

Research Article

Some Relationships between the Analogs of Euler Numbers and Polynomials

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We construct new twisted Euler polynomials and numbers. We also study the generating functions of the twisted Euler numbers and polynomials associated with their interpolation functions. Next we construct twisted Euler zeta function, twisted Hurwitz zeta function, twisted Dirichlet l -Euler numbers and twisted Euler polynomials at non-positive integers, respectively. Furthermore, we find distribution relations of generalized twisted Euler numbers and polynomials. By numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the twisted q -Euler polynomials. Finally, we give a table for the solutions of the twisted q -Euler polynomials.

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1. Introduction and notations

Throughout this paper, we use the following notations. By \mathbb{Z}_p we denote the ring of p -adic rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{Q}_p denotes the field of p -adic rational numbers, \mathbb{C} denotes the complex numbers field, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$, for $|x|_p \leq 1$.

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q} \quad (\text{cf. [1-18]}). \quad (1.1)$$

Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N , we set

$$\begin{aligned} X &= \varprojlim_N \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N ,

$$\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]_q} \tag{1.3}$$

is known to be a distribution on X (cf. [1–18]).

For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is a uniformly differentiable function}\}, \tag{1.4}$$

the p -adic q -integral was defined by [1, 2, 6–18]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} g(x) q^x. \tag{1.5}$$

Note that

$$I_1(g) = \lim_{q \rightarrow 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} g(x) \tag{1.6}$$

(see [1, 2, 6–18]). For $q \in [0, 1]$, certain q -deformed bosonic operators may be introduced which generalize the undeformed bosonic ones (corresponding $q = 1$); see [1, 2, 6–18].

For $g \in UD(\mathbb{Z}_p)$

$$\int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \int_X g(x) d\mu_1(x) \tag{1.7}$$

(see [6–18] for details).

We assume that $q \in \mathbb{C}$ with $|1 - q|_p < 1$. Using definition, we note that $I_1(g_1) = I_1(g) + g'(x)$, where $g_1(x) = g(x + 1)$.

Let

$$T_p = \bigcup_{m \geq 1} C_{p^m} = \lim_{m \rightarrow \infty} C_{p^m}, \tag{1.8}$$

where $C_{p^m} = \{w \mid w^{p^m} = 1\}$ is the cyclic group of order p^m . For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ the locally constant function $x \mapsto w^x$. If we take $f(x) = \phi_w(x)e^{tx}$, then we

easily see that

$$\int_{\mathbb{Z}_p} \phi_w(x) e^{tx} d\mu_1(x) = \frac{t}{we^t - 1}. \tag{1.9}$$

Kim [8] treated the analog of Bernoulli numbers, which is called twisted Bernoulli numbers. We define the twisted Bernoulli polynomials $B_{n,w}(x)$

$$e^{xt} \frac{t}{we^t - 1} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^n}{n!}. \tag{1.10}$$

Using Taylor series of e^{tx} in the above equation, we obtain

$$\int_{\mathbb{Z}_p} x^n \phi_w(x) d\mu_1(x) = B_{n,w}, \tag{1.11}$$

where $B_{n,w} = B_{n,w}(0)$.

The Euler numbers E_n are usually defined by means of the following generating function:

$$e^{Et} = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (\text{cf. [1-18]}), \tag{1.12}$$

where the symbol E_n is interpreted to mean that E^n must be replaced by E_n when we expand the one on the left. These numbers are classical and important in mathematics and in various places like analysis, number theory. Frobenius extended such numbers as E_n to the so-called Frobenius-Euler numbers $H_n(u)$ belonging to an algebraic number u with $|u| > 1$. Let u be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler numbers $H_n(u)$ belonging to u are defined by the generating function

$$e^{H(u)t} = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \quad (\text{cf. [6-10]}), \tag{1.13}$$

with the usual convention of symbolically replacing H^n by H_n . The Euler polynomials $E_n(x)$ are defined by

$$e^{E(x)t} = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (\text{cf. [6-16]}). \tag{1.14}$$

For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler polynomials $H_n(u, x)$ belonging to u are defined by

$$e^{H(u,x)t} = \frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(u, x) \frac{t^n}{n!} \quad (\text{cf. [6-18]}). \tag{1.15}$$

Kim gave a relation between $B_{n,w}$ and $H_n(u)$, with n th Euler numbers as follows:

$$B_{n,w} = \frac{n}{w-1} H_{n-1}(w^{-1}), \quad \text{if } w \neq 1. \tag{1.16}$$

Now, we consider the case $q \in (-1, 0)$ corresponding to q -deformed fermionic certain and annihilation operators and the literature given therein [6–18]. The expression for the $I_q(g)$ remains the same, so it is tempting to consider the limit $q \rightarrow -1$. That is,

$$I_{-1}(g) = \lim_{q \rightarrow -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x. \tag{1.17}$$

Let $g_1(x)$ be translation with $g_1(x) = g(x + 1)$. Then we see that

$$I_{-1}(g_1) = - \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x + 2g(0) = -I_{-1}(f) + 2g(0). \tag{1.18}$$

Therefore, we obtain the following lemma.

LEMMA 1.1. For $g \in UD(\mathbb{Z}_p)$, one has

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.19}$$

From (1.19), we can easily derive the following theorem.

THEOREM 1.2. For $g \in UD(\mathbb{Z}_p)$, $n \in \mathbb{N}$, one has

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \tag{1.20}$$

where $g_n(x) = g(x + n)$.

