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Research Article

New Inequalities Similar to Hardy-Hilbert Inequality and their Applications

Lü Zhongxue and Xie Hongzheng

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Two classes of new inequalities similar to Hardy-Hilbert inequality are showed by introducing some parameters a,b,c and two real functions $\phi(x)$ and $\psi(x)$. Some applications are obtained.

1. Introduction

The following inequality is well known as Hardy-Hilbert inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.1}$$

where $\pi/\sin(\pi/p)$ is the best value (see Hardy et al. [1]).

Integral analogues of (1.1) are the following inequalities:

$$\begin{gathered}\iint_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy \leq \pi\left(\int_0^{\infty}f^2(x)dx\int_0^{\infty}g^2(y)dy\right)^{1/2},\\ \int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}dx\right)^2 dy \leq \pi^2\int_0^{\infty}f^2(x)dx,\end{gathered} \tag{1.2}$$

where π is the best value (cf., [1, Chapter 9]).

In recent years, Gao [2], Yang [3–5], Yang and Debnath [6], Kuang [7], and Kuang and Debnath [8] gave some distinct improvements and generalizations of (1.1)-(1.2).

Yang and Rassias [9] gave a new inequality with a best constant factor similar to (1.1) as

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn}<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty}n^{p-1}a_n^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}n^{q-1}b_n^q\right\}^{1/q}, \tag{1.3}$$

where $\pi/\sin(\pi/p)$ is the best possible.

In this paper, we have two major objectives. One is motivated by [10], to give a generalization of (1.3) by introducing two real functions $\phi(x)$ and $\psi(x)$. The other is to build a class of new inequalities similar to Hardy-Hilbert inequality (1.2) by introducing some parameters a, b, and c.

2. Some lemmas

First, we give the β function $B(m,n)$:

$$B\left(\frac{1}{p},\frac{1}{q}\right)=\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/q}du, \tag{2.1}$$

where $p>1$, $1/p+1/q=1$.

Lemma 2.1. *Let $b>a\geq 1-c$, and*

$$\omega(a,b,x)=\int_a^b\frac{1}{(y+c)\ln(x+c)(y+c)}\left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2}dy, \tag{2.2}$$

provided the generalized integral exists. Then

$$\omega(a,b,x)\leq\pi-4\arctan\sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}}; \tag{2.3}$$

$$\omega(0,b,x)=\lim_{a\to 0}\omega(a,b,x)\leq\pi-4\arctan\sqrt[4]{\frac{\ln c}{\ln(b+c)}}; \tag{2.4}$$

$$\omega(a,\infty,x)=\lim_{b\to\infty}\omega(a,b,x)\leq\pi-2\arctan\sqrt{\frac{\ln(a+c)}{\ln(x+c)}}. \tag{2.5}$$

Proof. Putting $u=\ln(y+c)/\ln(x+c)$, we have

$$\begin{aligned}\omega(a,b,x)&=\int_a^b\frac{1}{(y+c)\ln(x+c)(y+c)}\left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2}dy=\int_{\ln(a+c)/\ln(x+c)}^{\ln(b+c)/\ln(x+c)}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2}du\\&=\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2}du-\int_{\ln(b+c)/\ln(x+c)}^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2}du-\int_0^{\ln(a+c)/\ln(x+c)}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2}du\\&=\pi-\left(\int_0^{\ln(x+c)/\ln(b+c)}\frac{1}{1+v}\left(\frac{1}{v}\right)^{1/2}dv+\int_0^{\ln(a+c)/\ln(x+c)}\frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2}du\right)\\&=\pi-\left(2\arctan\sqrt{\frac{\ln(x+c)}{\ln(b+c)}}+2\arctan\sqrt{\frac{\ln(a+c)}{\ln(x+c)}}\right).\end{aligned} \tag{2.6}$$

