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Research Article

Some Characteristic Quantities Associated with Homogeneous P-Type and M-Type Functions

Ya-Ping Fang, Nan-Jing Huang, and Yeol Je Cho

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Several characteristic quantities associated with homogeneous P-type and M-type functions are introduced and studied in this paper. Further, the concepts of P-property and M-property for a couple of functions are introduced and some quantities for a pair of homogeneous functions having P-property and M-property are obtained, respectively. As an application, a bound for the solution of the homogeneous complementarity problem with a P-type function is derived.

1. Introduction

For any given P-matrix [1](see also [2–4]), $M \in \mathbb{R}^{n\times n}$, Mathias and Pang [5] introduced a quantity $\alpha(M)$ by

$$\alpha(M) = \min_{\|x\|_\infty = 1} \max_{1\le i\le n} x_i(Mx)_i. \tag{1.1}$$

In terms of $\alpha(M)$, a bound for the solution of the linear complementarity problem LCP(M,q) (see [2–4]) with a P-matrix M is established in [5]. Recently, Xiu and Zhang [6] further gave some new properties of $\alpha(M)$ and introduced a new quantity $\beta(M)$, which is defined by

$$\beta(M) = \max_{\|x\|_\infty = 1} \max_{1\le i\le n} x_i(Mx)_i. \tag{1.2}$$

Moreover, Xiu and Zhang [6] introduced a fundamental quantity $\alpha\{A,B\}$ associated with a pair $\{A,B\}$ having v-column P-property (see [2–4, 7]) by

$$\alpha\{A,B\} = \min_{\|x\|_\infty=1} \max_{1\le i\le n} (Ax)_i(Bx)_i, \tag{1.3}$$

where $A,B \in \mathbb{R}^{m\times n}$. They developed some characteristic quantities of $\alpha\{A,B\}$. By means of these quantities, Xiu and Zhang [6] established global error bounds for the vertical and horizontal linear complementarity problems.

Motivated by these works, in this paper, we introduce the concepts of P-type and M-type functions and give several quantities for homogeneous P-type and M-type functions. Furthermore, we give the concepts of P-property and M-property for a couple of functions, and obtain some quantities for homogeneous continuous pair with P-property and M-property, respectively. As an application, a bound of the solution to the homogeneous complementarity problem with a P-type function is obtained.

2. Characteristic quantities for P-Type and M-Type functions

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a function. We say that T is positively homogeneous with degree $\theta > 0$ if $T(\lambda x) = \lambda^\theta T(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Define $\mathcal{H}$ by

$$\mathcal{H} = \{T \mid T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is continuous and positively homogeneous}\}. \tag{2.1}$$

Given $T \in \mathcal{H}$, define

$$\|T\| = \max_{\|x\|=1} ||T(x)|| = \sup_{x\neq 0} \frac{||T(x)||}{\|x\|^\theta}, \tag{2.2}$$

where $\theta > 0$ is the positively homogeneous degree of T and $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Theorem 2.1. *Let $T,S : \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Then the following conclusions hold:*

(i) $\|T(x)\| \le \|T\| \cdot \|x\|^\theta$;

(ii) *if the inverse T^{-1} in $\mathcal{H}$ exists, then T^{-1} is positively homogeneous with degree $1/\theta$;*

(iii) $\|T\cdot S\| \le \|T\| \cdot \|S\|^\theta$.

Proof. (i) This follows directly from (2.2).

(ii) Since $T^{-1} \in \mathcal{H}$, we suppose the degree of T^{-1} is θ'. It follows that

$$(T^{-1}\cdot T)(\lambda x) = \lambda x = T^{-1}(\lambda^\theta T(x)) = \lambda^{\theta\theta'}(T^{-1}\cdot T)(x) = \lambda^{\theta\theta'} x. \tag{2.3}$$

Hence $\theta' = 1/\theta$.

(iii) It is easy to see that $T\cdot S$ is positively homogeneous with degree $\theta\rho$. By (2.2),

$$\begin{aligned}\|T\cdot S\| &= \sup_{x\neq 0}\frac{\|T(S(x))\|}{\|x\|^{\theta\rho}} \leq \sup_{x\neq 0}\frac{\|T\|\cdot\|S(x)\|^{\theta}}{\|x\|^{\theta\rho}} \\ &\leq \sup_{x\neq 0}\frac{\|T\|\cdot\|S\|^{\theta}\cdot\|x\|^{\theta\rho}}{\|x\|^{\theta\rho}} = \|T\|\cdot\|S\|^{\theta}.\end{aligned} \tag{2.4}$$

This completes the proof. □

Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a function. Recall that T is a P-function (see [3, 4]) if

$$\max_{1\leq i\leq n}\left(x_i-y_i\right)\left(T(x)-T(y)\right)_i>0 \tag{2.5}$$

for all $x\neq y$.