COROLLARY 1.3. For $g \in UD(\mathbb{Z}_p)$, $n(= \text{odd}) \in \mathbb{N}$, one has

$$I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{l=0}^{n-1} (-1)^l g(l). \tag{1.21}$$

By Lemma 1.1, we can consider twisted Euler numbers. If we take $g(z) = \phi_w(z)e^{tz}$, ($w \in T_p$), then we have

$$I_{-1}(\phi_w(z)e^{tz}) = \frac{2}{we^t + 1} = \sum_{n=0}^{\infty} E_{n,w} \frac{t^n}{n!}. \tag{1.22}$$

Now we define twisted Euler numbers $E_{n,w}$ as follows:

$$F_w(t) = \frac{2}{we^t + 1} = \sum_{n=0}^{\infty} E_{n,w} \frac{t^n}{n!}. \tag{1.23}$$

Using Taylor series of e^{zt} above, we obtain

$$E_{n,w} = \int_{\mathbb{Z}_p} w^z z^n d\mu_{-1}(z). \tag{1.24}$$

For $w \in T_p$, we introduce the twisted Euler polynomials $E_{n,w}(z)$. Twisted Euler polynomials $E_{n,w}(z)$ are defined by means of the generating function

$$F_w(t, z) = \frac{2}{we^{zt} + 1} e^{zt} = I_{-1}(\phi_w(x)e^{t(z+x)}) = \sum_{n=0}^{\infty} E_{n,w}(z) \frac{t^n}{n!}, \quad (1.25)$$

where $E_{n,w}(0) = E_{n,w}$. Using Taylor series of e^{tx} in the above equation, we have

$$E_{n,w}(x) = \int_{\mathbb{Z}_p} \phi_w(x)(x+z)^n d\mu_{-1}(x) = \sum_{k=0}^n \binom{n}{k} z^{n-k} \int_{\mathbb{Z}_p} \phi_w(x)x^k d\mu_{-1}(x). \quad (1.26)$$

Thus we easily see that

$$E_{n,w}(z) = \sum_{k=0}^n \binom{n}{k} z^{n-k} E_{k,w}. \quad (1.27)$$

Let χ be the Dirichlet character with conductor $f (= \text{odd}) \in \mathbb{N}$. Ryou et al. [16] studied the generalized Euler numbers and polynomials. The generalized Euler numbers associated with χ , $E_{n,\chi}$, were defined by means of the generating function

$$F_\chi(t) = \frac{2 \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}. \quad (1.28)$$

Generalized Euler polynomials, $E_{n,\chi}(x)$, were also defined by means of the generating function

$$F_\chi(t, z) = \frac{2 \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft} + 1} e^{zt} = \sum_{n=0}^{\infty} E_{n,\chi}(z) \frac{t^n}{n!}. \quad (1.29)$$

Substituting $g(x) = \chi(x)\phi_w(x)e^{tx}$ into (1.21), then the generalized twisted Euler numbers $E_{n,\chi,w}$ are defined by means of the generating functions

$$\begin{aligned} F_{\chi,w}(t) &= \int_X \phi_w(x)e^{tx}\chi(x)d\mu_{-1}(x) \\ &= \frac{2 \sum_{a=0}^{f-1} e^{ta}(-1)^a \chi(a)\phi_w(a)}{\phi_w(f)e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,w} \frac{t^n}{n!}. \end{aligned} \quad (1.30)$$

Using the above equation, $E_{n,\chi,w}$ are defined by

$$E_{n,\chi,w} = \int_X \phi_w(x)x^n \chi(x) d\mu_{-1}(x). \quad (1.31)$$

Generalized twisted Euler polynomials, $E_{n,\chi,w}(z)$, are defined by

$$\begin{aligned} F_{\chi,w}(t, z) &= F_{\chi,w}(t)e^{zt} = \int_X \phi_w(x)e^{tx}\chi(x)d\mu_{-1}(x)e^{tz} \\ &= \left(\frac{2 \sum_{a=0}^{f-1} e^{ta}(-1)^a \chi(a)\phi_w(a)}{\phi_w(f)e^{ft} + 1} \right) e^{zt} = \sum_{n=0}^{\infty} E_{n,\chi,w}(z) \frac{t^n}{n!}. \end{aligned} \quad (1.32)$$

We set

$$F_{\chi,w}(t, z) = \frac{2 \sum_{a=0}^{f-1} (-1)^a \chi(a) \phi_w(a) e^{(a+z)t}}{\phi_w(f) e^{ft} + 1}. \tag{1.33}$$

Using the above equation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,w}(z) \frac{t^n}{n!} &= \int_X \phi_w(x) e^{tx} \chi(x) d\mu_{-1}(x) e^{tz} \\ &= \int_X \phi_w(x) e^{t(x+z)} \chi(x) d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} \left(\int_X \phi_w(x) (x+z)^n \chi(x) d\mu_{-1}(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} z^{n-k} \int_X \phi_w(x) x^k \chi(x) d\mu_{-1}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{1.34}$$

Using the comparing coefficients $t^n/n!$, we easily see that

$$E_{n,\chi,w}(z) = \sum_{k=0}^n \binom{n}{k} z^{n-k} E_{k,\chi,w}. \tag{1.35}$$

We have the following remark.

Remark 1.4. Note that

(1) if $w \rightarrow 1$, then $F_{\chi,w}(t, z) \rightarrow F_{\chi}(t, z)$ and $E_{n,\chi,w}(z) \rightarrow E_{n,\chi}(z)$;

(2)

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,w}(z) \frac{t^n}{n!} &= \left(\frac{2 \sum_{a=0}^{f-1} e^{ta} (-1)^a \chi(a) \phi_w(a)}{\phi_w(f) e^{ft} + 1} \right) e^{zt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,w}(z) \frac{t^n}{n!} \sum_{n=0}^{\infty} z^n \frac{t^n}{n!}. \end{aligned} \tag{1.36}$$

Using the Cauchy product in the right-hand side of the above equation in (2), we obtain

$$\sum_{n=0}^{\infty} E_{n,\chi,w}(z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n E_{k,\chi,w} \frac{z^{n-k} t^n}{k!(n-k)!}. \tag{1.37}$$

Comparing the coefficients t^n on both sides of the above equation, we arrive at (1.35).

2. Twisted zeta function

In this section, we introduce the twisted Euler zeta function and twisted Hurwitz-Euler zeta function. We derive a new twisted Hurwitz-type l -function which interpolates the

generalized Euler polynomials $E_{n,\chi,w}(x)$. We give the relation between twisted Euler numbers and twisted l -functions at nonpositive integers. Let χ be the Dirichlet character with conductor $f (= \text{odd}) \in \mathbb{N}$. We set

$$F_{\chi,w}(t) = \frac{2 \sum_{a=0}^{f-1} e^{ta} (-1)^a \chi(a) \phi_w(a)}{\phi_w(f) e^{ft} + 1}, \quad \left(-\frac{\pi}{f} - \log w < t < \frac{\pi}{f} - \log \right). \quad (2.1)$$

By (2.1), we see that

$$F_{\chi,w}(t) = 2 \sum_{m=1}^{\infty} \chi(m) w^m (-1)^m e^{tm}. \quad (2.2)$$

From (1.30) and (2.2), we note that

$$\frac{d^k}{dt^k} F_{\chi,w}(t) |_{t=0} = 2 \sum_{m=1}^{\infty} \chi(m) w^m (-1)^m m^k \quad (k \in \mathbb{N}). \quad (2.3)$$

Therefore, we obtain the following theorem.