Since $\arctan x$ is strictly increasing, then

$$\begin{aligned}\omega(a,b,x) &= \pi - 2\arctan\frac{\sqrt{\ln(x+c)/\ln(b+c)}+\sqrt{\ln(a+c)/\ln(x+c)}}{1-\sqrt{\ln(a+c)/\ln(b+c)}}\\ &\le \pi - 2\arctan\frac{2\sqrt[4]{\ln(a+c)/\ln(b+c)}}{1-\sqrt{\ln(a+c)/\ln(b+c)}} = \pi - 4\arctan\sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}}.\end{aligned} \tag{2.7}$$

Relation (2.3) is valid. By (2.3) as $a\to 0$, we have

$$\omega(0,b,x) = \lim_{a\to 0}\omega(a,b,x) \le \pi - 4\arctan\sqrt[4]{\frac{\ln c}{\ln(b+c)}}. \tag{2.8}$$

Relation (2.4) is valid. Similarly, (2.5) is also valid. The lemma is proved. □

Lemma 2.2. *Let* $0<\alpha<1$, $0\le c<1$, $g(s)\in C^1[c,1]$, $g(s)>0$, $g'(s)>0$ *for all* $s\in[c,1]$, *and* $F(x)=\int_c^x(s^{-\alpha}/g(s))ds$ *for all* $x\in[c,1]$. *Then*

$$F(x) \ge \frac{x^{1-\alpha}-c^{1-\alpha}}{1-c^{1-\alpha}}F(1). \tag{2.9}$$

Proof. Let $\tau = s^{1-\alpha}$, then

$$F(x) = \int_c^x \frac{s^{-\alpha}}{g(s)}ds = \frac{1}{1-\alpha}\int_{c^{1-\alpha}}^{x^{1-\alpha}}\frac{1}{g\left(\tau^{1/(1-\alpha)}\right)}d\tau. \tag{2.10}$$

Let $G(y) = (1/1-\alpha)\int_{c^{1-\alpha}}^{y}(1/g(\tau^{11-\alpha}))d\tau$. Since $G'(y)>0$, $G''(x)\le 0$ in $[c^{1-\alpha},1]$, and $G(y)$ is concave in $[c^{1-\alpha},1]$, then

$$\begin{aligned}G(y) &= G\left(\frac{1-y}{1-c^{1-\alpha}}c^{1-\alpha}+\frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}\right) = G\left((1-\lambda)c^{1-\alpha}+\lambda\right) \quad \left(\lambda = \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}\right)\\ &\ge \frac{1-y}{1-c^{1-\alpha}}G\left(c^{1-\alpha}\right)+\frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}G(1) = \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}G(1).\end{aligned} \tag{2.11}$$

Thus

$$F(x) = G\left(x^{1-\alpha}\right) \ge \frac{x^{1-\alpha}-c^{1-\alpha}}{1-c^{1-\alpha}}F(1). \tag{2.12}$$

The lemma is proved. □

Let

$$F_{1,r}(x) = \int_0^{\ln(a+c)/\ln(x+c)}\frac{u^{-1/r}}{1+u}du, \qquad F_{2,r}(x) = \int_0^{\ln(x+c)/\ln(b+c)}\frac{u^{-1/r}}{1+u}du, \tag{2.13}$$

where $r>1$, $1-c\le a\le x\le b$.

If $g(s)=1+s$ and $\alpha = 1/r$ in Lemma 2.2, we get the following.

Lemma 2.3. *Let* $1-c<a\le x\le b<+\infty$, $p>1$, $1/p+1/q=1$. *Then*

$$\begin{aligned} F_{1,q}(x)+F_{2,p}(x) &\ge \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p}\Phi(q)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q}\Phi(p) \\ &\ge \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq}(q\Phi(q))^{1/q}(p\Phi(p))^{1/p}; \end{aligned} \tag{2.14}$$

$$\begin{aligned} F_{1,p}(x)+F_{2,q}(x) &\ge \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/q}\Phi(p)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/p}\Phi(q) \\ &\ge \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq}(q\Phi(q))^{1/q}(p\Phi(p))^{1/p}, \end{aligned} \tag{2.15}$$

where $\Phi(r)=\int_0^1(u^{-1/r}/1+u)du$.

