpha \geq 0$, *and* $\beta \geq 0$. *Suppose that* (E_h^λ) *with* $\lambda = 1 + \alpha + \beta$ *and* $h = -\tau$ *has no eventually positive solution of degree 0 whenever* n *is odd. Then every solution* $x(t)$ *of (1.1) is oscillatory.*

Theorem 2.8. *Let* $\delta = -1$, $\alpha \leq 0$, $\beta \leq 0$, *and* $\alpha + \beta < 0$. *Suppose that*

(a) *equation* (E_k^μ) *with* $\mu = -(\alpha + \beta)$ *and* $k = -\tau$ *has no eventually positive solution of degree* n;

(b) *equation* (E_h^λ) *with* $\lambda = 1$ *and* $h = 0$ *has no eventually positive solution of degree 0 whenever* n *is odd;*

(c) *equation* (E_k^μ) *with* $\mu = -(\alpha + \beta)$ *and* $k = \tau$ *has no eventually positive solution of degree 0 whenever* n *is even.*

Then every solution $x(t)$ *of (1.1) is oscillatory.*

3. Oscillation criteria

The comparison type oscillation criteria derived in Section 2 are based upon the nonexistence of certain eventually positive solutions of (E_h^λ) and (E_k^μ) which are in general not easy to verify. Therefore there is a need to provide conditions in terms of the coefficients appearing in (1.1). Our aim is to obtain such oscillation criteria in this section. The results in certain special cases extend to (1.1) all the results established by Agarwal and Grace in [3].

Let $q : [t_0, \infty) \to R$ be continuous and eventually nonnegative. Following Agarwal and Grace, we define

$$I_i(\sigma, q) = \limsup_{t\to\infty} \int_{t-\sigma}^{t} \frac{(t-s)^i (s-t+\sigma)^{n-i-1}}{i!(n-i-1)!} q(s)\mathrm{d}s,$$
$$J_i(\sigma, q) := \limsup_{t\to\infty} \int_{t}^{t+\sigma} \frac{(s-t)^i (t-s+\sigma)^{n-i-1}}{i!(n-i-1)!} q(s)\,\mathrm{d}s. \tag{3.1}$$

We will also make use of the notation that $N_0 = \{0, 1, 2, \ldots, n-1\}$.

Lemma 3.1 (see [2, 3, 15]). *If* $I_i(\sigma, q) > 1$ *for some* $\sigma > 0$ *and for some* $i \in N_0$, *then*

$$(-1)^n y^{(n)}(t) - q(t)y(t-\sigma) \geq 0 \tag{3.2}$$

has no eventually positive solution of degree 0, and if $J_i(\sigma, q) > 1$ *for some* $\sigma > 0$ *and for some* $i \in N_0$, *then*

$$y^{(n)}(t) - q(t)y(t+\sigma) \geq 0 \tag{3.3}$$

has no eventually positive solution of degree n.

In what follows we set

$$Q_1(t) = \int_a^b \mathrm{d}_s q_1(t,s), \qquad Q_2(t) = \int_c^d \mathrm{d}_s q_2(t,s). \tag{3.4}$$

Theorem 3.2. *Let $\delta = 1$, $\alpha \geq 0$, $\beta < 0$, and $1+\alpha+\beta > 0$. Suppose that*
(a) *$J_i(c,Q_2) > 1+\alpha+\beta$ for some $i \in N_0$;*
(b) *if n is odd, then either $I_i(\tau+a,Q_1) > -\beta$ for some $i \in N_0$ or $I_i(\tau-d,Q_2) > -\beta$ for some $\tau > d$ and for some $i \in N_0$;*
(c) *if n is even, then $I_i(a-\tau,Q_1) > 1+\alpha$ for some $a > \tau$ and for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Proof. It suffices to show that the conditions of Theorem 2.1 are satisfied.

Let us first suppose on the contrary that the condition (a) of Theorem 2.1 fails to hold, that is, there is an eventually positive solution of degree n of

$$w^{(n)}(t) - \int_a^b \frac{w(t-s)}{1+\alpha+\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t+s)}{1+\alpha+\beta}\mathrm{d}_s q_2(t,s) \geq 0. \tag{3.5}$$

It follows from (3.5) and (H3) that $w(t)$ is also a solution of

$$w^{(n)}(t) - \frac{Q_2(t)}{1+\alpha+\beta}w(t+c) \geq 0. \tag{3.6}$$

Due to our assumption (b) combined with the second part of Lemma 3.1, we see that (3.6) cannot have an eventually positive solution of degree n, which is a contradiction with (3.5).