THEOREM 2.1. *For $k \in \mathbb{N}$, one has*

$$E_{k,\chi,w} = 2 \sum_{m=1}^{\infty} \chi(m) w^m (-1)^m m^k \quad (k \in \mathbb{N}). \quad (2.4)$$

Thus we define the twisted Dirichlet-type l -series as follows.

Definition 2.2. For $s \in \mathbb{C}$, define the Dirichlet-type l -series related to twisted Euler numbers,

$$l_w(s, \chi) = 2 \sum_{n=1}^{\infty} \frac{\chi(n) (-1)^n w^n}{n^s}. \quad (2.5)$$

THEOREM 2.3. *For $k \in \mathbb{N}$, one has*

$$l_w(-k, \chi) = E_{k,\chi,w}. \quad (2.6)$$

Next, we introduce the Hurwitz-type twisted Euler zeta function. Since

$$F_w(t, z) = \frac{2}{w e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_{n,w}(z) \frac{t^n}{n!}, \quad (2.7)$$

we obtain

$$F_w(t, z) = 2 \sum_{n=0}^{\infty} (-1)^n w^n e^{(n+z)t}. \quad (2.8)$$

From (2.8), we note that

$$\frac{d^k}{dt^k} F_w(t, z) |_{t=0} = 2 \sum_{n=0}^{\infty} (-1)^n w^n (n+z)^k \quad (k \in \mathbb{N}). \quad (2.9)$$

Therefore, we have the following theorem.

THEOREM 2.4. For $k \in \mathbb{N}$, one has

$$E_{k,w}(z) = 2 \sum_{n=0}^{\infty} (-1)^n w^n (n+z)^k \quad (k \in \mathbb{N}). \tag{2.10}$$

Thus the twisted Hurwitz-Euler zeta function is defined as follows.

Definition 2.5. Let $s \in \mathbb{C}$. Then

$$\zeta_{E,w}(s, z) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{(n+z)^s}. \tag{2.11}$$

By Theorem 2.4 and Definition 2.5, we have the following theorem.

THEOREM 2.6. For $k \in \mathbb{N}$, one obtains

$$\zeta_{E,w}(-k, z) = E_{k,w}(z). \tag{2.12}$$

Let us define two-variable twisted Euler numbers attached to χ as follows. By (1.33), we see that

$$F_{\chi,w}(t, z) = 2 \sum_{n=0}^{\infty} (-1)^n \chi(n) w^n e^{(n+z)t}. \tag{2.13}$$

From (2.13), we note that

$$\frac{d^k}{dt^k} F_{\chi,w}(t, z) |_{t=0} = 2 \sum_{n=0}^{\infty} (-1)^n \chi(n) w^n (n+z)^k \quad (k \in \mathbb{N}). \tag{2.14}$$

Therefore, we obtain the following theorem.

THEOREM 2.7. For $k \in \mathbb{N}$, one has

$$E_{k,\chi,w}(z) = 2 \sum_{n=0}^{\infty} (-1)^n \chi(n) w^n (n+z)^k \quad (k \in \mathbb{N}). \tag{2.15}$$

Hence we define two-variable twisted l -series as follows.

Definition 2.8. For $s \in \mathbb{C}$. Then

$$l_w(s, \chi | z) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \chi(n) w^n}{(n+z)^s}. \tag{2.16}$$

The relation between $l_w(-k, \chi | z)$ and $E_{k,\chi,w}(z)$ is given by the following theorem.

THEOREM 2.9. For $k \in \mathbb{N}$, one obtains

$$l_w(-k, \chi | z) = E_{k,\chi,w}(z). \tag{2.17}$$

Note that

$$\begin{aligned} l_w(s, \chi | z) &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n \chi(n) w^n}{(n+z)^s} \\ &= f^{-s} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \zeta_{E, w^f} \left(s, \frac{a+z}{f} \right). \end{aligned} \quad (2.18)$$

The relation between $l_w(s, \chi | z)$ and $\zeta_{E, w}(s, z)$ is given by the following theorem.

THEOREM 2.10. For $s \in \mathbb{C}$,

$$l_w(s, \chi | z) = \frac{1}{f^s} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \zeta_{E, w^f} \left(s, \frac{a+z}{f} \right). \quad (2.19)$$

Observe that, substituting $z = 0$ into (2.19),

$$l_w(s, \chi) = f^{-s} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \zeta_{E, w^f} \left(s, \frac{a}{f} \right). \quad (2.20)$$

Substituting $s = -n$ with $n \in \mathbb{N}$,

$$l_w(-n, \chi | z) = f^{-n} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \zeta_{E, w^f} \left(-n, \frac{a+z}{f} \right). \quad (2.21)$$

By Theorem 2.6 and (2.21), we have

$$l_w(-n, \chi | z) = f^{-n} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) E_{n, w^f} \left(\frac{a+z}{f} \right). \quad (2.22)$$

Using (1.27), we get

$$\begin{aligned} l_w(-n, \chi | z) &= f^{-n} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \sum_{k=0}^n \binom{n}{k} f^{k-n} (a+z)^{n-k} E_{k, w^f} \\ &= f^{-n} \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) \sum_{k=0}^n \binom{n}{k} f^{k-n} \sum_{j=0}^{n-k} \binom{n-k}{j} z^{n-k-j} a^j E_{k, w^f} \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} f^k z^{n-k-j} E_{k, w^f} S_{w, \chi}(j), \end{aligned} \quad (2.23)$$

where

$$S_{w, \chi}(j) = \sum_{a=0}^{f-1} (-1)^a w^a \chi(a) a^j. \quad (2.24)$$

By Theorem 2.9 and (2.23), we arrive at the following theorem.