Proof. For $1-c<a\le x\le b<+\infty$, by Lemma 2.2, we have

$$\begin{aligned} F_{1,q}(x)+F_{2,p}(x) &\ge \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p}\Phi(q)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q}\Phi(p), \\ F_{1,p}(x)+F_{2,q}(x) &\ge \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/q}\Phi(p)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/p}\Phi(q). \end{aligned} \tag{2.16}$$

Let $\alpha=1/p$, $\beta=1/q$, $p_1=1+\alpha/\beta$, $q_1=1+\beta/\alpha$, then

$$\frac{1}{p_1}+\frac{1}{q_1}=1,\qquad \frac{\alpha}{p_1}+\frac{\beta}{q_1}=\frac{2\alpha\beta}{\alpha+\beta},\quad \alpha+\beta=1. \tag{2.17}$$

By Young inequality, we get

$$\begin{aligned} &\left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p}\Phi(q)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q}\Phi(p) \\ &\quad=\left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{\alpha}\Phi(q)+\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{\beta}\Phi(p) \\ &\quad=\frac{1}{p_1}\left(p_1^{1/p_1}\left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{\alpha/p_1}(\Phi(q))^{1/p_1}\right)^{p_1}+\frac{1}{q_1}\left(q_1^{1/q_1}\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{\beta/q_1}(\Phi(p))^{1/q_1}\right)^{q_1} \\ &\quad\ge\left(p_1^{1/p_1}\left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{\alpha/p_1}(\Phi(q))^{1/p_1}\right)\left(q_1^{1/q_1}\left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{\beta/q_1}(\Phi(p))^{1/q_1}\right) \\ &\quad=\left(1+\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)}\left(1+\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}\left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{\alpha\beta/(\alpha+\beta)}\times(\Phi(q))^{\beta/(\alpha+\beta)}(\Phi(p))^{\alpha/(\alpha+\beta)} \\ &\quad=\left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq}(q\Phi(q))^{1/q}(p\Phi(p))^{1/p}. \end{aligned} \tag{2.18}$$

Then (2.14) is valid.

In the same way, (2.15) can be obtained. This completes the proof. □

Lemma 2.4. *Let* $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ *and* $\psi(x)$ *are continuously differentiable functions on* (a,b), $\phi(a) \geq 1$, $\phi'(x) > 0$, $\psi(a) \geq 1$, $\psi'(x) > 0$, $\inf_x \phi'(x) \neq 0$, *and* $\inf_x \psi'(x) \neq 0$, *provided that the generalized integral exists. Then*

$$\begin{aligned} &\int_a^b \frac{1}{\psi(y)\ln\phi(x)\psi(y)} \left(\frac{\ln\phi(x)}{\ln\psi(y)}\right)^{1/q} dy \\ &\quad \leq \frac{1}{\inf\{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(a)}{\ln\psi(b)}\right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q}\right), \end{aligned} \tag{2.19}$$

where Φ *is as in Lemma 2.3.*

Proof. Putting $u = \ln\psi(y)/\ln\phi(x)$, by Lemma 2.2 and the proof of Lemma 2.3, we have

$$\begin{aligned} &\int_a^b \frac{1}{\psi(y)\ln\phi(x)\psi(y)} \left(\frac{\ln\phi(x)}{\ln\psi(y)}\right)^{1/q} dy \\ &\quad = \int_{\ln\psi(a)/\ln\phi(x)}^{\ln\psi(b)/\ln\phi(x)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/q} \frac{1}{\psi'(y)} du \\ &\quad \leq \frac{1}{\inf\{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \int_0^{\ln\psi(a)/\ln\phi(x)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/q} du - \int_0^{\ln\phi(x)/\ln\psi(b)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/p} du\right) \\ &\quad \leq \frac{1}{\inf\{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(a)}{\ln\phi(x)}\right)^{1/p} \Phi(q) - \left(\frac{\ln\phi(x)}{\ln\psi(b)}\right)^{1/q} \Phi(p)\right) \\ &\quad \leq \frac{1}{\inf\{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(a)}{\ln\psi(b)}\right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q}\right). \end{aligned} \tag{2.20}$$