Similarly, if the condition (b) of Theorem 2.1 fails, then there would exist an eventually positive solution of degree 0 of

$$w^{(n)}(t) - \int_a^b \frac{w(t-\tau-s)}{\beta}\mathrm{d}_s q_1(t,s) - \int_c^d \frac{w(t-\tau+s)}{\beta}\mathrm{d}_s q_2(t,s) \leq 0. \tag{3.7}$$

It is easy to see from (3.7) and (H3) that

$$w^{(n)}(t) - \frac{Q_1(t)}{\beta}w(t-\tau-a) \leq 0, \tag{3.8}$$

$$w^{(n)}(t) - \frac{Q_2(t)}{\beta}w(t-\tau+d) \leq 0, \tag{3.9}$$

where we have used the fact that $w(t)$ is eventually increasing. On the other hand, in view of our assumption (a) in this theorem, applying the first part of Lemma 3.1 we see that neither (3.8) nor (3.9) can have an eventually positive solution of degree 0, which is a contradiction.

Lastly, if the condition (c) of Theorem 2.1 was not true, then we would arrive at

$$w^{(n)}(t) - \frac{Q_1(t)}{1+\alpha}w(t+\tau-a) \geq 0, \tag{3.10}$$

where n is even, and hence obtain a contradiction in view of our assumption (c) and the first part of Lemma 3.1. □

The following theorems are obtained in a similar manner by applying the theorems in the the previous section, respectively. The proofs are very much like the same as that of Theorem 3.2, and therefore we only state them without proof.

Theorem 3.3. *Let $\delta = 1$, $\alpha < 0$, $\beta \geq 0$, and $1+\alpha+\beta > 0$. Suppose that*
(a) *$J_i(c-\tau, Q_2) > 1+\beta$ for some $\tau < c$ and for some $i \in N_0$;*
(b) *if n is odd, then $I_i(a-\tau, Q_1) > -\alpha$ for some $\tau < a$ and for some $i \in N_0$;*
(c) *if n is even, then $J_i(a-\tau, Q_1) > 1+\alpha+\beta$ for some $\tau < a$ and for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 3.4. *Let $\delta = 1$, $\alpha \geq 0$, and $\beta \geq 0$. Suppose that*
(a) *$J_i(c-\tau, Q_2) > 1+\alpha+\beta$ for some $\tau < c$ and for some $i \in N_0$;*
(b) *if n is even, then $J_i(a-\tau, Q_1) > 1+\alpha$ for some $\tau < a$ and for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 3.5. *Let $\delta = 1$, $\alpha \leq 0$, $\beta \leq 0$, and $\alpha+\beta < 0$. Suppose that*
(a) *$J_i(c, Q_2) > 1$ for some $i \in N_0$;*
(b) *if n is odd, then $I_i(a-\tau, Q_1) > -(\alpha+\beta)$ for some $\tau < a$ and for some $i \in N_0$;*
(c) *if n is even, then $J_i(a, Q_1) > 1$ for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 3.6. *Let $\delta = -1$, $\alpha \geq 0$, and $\beta < 0$. Suppose that*
(a) *$J_i(c, Q_2) > -1/\beta$ for some $i \in N_0$;*
(b) *if n is odd, then $I_i(\tau+a, Q_1) > 1+\alpha$ for some $i \in N_0$;*
(c) *if n is even, then either $I_i(a+\tau, Q_1) > -\beta$ or $J_i(c-\tau, Q_1) > -\beta$ for some $\tau < c$ and for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 3.7. *Let $\delta = -1$, $\alpha < 0$, and $\beta \geq 0$. Suppose that*
(a) *$J_i(c+\tau, Q_2) > -\alpha$ for some $i \in N_0$;*
(b) *if n is odd, then either $I_i(a+\tau, Q_1) > 1+\beta$ for some $\tau < a$ and for some $i \in N_0$ or $I_i(\tau-d, Q_1) > 1+\beta$ for some $\tau < d$ and for some $i \in N_0$;*
(c) *if n is even, then $J_i(a-\tau, Q_1) > -\alpha$ for some $\tau < a$ and for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Theorem 3.8. *Let $\delta = -1$, $\alpha \geq 0$, and $\beta \geq 0$. Suppose that if n is odd, then either $I_i(a+\tau, Q_1) > 1+\alpha+\beta$ for some $i \in N_0$ or $I_i(\tau-c, Q_2) > 1+\alpha+\beta$ for some $\tau > c$ and for some $i \in N_0$. Then every solution $x(t)$ of (1.1) is oscillatory.*

Theorem 3.9. *Let $\delta = -1$, $\alpha \leq 0$, $\beta \leq 0$, and $\alpha+\beta < 0$. Suppose that*
(a) *$J_i(c-\tau, Q_2) > -(\alpha+\beta)$ for some $\tau < c$ and for some $i \in N_0$;*
(b) *if n is odd, then $I_i(a, Q_1) > 1$ for some $i \in N_0$;*
(c) *if n is even, then $J_i(c-\tau, Q_1) > -(\alpha+\beta)$ for some $i \in N_0$.*
Then every solution $x(t)$ of (1.1) is oscillatory.