THEOREM 2.11. For $n \in \mathbb{Z}^+$, one obtains

$$E_{n,\chi,w}(z) = \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} f^k z^{n-k-j} E_{k,w^f} S_{w,\chi}(j). \tag{2.25}$$

3. Twisted q -Euler zeta function and twisted q -analog Dirichlet l -function

Our primary goal of this section is to define generating functions of the twisted q -Euler numbers and polynomials. Using these functions, twisted q -zeta function and twisted q - l -functions are defined. These functions interpolate twisted q -Euler numbers and generalized twisted q -Euler numbers, respectively. Now, we introduce the generating functions $F_q(t)$ and $F_q(x, t)$. Ryoo et al. [15] treated the analog of Euler numbers, which is called q -Euler numbers in this paper. Using p -adic q -integral, we defined the q -Euler numbers as follows:

$$E_{n,q} = \int_{\mathbb{Z}_p} [t]_q^n d\mu_{-q}(t), \quad \text{for } n \in \mathbb{N}. \tag{3.1}$$

Thus we obtain

$$E_{n,q} = [2]_q \left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{l+1}}, \tag{3.2}$$

where $\binom{n}{i}$ is the binomial coefficient (see [1–18]). Using the above equation, we have

$$F_q(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_q t}. \tag{3.3}$$

Thus q -Euler numbers, $E_{n,q}$, are defined by means of the generating function

$$F_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}. \tag{3.4}$$

Note that

$$\sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{[x]_q^n}{n!} t^n d\mu_{-q}(x) = \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q}(x). \tag{3.5}$$

Thus we have

$$\int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}. \tag{3.6}$$

Similarly, the generating function $F_q(t, z)$ of the q -Euler polynomials $E_{n,q}(z)$ is defined analogously as follows:

$$F_q(t, z) = \sum_{n=0}^{\infty} E_{n,q}(z) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+z]_q t}. \tag{3.7}$$

Now, we introduce twisted q -Euler numbers $E_{n,q,w}$. For $w \in T_p$, we define twisted q -Euler numbers as follows:

$$E_{n,q,w} = \int_{\mathbb{Z}_p} [x]_q^n w^x d\mu_{-q}(x), \quad \text{for } n \in \mathbb{N}. \quad (3.8)$$

Using (3.8), we obtain

$$E_{n,q,w} = [2]_q \left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{l+1}}. \quad (3.9)$$

Note that

$$\begin{aligned} F_{q,w}(t) &= \sum_{n=0}^{\infty} E_{n,q,w} \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{l+1}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^m w^m [m]_q^n \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m w^m e^{[m]_q t}. \end{aligned} \quad (3.10)$$

Thus we obtain the generating function of twisted q -Euler numbers $E_{n,q,w}$ as follows:

$$F_{q,w}(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q,w} \frac{t^n}{n!}. \quad (3.11)$$

Observe that $\lim_{q \rightarrow 1} E_{n,q,w} = E_{n,w}$. Using (3.11), we easily see that

$$\begin{aligned} F_{q,w}(t) &= \sum_{n=0}^{\infty} E_{n,q,w} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} w^x [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{[x]_q^n}{n!} t^n d\mu_{-q}(x) = \int_{\mathbb{Z}_p} w^x e^{[x]_q t} d\mu_{-q}(x). \end{aligned} \quad (3.12)$$

Hence we obtain

$$\int_{\mathbb{Z}_p} w^x e^{[x]_q t} d\mu_{-q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n e^{[n]_q t}. \quad (3.13)$$

From (3.11), we note that

$$\frac{d^k}{dt^k} F_{q,w}(t, z) |_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n [n]_q^k \quad (k \in \mathbb{N}). \quad (3.14)$$

Using the above equation, we are now ready to define twisted q -Euler zeta functions.

Definition 3.1. Let $s \in \mathbb{C}$. Then

$$\zeta_{q,w}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^n}{[n]_q^s}. \tag{3.15}$$

Note that $\zeta_{E,w}(s)$ is a meromorphic function on \mathbb{C} . The relation between $\zeta_{q,w}(s)$ and $E_{k,q,w}$ is given by the following theorem.

THEOREM 3.2. For $k \in \mathbb{N}$, one has

$$\zeta_{q,w}(-k) = E_{k,q,w}. \tag{3.16}$$

Observe that $\zeta_{q,w}(s)$ function interpolates $E_{k,q,w}$ numbers at nonnegative integers. Using p -adic q -integral, we defined the twisted q -Euler polynomials as follows:

$$E_{n,q,w}(z) = \int_{\mathbb{Z}_p} w^x [x+z]_q^n d\mu_{-q}(x), \quad \text{for } n \in \mathbb{N}. \tag{3.17}$$

Since $[x+z]_q = [z]_q + q^z [x]_q$, we have

$$\begin{aligned} E_{n,q,w}(z) &= \int_{\mathbb{Z}_p} w^x [x+z]_q^n d\mu_{-q}(x) \\ &= \int_{\mathbb{Z}_p} w^x \sum_{l=0}^n \binom{n}{l} q^{lz} [x]_q^l [z]_q^{n-l} d\mu_{-q}(x) \\ &= \sum_{l=0}^n \binom{n}{l} q^{lz} [z]_q^{n-l} \int_{\mathbb{Z}_p} w^x [x]_q^l d\mu_{-q}(x) \\ &= \sum_{l=0}^n \binom{n}{l} q^{lz} E_{l,q,w} [z]_q^{n-l} \\ &= (q^z E_{q,w} + [z]_q)^n, \end{aligned} \tag{3.18}$$

where the symbol $E_{k,q,w}$ is interpreted to mean that $E_{q,w}^k$ must be replaced by $E_{k,q,w}$. Using (3.17), we have

$$E_{n,q,w}(z) = [2]_q \left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lz} \frac{1}{1+wq^{l+1}}. \tag{3.19}$$

Note that

$$\begin{aligned}
 F_{q,w}(t,z) &= \sum_{n=0}^{\infty} E_{n,q,w}(z) \frac{t^n}{n!} \\
 &= [2]_q \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lz} \frac{1}{1+wq^{l+1}} \right) \frac{t^n}{n!} \\
 &= [2]_q \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^m w^m [m+z]_q^n \frac{t^n}{n!} \\
 &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m w^m e^{[m+z]_q t}.
 \end{aligned} \tag{3.20}$$

Hence we have the generating function of twisted q -Euler polynomials $E_{n,q,w}(z)$ as follows:

$$F_{q,w}(t,z) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m w^m e^{[m+z]_q t}. \tag{3.21}$$

Observe that $\lim_{q \rightarrow 1} E_{n,q,w}(z) = E_{n,w}(z)$. From (3.21), we note that

$$\frac{d^k}{dt^k} F_{q,w}(t,z) \Big|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n [n+z]_q^k \quad (k \in \mathbb{N}). \tag{3.22}$$

Using the above equation, we are now ready to define the twisted Hurwitz q -Euler zeta functions.