The lemma is proved. □

Remark 2.5. When $a = 1$, and $b = \infty$, we get

$$\begin{aligned} \int_1^\infty \frac{1}{\psi(y)\ln\phi(x)\psi(y)} \left(\frac{\ln\phi(x)}{\ln\psi(y)}\right)^{1/q} dy &\leq \frac{1}{\inf\{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(1)}{\ln\phi(x)}\right)^{1/p} \Phi(q)\right) \\ &\leq \frac{1}{\inf\{\psi'(y)\}} \frac{\pi}{\sin(\pi/p)}. \end{aligned} \tag{2.21}$$

3. Main results

Now, we introduce main results.

Theorem 3.1. *Let* $-c \leq a < b < +\infty$, f, g *are integrable nonnegative functions on* $[a,b]$ *such that* $0 < \int_a^b (x+c)f^2(x)dx < \infty$ *and* $0 < \int_a^b (y+c)g^2(y)dy < \infty$. *Then*

$$\iint_a^b \frac{f(x)g(y)}{\ln(x+c)(y+c)}dx\,dy \leq \left(\pi - 4\arctan\sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}}\right)\left(\int_a^b (x+c)f^2(x)dx\int_a^b (y+c)g^2(y)dy\right)^{1/2}. \tag{3.1}$$

Proof. By Cauchy-Schwarz inequality and (2.3), we have

$$\begin{aligned}
&\iint_a^b \frac{f(x)g(y)}{\ln(x+c)\ln(y+c)}dx\,dy \\
&\quad= \iint_a^b \frac{f(x)}{(\ln(x+c)(y+c))^{1/2}}\left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/4}\left(\frac{x+c}{y+c}\right)^{1/2} \\
&\quad\quad\times \frac{g(y)}{(\ln(x+c)(y+c))^{1/2}}\left(\frac{\ln(y+c)}{\ln(x+c)}\right)^{1/4}\left(\frac{y+c}{x+c}\right)^{1/2}dx\,dy \\
&\quad\leq \left[\iint_a^b \frac{f^2(x)}{\ln(x+c)(y+c)}\left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2}\frac{x+c}{y+c}dx\,dy\right]^{1/2} \\
&\quad\quad\times \left[\iint_a^b \frac{g^2(y)}{\ln(x+c)(y+c)}\left(\frac{\ln(y+c)}{\ln(x+c)}\right)^{1/2}\frac{y+c}{x+c}dx\,dy\right]^{1/2} \\
&\quad= \left[\int_a^b (x+c)f^2(x)\left(\int_a^b \frac{1}{(y+c)\ln(x+c)(y+c)}\left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2}dy\right)dx\right]^{1/2} \\
&\quad\quad\times \left[\int_a^b (y+c)g^2(y)\left(\int_a^b \frac{1}{(x+c)\ln(x+c)(y+c)}\left(\frac{\ln(y+c)}{\ln(x+c)}\right)^{1/2}dx\right)dy\right]^{1/2} \\
&\quad\leq \left(\pi - 4\arctan\sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}}\right)\left(\int_a^b (x+c)f^2(x)dx\int_a^b (y+c)g^2(y)dy\right)^{1/2}.
\end{aligned} \tag{3.2}$$

Then relation (3.1) is valid. Theorem 3.1 is proved. □

In a similar way to the proof of Theorem 3.1, we can prove the following theorem.

THEOREM 3.2. *Let $1-c \le a < b < +\infty$, f is an integrable nonnegative function on $[a,b]$, such that $0 < \int_a^b (x+c)f^2(x)dx < \infty$, then*

$$\int_a^b \left(\int_a^b \frac{f(x)}{\ln(x+c)(y+c)} dx \right)^2 dy \le \left(\pi - 4\arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}} \right)^2 \int_a^b (x+c)f^2(x)dx. \tag{3.3}$$