Remark 3.10. Let $p,q:[t_0,\infty)\to[0,\infty)$ be continuous and τ periodic, and let $g\in[a,b]$, $h\in[c,d]$ be positive real numbers. If we set

$$q_1(t,s)=q(t)H(s-g),\qquad q_2(t,s)=p(t)H(s-h), \tag{3.11}$$

where H is the Heaviside function, then (1.1) takes the form

$$\big[x(t)+\alpha x(t-\tau)+\beta x(t+\tau)\big]^{(n)}=\delta q(t)x(t-g)+\delta p(t)x(t+h)=0, \tag{3.12}$$

which was studied by Agarwal and Grace [3]. One can easily see that the oscillation criteria established in [3] can be recovered from the above theorems. Moreover, we have improved some of the results in this special case as well. For instance, with $a=g$ and $c=h$ our condition $J_i(c,Q_2)=J_i(h,p)>1+\alpha+\beta$ in Theorem 3.2 is weaker than $J_i(h,p)>1+\alpha$ imposed in [3, Theorem 3.1].

Example 3.11. Consider

$$\left[x(t)+6x\left(t-\frac{\pi}{2}\right)-4x\left(t+\frac{\pi}{2}\right)\right]''=10x\left(t-\frac{3\pi}{2}\right)+x(t+\pi)=0 \tag{3.13}$$

so that

$$\alpha=6,\qquad \beta=-4,\qquad q(t)\equiv 10,\qquad p(t)\equiv 1,\qquad \tau=\frac{\pi}{2},\qquad g=\frac{3\pi}{2},\qquad h=\pi. \tag{3.14}$$

It is easy to see that

$$J_i(h,p)=\frac{ph^2}{2}=\frac{\pi^2}{2}<1+\alpha=7,\quad (i=0,1). \tag{3.15}$$

Therefore, Theorem 3.2 given by Agarwal and Grace in [3] is not applicable for (3.13). However, since

$$\begin{gathered} J_i(h,p)=\frac{\pi^2}{2}>1+\alpha+\beta=3,\quad (i=0,1),\\ I_i(g-\tau,q)=\frac{q(g-\tau)^2}{2}=10\pi^2>1+\alpha=7,\quad (i=0,1), \end{gathered} \tag{3.16}$$

we may apply Theorem 3.2 to deduce that every solution of (3.13) is oscillatory. Indeed, $x(t)=\sin t$ is such a solution of the equation.

Example 3.12. Consider

$$\begin{aligned} &\big[x(t)+\alpha x(t-\pi)+\beta x(t+\pi)\big]''' \\ &\quad=\int_a^b\big[s-\sin^2(t+s)\big]x(t-s)\mathrm{d}s+(1-\cos 2t)\int_{2\pi}^{2\pi+k}x(t+s)\mathrm{d}s=0, \end{aligned} \tag{3.17}$$

where $\alpha,\beta,\gamma \geq 0$, $b > a \geq 0$, and $k > 0$ are real constants. Note that we have $\tau = \pi$, $c = 2\pi$, $d = 2\pi + k$, $q_1(t,s) = s - \sin^2(t+s)$, and $q_2(t,s) = s(1-\cos 2t)$. It follows that $Q_2(t) = k(1-\cos 2t)$ and hence

$$\begin{aligned} J_i(c-\tau,Q_2) &= J_i(\pi,k(1-\cos 2t)) \\ &= k\limsup_{t\to\infty}\int_t^{t+\pi}\frac{(s-t)^i(t-s+\pi)^{2-i}}{i!(2-i)!}(1-\cos 2s),\mathrm{d}s \\ &\geq 7.75k, \quad (i=0,1,2). \end{aligned} \tag{3.18}$$

Therefore, by Theorem 3.4 we may conclude that every solution of (3.17) is oscillatory if $1+\alpha+\beta < 7.75k$. Note that if k is sufficiently large then every solution of (3.17) becomes oscillatory.

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