Definition 3.3. Let $s \in \mathbb{C}$. Then

$$\zeta_{q,w}(s,z) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^n}{[n+z]_q^s}. \tag{3.23}$$

Note that $\zeta_{E,w}(s,z)$ is a meromorphic function on \mathbb{C} . The relation between $\zeta_{q,w}(s,z)$ and $E_{k,q,w}(z)$ is given by the following theorem.

THEOREM 3.4. For $k \in \mathbb{N}$, one has

$$\zeta_{q,w}(-k,z) = E_{k,q,w}(z). \tag{3.24}$$

Observe that $\zeta_{q,w}(-k,z)$ function interpolates $E_{k,q,w}(z)$ numbers at nonnegative integers.

4. Distribution and structure of the zeros

In this section, we investigate the zeros of the twisted q -Euler polynomials $E_{n,q,w}(z)$ by using computer. Let $w = e^{2\pi i/N}$ in \mathbb{C} . We plot the zeros of $E_{n,q,w}(x)$, $x \in \mathbb{C}$, for $N = 1$, $q = 1/2$ (see Figures 4.1, 4.2, 4.3, and 4.4).

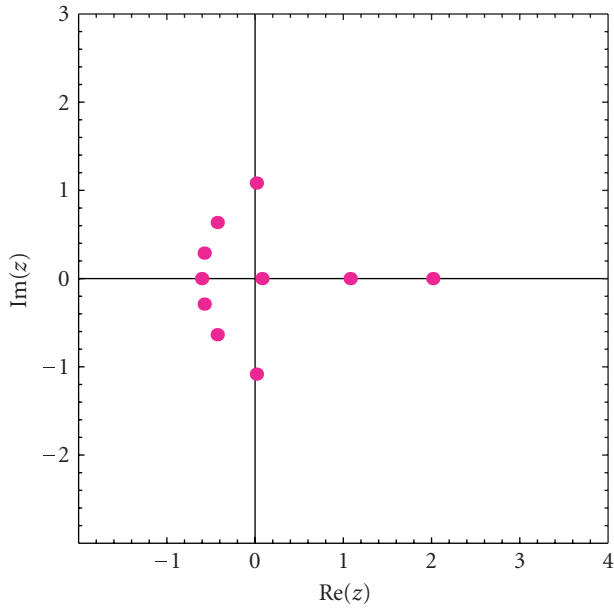


FIGURE 4.1. Zeros of $E_{10,q,w}(x)$.

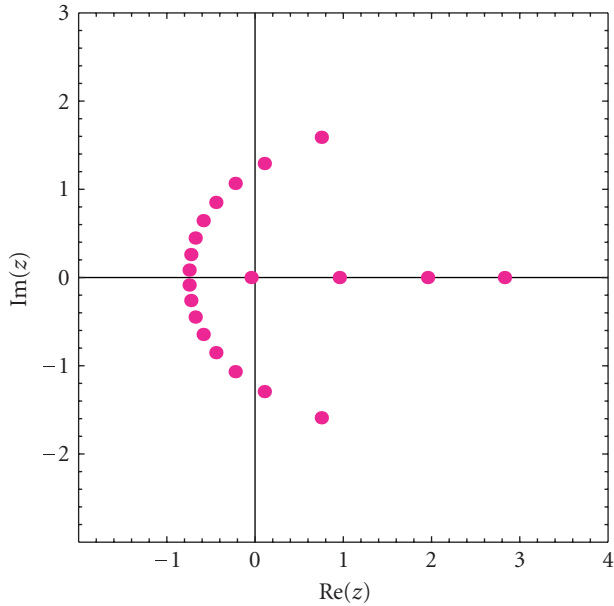


FIGURE 4.2. Zeros of $E_{20,q,w}(x)$.

Next, we plot the zeros of $E_{n,q,w}(x)$, $x \in \mathbb{C}$, for $N = 2$, $q = 1/2$ (see Figures 4.5, 4.6, 4.7, and 4.8).

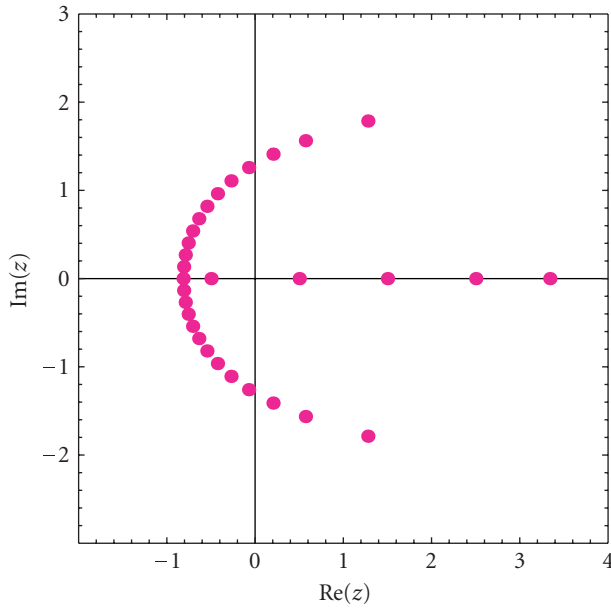


FIGURE 4.3. Zeros of $E_{30,q,w}(x)$.