Remark 3.3. Specially, when $a=0$, $c=1$, and $b=\infty$ in Theorems 3.1 and 3.2, we get

$$\iint_0^\infty \frac{f(x)g(y)}{\ln(x+1)(y+1)} dx\,dy \le \pi \left(\int_0^\infty (x+1)f^2(x)dx \right)^{1/2} \left(\int_0^\infty (y+1)g^2(y)dy \right)^{1/2};$$
$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{\ln(x+1)(y+1)} dx \right)^2 dy \le \pi^2 \int_0^\infty (x+1)f^2(x)dx. \tag{3.4}$$

THEOREM 3.4. *Let $p>1$, $1/p+1/q=1$, f, g are integrable nonnegative functions on $[a,b]$, such that $0 < \int_a^b \phi^{p-1}(x)f^p(x)dx < \infty$ and $0 < \int_a^b \psi^{q-1}(y)g^q(y)dy < \infty$. Then*

$$\iint_a^b \frac{f(x)g(y)}{\ln\phi(x)\psi(y)} dx\,dy \le \frac{[\pi/\sin(\pi/p)-\phi_1]^{1/p}[\pi/\sin(\pi/p)-\phi_2]^{1/q}}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \times \left\{ \int_a^b \phi^{p-1}(x)f^p(x)dx \right\}^{1/p} \left\{ \int_a^b \psi^{q-1}(y)g^q(y)dy \right\}^{1/q}; \tag{3.5}$$

$$\int_a^b \frac{1}{\psi(y)} \left(\int_a^b \frac{f(x)}{\ln\phi(x)\psi(y)} dx \right)^p dy \le \left(\frac{[\pi/\sin(\pi/p)-\phi_1]^{1/p}[\pi/\sin(\pi/p)-\phi_2]^{1/q}}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \right)^p \times \int_a^b \phi^{p-1}(x)f^p(x)dx, \tag{3.6}$$

where the $\phi(x)$ and $\psi(y)$ are as in Lemma 2.4 ($\phi_1 = (\ln\psi(a)/\ln\psi(b))^{1/pq}(p\Phi(p))^{1/p} \times (q\Phi(q))^{1/q}$, $\phi_2 = (\ln\phi(a)/\ln\phi(b))^{1/pq}(p\Phi(p))^{1/p}(q\Phi(q))^{1/q}$).

Proof. By Hölder inequality and (2.19), we have

$$\iint_a^b \frac{f(x)g(y)}{\ln\phi(x)\psi(y)} dx\,dy$$
$$= \iint_a^b \left[\frac{f(x)}{[\ln\phi(x)\psi(y)]^{1/p}} \left(\frac{\ln\phi(x)}{\ln\psi(y)} \right)^{1/pq} \frac{\phi(x)^{1/q}}{\psi(y)^{1/p}} \right]$$
$$\times \left[\frac{g(y)}{[\ln\phi(x)\psi(y)]^{1/q}} \left(\frac{\ln\psi(y)}{\ln\phi(x)} \right)^{1/pq} \frac{\psi(y)^{1/p}}{\phi(x)^{1/q}} \right] dxdy$$

$$\begin{aligned}
&\leq \left\{\iint_a^b \frac{f^p(x)}{\ln\phi(x)\psi(y)}\left(\frac{\ln\phi(x)}{\ln\psi(y)}\right)^{1/q}\frac{\phi(x)^{p-1}}{\psi(y)}dx\,dy\right\}^{1/p} \\
&\quad\times\left\{\iint_a^b \frac{g^q(y)}{\ln\phi(x)\psi(y)}\left(\frac{\ln\psi(y)}{\ln\phi(x)}\right)^{1/p}\frac{\psi(y)^{q-1}}{\phi(x)}dxdy\right\}^{1/q} \\
&= \left\{\int_a^b \omega(\phi,\psi,q,x)f^p(x)dx\right\}^{1/p}\left\{\int_a^b \omega(\psi,\phi,p,y)g^q(y)dy\right\}^{1/q} \\
&\leq \frac{1}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}} \\
&\quad\times\left\{\int_a^b\left[\frac{\pi}{\sin(\pi/p)}-\left(\frac{\ln\psi(a)}{\ln\psi(b)}\right)^{1/pq}(p\Phi(p))^{1/p}(q\Phi(q))^{1/q}\right]\phi^{p-1}(x)f^p(x)dx\right\}^{1/p} \\
&\quad\times\left\{\int_a^b\left[\frac{\pi}{\sin(\pi/p)}-\left(\frac{\ln\phi(a)}{\ln\phi(b)}\right)^{1/pq}(p\Phi(p))^{1/p}(q\Phi(q))^{1/q}\right]\psi^{q-1}(y)g^q(y)dy\right\}^{1/q} \\
&\leq \frac{\left[\pi/\sin(\pi/p)-\phi_3\right]^{1/p}\left[\pi/\sin(\pi/p)-\phi_4\right]^{1/q}}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}} \\
&\quad\times\left\{\int_a^b \phi^{p-1}(x)f^p(x)dx\right\}^{1/p}\left\{\int_a^b \psi^{q-1}(y)g^q(y)dy\right\}^{1/q}
\end{aligned} \tag{3.7}$$