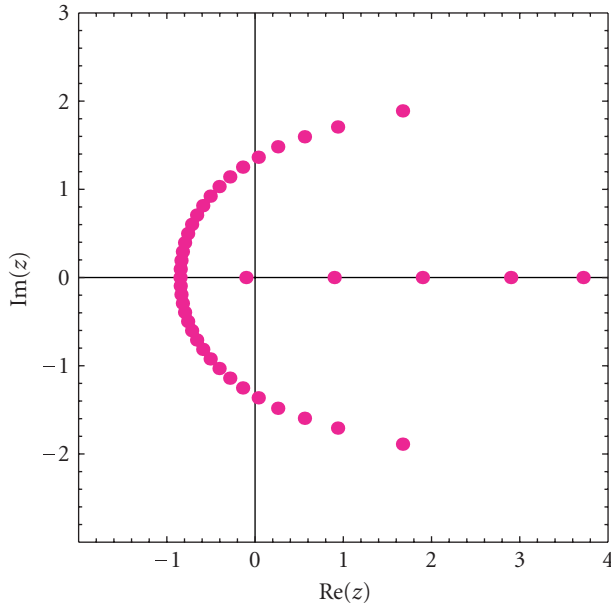


FIGURE 4.4. Zeros of $E_{40,q,w}(x)$.

Finally, we plot the zeros of $E_{n,q,w}(x)$, $x \in \mathbb{C}$, for $N = 4$, $q = 1/3$ (see Figures 4.9, 4.10, 4.11, and 4.12).

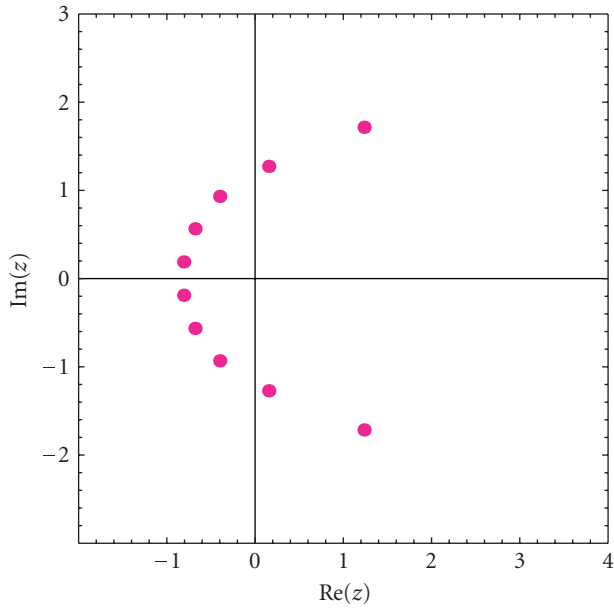


FIGURE 4.5. Zeros of $E_{10,q,w}(x)$.

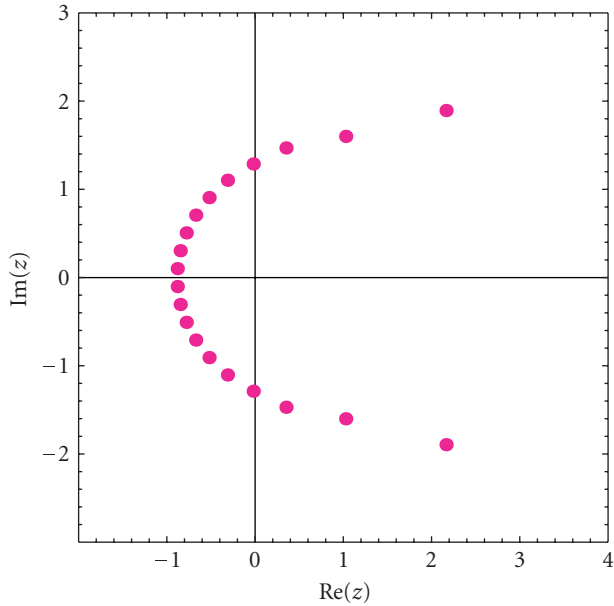


FIGURE 4.6. Zeros of $E_{20,q,w}(x)$.

Our numerical results for numbers of real and complex zeros of $E_{n,q,w}(x)$, $q = 1/2$ are displayed in Table 4.1.

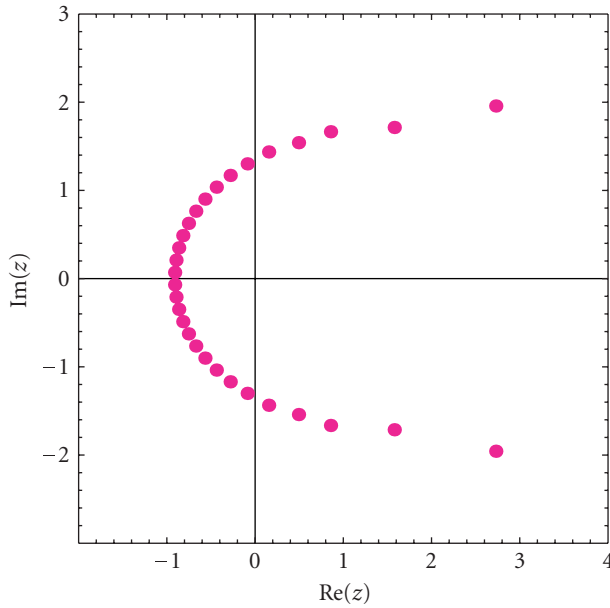


FIGURE 4.7. Zeros of $E_{30,q,w}(x)$.

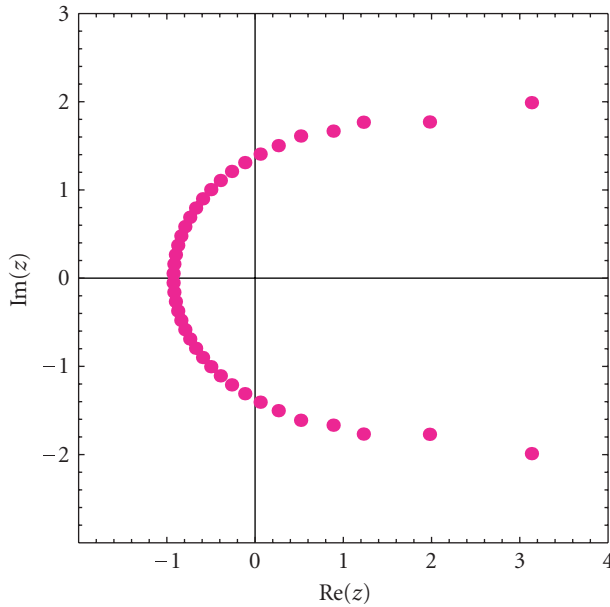


FIGURE 4.8. Zeros of $E_{40,q,w}(x)$.

We will consider the more general open problem. In general, how many roots does $E_{n,q,w}(x)$ have? Prove or disprove: $E_{n,q}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x)}$ of $E_{n,q,w}(x)$, $\text{Im}(x) \neq 0$. Prove or give a counterexample.