Hence (3.5) is valid.

Let $g(y) = (1/\psi(y))(\int_a^b(f(x)/\ln\phi(x)\psi(y))dx)^{p-1} > 0$ $(y \in (a,b))$. By (4.2), we have

$$\begin{aligned}
0 &< \int_a^b \psi(y)^{q-1}g^q(y)dy = \int_a^b \frac{1}{\psi(y)}\left(\int_a^b \frac{f(x)}{\ln\phi(x)\psi(y)}dx\right)^p dy = \iint_a^b \frac{f(x)g(y)}{\ln\phi(x)\psi(y)}dx\,dy \\
&\leq \frac{\pi/\sin(\pi/p)}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}}\left\{\int_a^b \phi^{p-1}(x)f^p(x)dx\right\}^{1/p}\left\{\int_a^b \psi^{q-1}(y)g^q(y)dy\right\}^{1/q}.
\end{aligned} \tag{3.8}$$

Then we find

$$\begin{aligned}
&\int_a^b \frac{1}{\psi(y)}\left(\int_a^b \frac{f(x)}{\ln\phi(x)\psi(y)}dx\right)^p dy \\
&\quad= \int_a^b \psi(y)^{q-1}g^q(y)dy \leq \left(\frac{\pi/\sin(\pi/p)}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}}\right)^p\int_a^b \phi^{p-1}(x)f^p(x)dx.
\end{aligned} \tag{3.9}$$

Since $0 < \int_a^b \phi^{p-1}(x)f^p(x)dx$, it follows that $0 < \int_a^b \psi(y)^{q-1}g^q(y)dy < \infty$. Still by (3.5), we have (3.6). The theorem is proved. □

Remark 3.5. Specially when $a = 1$ and $b = \infty$, we get

$$\begin{aligned}
&\iint_1^{\infty} \frac{f(x)g(y)}{\ln \phi(x)\psi(y)} dx\,dy \\
&\quad\leq \frac{1}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \\
&\quad\quad\times \left\{\int_1^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(1)}{\ln\phi(x)}\right)^{1/p}\Phi(q)\right]\phi^{p-1}(x)f^p(x)dx\right\}^{1/p} \\
&\quad\quad\times \left\{\int_1^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\phi(1)}{\ln\psi(y)}\right)^{1/q}\Phi(p)\right]\psi^{q-1}(y)g^q(y)dy\right\}^{1/q} \\
&\quad\leq \frac{\pi/\sin(\pi/p)}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \\
&\quad\quad\times \left\{\int_1^{\infty}\phi^{p-1}(x)f^p(x)dx\right\}^{1/p}\left\{\int_1^{\infty}\psi^{q-1}(y)g^q(y)dy\right\}^{1/q};
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\int_1^{\infty}\frac{1}{\psi(y)}\left(\int_1^{\infty}\frac{f(x)}{\ln\phi(x)\psi(y)}dx\right)^p dy \\
&\quad\leq \left(\frac{\pi/\sin(\pi/p)}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}}\right)^p \int_1^{\infty}\phi^{p-1}(x)f^p(x)dx,
\end{aligned} \tag{3.11}$$

where Φ is as in Lemma 2.3.