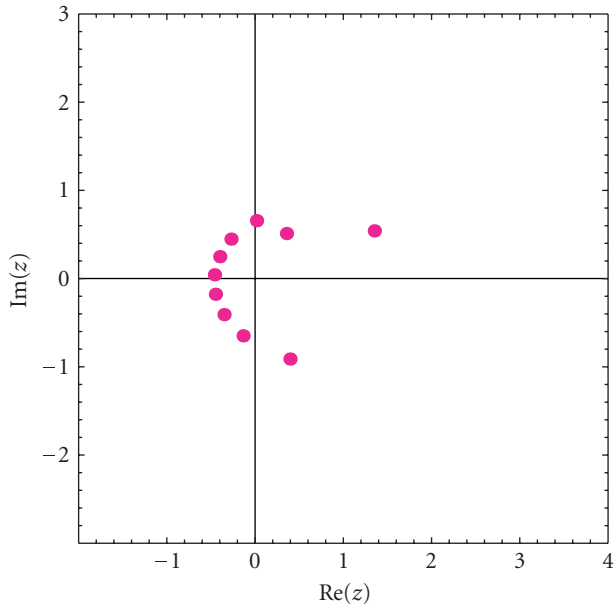


FIGURE 4.9. Zeros of $E_{10,q,w}(x)$.

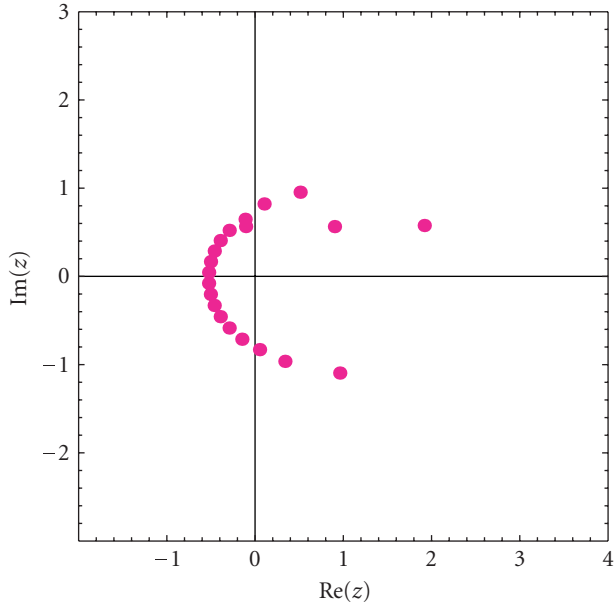


FIGURE 4.10. Zeros of $E_{20,q,w}(x)$.

Conjecture. Since n is the degree of the polynomial $E_{n,q,w}(x)$, the number of real zeros $R_{E_{n,q,w}(x)}$ lying on the real plane $\text{Im}(x) = 0$ is then $R_{E_{n,q,w}(x)} = n - C_{E_{n,q,w}(x)}$, where $C_{E_{n,q,w}(x)}$

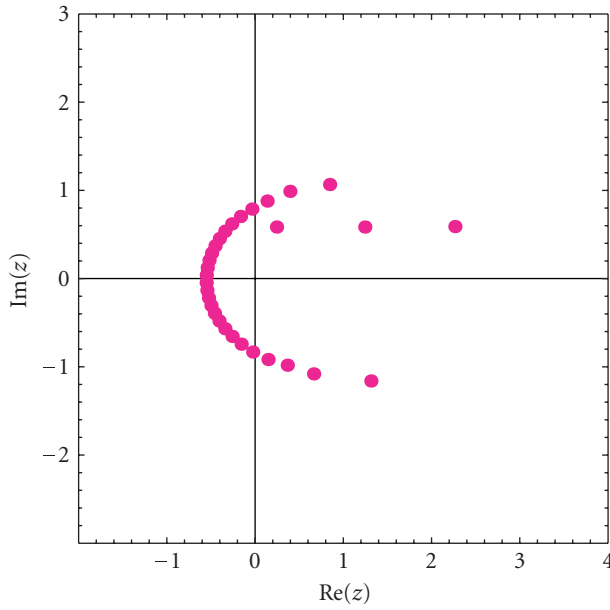


FIGURE 4.11. Zeros of $E_{30,q,w}(x)$.

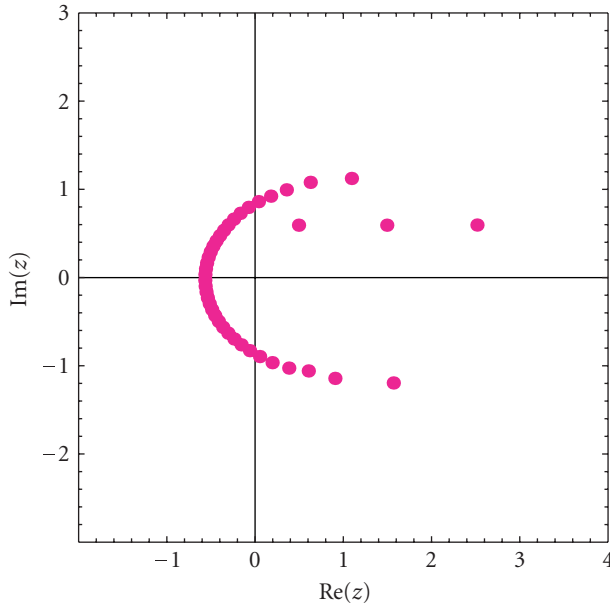


FIGURE 4.12. Zeros of $E_{40,q,w}(x)$.

denotes complex zeros. See Table 4.1 for tabulated values of $R_{E_{n,q,w}(x)}$ and $C_{E_{n,q,w}(x)}$. The authors have no doubt that investigation along this line will lead to a new approach

TABLE 4.1. Numbers of real and complex zeros of $E_{n,q,w}(x)$.

Degree n	$w = e^{2\pi i/2}$		$w = e^{2\pi i/4}$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	0	1
2	0	2	0	2
3	1	2	0	3
4	0	4	0	4
5	1	4	0	5
6	0	6	0	6
7	1	6	0	7
8	0	8	0	8
9	1	8	0	9
10	0	10	0	10

TABLE 4.2. Approximate solutions of $E_{n,q,w}(x) = 0$, $w = e^{2\pi i/2}$.

Degree n	x
1	-0.584963
2	-0.403677 - 0.708194i, -0.403677 + 0.708194i
3	-0.683972, -0.111459 - 1.05549i, -0.111459 + 1.05549i
4	-0.635616 - 0.40476i, -0.635616 + 0.40476i, 0.158518 - 1.26338i, 0.158518 + 1.26338i
5	-0.746907, -0.510667 - 0.673i, -0.510667 + 0.673i, 0.39548 - 1.40175i, 0.39548 + 1.40175i
6	-0.728973 - 0.291464i, -0.728973 + 0.291464i, -0.368723 - 0.86325i, -0.368723 + 0.86325i, 0.603353 - 1.50045i, 0.603353 + 1.50045i

employing numerical method in the field of research of the $E_{n,q,w}(x)$ to appear in mathematics and physics. The reader may refer to [11, 14–16] for the details. We calculated an approximate solution satisfying $E_{n,q,w}(x)$, $N = 2, 4$, $q = 1/2$, $x \in \mathbb{C}$. The results are given in Tables 4.2 and 4.3.

5. Further remarks and observations

Using p -adic q -fermionic integral, Rim and Kim [13] studied explicit p -adic expansion for alternating sums of powers. In the recent paper [10], Kim and Rim constructed (h, q) -extensions of the twisted Euler numbers and polynomials. They also defined (h, q) -generalizations of the twisted zeta function and L -series. These numbers and polynomials are considered as the (h, q) -extensions of their previous results. However, these (h, q) -Euler numbers and generating functions do not seem to be natural extension of Euler numbers and polynomials. By this reason, we consider the natural q -extension of Euler numbers and polynomials. In this paper, we include the numerical computations for our

TABLE 4.3. Approximate solutions of $E_{n,q,w}(x) = 0$, $w = e^{2\pi i/4}$.

Degree n	x
1	$0.117233 + 0.315473i$
2	$-0.349486 - 0.000392286i, \quad 0.499266 + 0.489888i$
3	$-0.434656 - 0.317986i, \quad -0.214323 + 0.325767i, \quad 0.807131 + 0.571069i$
4	$-0.491443 + 0.120881i, \quad -0.384977 - 0.598516i, \quad -0.0235844 + 0.482636i$ $1.06026 + 0.618831i$
5	$-0.573494 - 0.129254i, \quad -0.422702 + 0.36838i, \quad -0.282405 - 0.820568i,$ $0.164523 + 0.577386i, \quad 1.27487 + 0.650419i$
6	$-0.578686 - 0.346326i, \quad -0.569935 + 0.151674i, \quad -0.330723 + 0.532678i,$ $-0.160255 - 0.9957i, \quad 0.339376 + 0.642424i, \quad 1.46114 + 0.672881i$

twisted q -Euler numbers and polynomials and the Euler numbers and polynomials which are treated in this paper. In [9], many interesting integral equations related to fermionic p -adic integrals on \mathbb{Z}_p are known. We proceed by first constructing generating functions of the twisted q -Euler polynomials and numbers. Then, by applying Mellin transformation to these generating functions, integral representations of the twisted q -Euler zeta function (and l -functions) are obtained, which interpolate the (generalized) twisted q -Euler numbers at nonpositive integers.

References

- [1] M. Cenkci and M. Can, "Some results on q -analogue of the Lerch zeta function," *Advanced Studies in Contemporary Mathematics*, vol. 12, no. 2, pp. 213–223, 2006.
- [2] M. Cenkci, M. Can, and V. Kurt, " p -adic interpolation functions and Kummer-type congruences for q -twisted and q -generalized twisted Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 203–216, 2004.
- [3] A. Kudo, "A congruence of generalized Bernoulli number for the character of the first kind," *Advanced Studies in Contemporary Mathematics*, vol. 2, pp. 1–8, 2000.
- [4] Q.-M. Luo, "Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order," *Advanced Studies in Contemporary Mathematics*, vol. 10, no. 1, pp. 63–70, 2005.
- [5] Q.-M. Luo and F. Qi, "Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 7, no. 1, pp. 11–18, 2003.
- [6] T. Kim, " q -Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
- [7] T. Kim, "A note on q -Volkenborn integration," *Proceedings of the Jangjeon Mathematical Society*, vol. 8, no. 1, pp. 13–17, 2005.
- [8] T. Kim, " q -Euler numbers and polynomials associated with p -adic q -integrals," *Journal of Non-linear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.
- [9] T. Kim, "A note on some formulas for the q -Euler numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 9, pp. 227–232, 2006.
- [10] T. Kim and S.-H. Rim, "On the twisted q -Euler numbers and polynomials associated with basic q - l -functions," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 738–744, 2007.

- [11] T. Kim, C. S. Ryoo, L. C. Jang, and S.-H. Rim, "Exploring the q -Riemann zeta function and q -Bernoulli polynomials," *Discrete Dynamics in Nature and Society*, vol. 2005, no. 2, pp. 171–181, 2005.
- [12] H. Ozden, Y. Simsek, S.-H. Rim, and I. N. Cangul, "A note on p -adic q -Euler measure," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 233–239, 2007.
- [13] S.-H. Rim and T. Kim, "Explicit p -adic expansion for alternating sums of powers," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 241–250, 2007.
- [14] C. S. Ryoo and T. Kim, "A numerical computation of the structure of the roots of q -Bernoulli polynomials," to appear in *Journal of Computational and Applied Mathematics*.
- [15] C. S. Ryoo, T. Kim, and R. P. Agarwal, "A numerical investigation of the roots of q -polynomials," *International Journal of Computer Mathematics*, vol. 83, no. 2, pp. 223–234, 2006.
- [16] C. S. Ryoo, T. Kim, and L. C. Jang, "A note on generalized Euler numbers and polynomials," *International Journal of Computer Mathematics*, vol. 84, no. 7, pp. 1099–1111, 2007.
- [17] Y. Simsek, "Theorems on twisted L -function and twisted Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 11, no. 2, pp. 205–218, 2005.
- [18] Y. Simsek and S. Yang, "Transformation of four Titchmarsh-type infinite integrals and generalized Dedekind sums associated with Lambert series," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 195–202, 2004.

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