By Theorem 3.4, we have the following corollary.

Corollary 3.6. *Let $1 - c \leq a < b < +\infty$, $p > 1$, $1/p + 1/q = 1$, f, g are integrable nonnegative functions on $[a,b]$, such that $0 < \int_a^b (x+c)^{p-1}f^p(x)dx < \infty$ and $0 < \int_a^b (y+c)^{q-1}g^q(y)dy < \infty$. Then*

$$\begin{aligned}
\iint_a^b \frac{f(x)g(y)}{\ln(x+c)(y+c)}dx\,dy \leq{}& \left(B\left(\frac{1}{p},\frac{1}{q}\right) - \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq}(q\Phi(q))^{1/q}(p\Phi(p))^{1/p}\right) \\
&\times\left(\int_a^b (x+c)^{p-1}f^p(x)dx\right)^{1/p}\left(\int_a^b (y+c)^{q-1}g^q(y)dy\right)^{1/q},
\end{aligned} \tag{3.12}$$

where Φ is as in Lemma 2.3.

In what follows, we give the associated discrete inequalities. The proofs should be omitted.

Theorem 3.7. *Let* $p>1$, $1/p+1/q=1$, $\{a_m\}$, $\{b_n\}$ *are nonnegative real sequences, such that* $0<\sum_{n=2}^{\infty}\phi^{p-1}(n)a_n^p<\infty$, $0<\sum_{n=2}^{\infty}\psi^{q-1}(n)b_n^q<\infty$. *Then*

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln\phi(m)\psi(n)} &\leq \frac{1}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}}\\ &\quad\times\left\{\sum_{m=2}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\left(\frac{\ln\psi(1)}{\ln\phi(m)}\right)^{1/p}\Phi(q)\right]\phi^{p-1}(m)a_m^p\right\}^{1/p}\\ &\quad\times\left\{\sum_{n=2}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\left(\frac{\ln\phi(1)}{\ln\psi(n)}\right)^{1/q}\Phi(p)\right]\psi^{q-1}(n)b_n^q\right\}^{1/q}\\ &\leq \frac{\pi/\sin(\pi/p)}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}}\left\{\sum_{m=2}^{\infty}\phi^{p-1}(m)a_m^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\psi^{q-1}(n)b_n^q\right\}^{1/q}.\end{aligned}\tag{3.13}$$

where $\phi(x)$ *and* $\psi(y)$ *are as in Lemma 2.4, and* Φ *is as in Lemma 2.3.*

Theorem 3.8. *Let* $p>1$, $1/p+1/q=1$, $\{a_m\}$ *is nonnegative real sequence, such that* $0<\sum_{n=2}^{\infty}\phi^{p-1}(n)a_n^p<\infty$. *Then*

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{\psi(n)}\left(\sum_{m=2}^{\infty}\frac{a_m}{\ln\phi(m)\psi(n)}\right)^p &\leq \left(\frac{\pi/\sin(\pi/p)}{\left(\inf\{\psi'(y)\}\right)^{1/p}\left(\inf\{\phi'(x)\}\right)^{1/q}}\right)^p\sum_{m=2}^{\infty}\phi^{p-1}(m)a_m^p,\end{aligned}\tag{3.14}$$

where $\phi(x)$ *and* $\psi(y)$ *are as in Lemma 2.4.*

Remark 3.9. When $\phi(x)=x$ and $\psi(y)=y$, then inequalities (3.10), (3.11), (3.13), and (3.14) change to (2.4), (2.10), (3.3), and (3.4) in [10], respectively, hence inequalities (3.10), (3.11), (3.13), and (3.14) are generalizations of related results in [10].

4. Some corollaries

By Theorems 3.4, 3.7, and 3.8, some inequalities can also be obtained.

For example, we take $\phi(x)$ and $\psi(y)$ as

$$\phi(x)=e^x,\qquad \psi(y)=e^y,\tag{4.1}$$

then by Theorems 3.4, 3.7, and 3.8, we get the following corollaries.

Corollary 4.1. *Let* $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$, $\{b_n\}$ *are nonnegative real sequences, such that* $0 < \sum_{n=2}^{\infty} e^{(p-1)n} a_n^p < \infty$, $0 < \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q < \infty$. *Then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi/\sin(\pi/p)}{e}\left\{\sum_{m=2}^{\infty} e^{(p-1)m} a_m^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty} e^{(q-1)n} b_n^q\right\}^{1/q};$$
$$\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{m+n}\right)^p \leq \left(\frac{\pi/\sin(\pi/p)}{e}\right)^p \sum_{m=2}^{\infty} e^{(p-1)m} a_m^p. \tag{4.2}$$

Corollary 4.2. *Let* $p > 1$, $1/p + 1/q = 1$, f, g *are integrable nonnegative functions on* $[a,b]$, *such that* $0 < \int_0^{\infty} e^{(p-1)t} f^p(t)dt < \infty$, $0 < \int_0^{\infty} e^{(q-1)t} g^q(t)dt < \infty$. *Then*

$$\iint_1^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy \leq \frac{\pi/\sin(\pi/p)}{e}\left\{\int_1^{\infty} e^{(p-1)x} f^p(x)dx\right\}^{1/p}\left\{\int_1^{\infty} e^{(q-1)y} g^q(y)dy\right\}^{1/q};$$
$$\int_1^{\infty}\frac{1}{y}\left(\int_1^{\infty}\frac{f(x)}{x+y}dx\right)^p dy \leq \left(\frac{\pi/\sin(\pi/p)}{e}\right)^p \int_1^{\infty} e^{(p-1)x} f^p(x)dx. \tag{4.3}$$

We take $\phi(x)$ *and* $\psi(y)$ *as*

$$\phi(x) = x^2, \qquad \psi(y) = e^y. \tag{4.4}$$

Then we have the following corollary.

Corollary 4.3. *Let* $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$, $\{b_n\}$ *are nonnegative real sequences, such that* $0 < \sum_{n=2}^{\infty} n^{2(p-1)} a_n^p < \infty$, $0 < \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q < \infty$. *Then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{2\ln m+n} \leq \frac{\pi/\sin(\pi/p)}{2^{1/q}e^{1/p}}\left\{\sum_{m=2}^{\infty} m^{2(p-1)} a_m^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty} e^{(q-1)n} b_n^q\right\}^{1/q};$$
$$\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{2\ln m+n}\right)^p \leq \left(\frac{\pi/\sin(\pi/p)}{2^{1/q}e^{1/p}}\right)^p \sum_{m=2}^{\infty} m^{2(p-1)} a_m^p. \tag{4.5}$$

Corollary 4.4. *Let* $p > 1$, $1/p + 1/q = 1$, f, g *are integrable nonnegative functions on* $[a,b]$, *such that* $0 < \int_1^{\infty} x^{2(p-1)} f^p(x)dx < \infty$, $0 < \int_0^{\infty} e^{(q-1)x} g^q(x)dx < \infty$. *Then*

$$\iint_1^{\infty}\frac{f(x)g(y)}{2\ln x+y}dx\,dy \leq \frac{\pi/\sin(\pi/p)}{2^{1/q}e^{1/p}}\left\{\int_1^{\infty} x^{2(p-1)} f^p(x)dx\right\}^{1/p}\left\{\int_1^{\infty} e^{(q-1)y} g^q(y)dy\right\}^{1/q};$$
$$\int_1^{\infty}\frac{1}{y}\left(\int_1^{\infty}\frac{f(x)}{2\ln x+y}dx\right)^p dy \leq \left(\frac{\pi/\sin(\pi/p)}{2^{1/q}e^{1/p}}\right)^p \int_1^{\infty} x^{2(p-1)} f^p(x)dx. \tag{4.6}$$

Remark 4.5. Inequalities (4.2)–(4.6) are also new results.

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Lü Zhongxue: School of Mathematical Sciences, Xuzhou Normal University, Xuzhou, Jiangsu 221116, China
Email address: lvzx1@tom.com

Xie Hongzheng: Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
Email address: xds@mail.edu.cn