We now introduce the concepts of M-type and P-type functions as follows.

Definition 2.2. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a function. T is said to be

(i) M-type if

$$\min_{1\leq i\leq n}x_i\left(T(x)\right)_i>0,\quad \forall x\neq 0; \tag{2.6}$$

(ii) P-type if

$$\max_{1\leq i\leq n}x_i\left(T(x)\right)_i>0,\quad \forall x\neq 0. \tag{2.7}$$

Note that a P-function T with $T(0)=0$ is P-type and a function $T:\mathbb{R}^n\to\mathbb{R}^n$ is M-type if $T(0)=0$ and $T_i:\mathbb{R}^n\to\mathbb{R}$ is strictly monotone for each i, where $T_i(x)=[T(x)]_i$.

For any given P-type and positively homogeneous function T with degree $\theta>0$, we define $\alpha(T)$ and $\beta(T)$ by

$$\alpha(T)=\min_{\|x\|_\infty=1}\max_{1\leq i\leq n}x_i\left(T(x)\right)_i=\inf_{x\neq 0}\frac{\max_{1\leq i\leq n}x_i\left(T(x)\right)_i}{\|x\|_\infty^{\theta+1}}, \tag{2.8}$$

$$\beta(T)=\max_{\|x\|_\infty=1}\max_{1\leq i\leq n}x_i(T(x))_i=\sup_{x\neq 0}\frac{\max_{1\leq i\leq n}x_i(T(x))_i}{\|x\|_\infty^{\theta+1}}, \tag{2.9}$$

where $\|x\|_\infty=\max_{1\leq i\leq n}\{|x_i|\}$. In addition, if T is M-type, we can further define $\alpha'(T)$ and $\beta'(T)$ by

$$\alpha'(T)=\max_{\|x\|_\infty=1}\min_{1\leq i\leq n}x_i(T(x))_i=\sup_{x\neq 0}\frac{\min_{1\leq i\leq n}x_i\left(T(x)\right)_i}{\|x\|_\infty^{\theta+1}}, \tag{2.10}$$

$$\beta'(T)=\min_{\|x\|_\infty=1}\min_{1\leq i\leq n}x_i\left(T(x)\right)_i=\inf_{x\neq 0}\frac{\min_{1\leq i\leq n}x_i(T(x))_i}{\|x\|_\infty^{\theta+1}}. \tag{2.11}$$

Obviously, $\alpha(T)$, $\beta(T)$, $\alpha'(T)$, and $\beta'(T)$ are well defined, finite, and positive.

Remarks 2.3. The definitions of $\alpha(T)$, $\beta(T)$ associated with a P-type positively homogeneous function T generalize the definitions of $\alpha(M)$, $\beta(M)$ associated with a P-matrix in [5, 6], respectively.

By (2.8)–(2.11), we can obtain the following proposition.

Proposition 2.4. *Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a positively homogeneous function with degree θ. Then the following conclusions hold:*

(i) *if T is P-type, then*

$$\alpha(T)\|x\|_\infty^{\theta+1}\le\max_{1\le i\le n}x_i\big(T(x)\big)_i\le\beta(T)\|x\|_\infty^{\theta+1};\tag{2.12}$$

(ii) *if T is M-type, then*

$$\beta'(T)\|x\|_\infty^{\theta+1}\le\min_{1\le i\le n}x_i\big(T(x)\big)_i\le\alpha'(T)\|x\|_\infty^{\theta+1},\tag{2.13}$$

$$\beta'(T)\le\alpha'(T)\le\alpha(T)\le\beta(T).\tag{2.14}$$

Theorem 2.5. *Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a P-type and positively homogeneous function with degree θ and have inverse T^{-1} in $\mathcal{H}$. Then the following conclusions hold:*

(a) $\beta(T)\le\|T\|_\infty$;
(b) $\alpha(T)\le 1/\|T^{-1}\|_\infty^\theta$;
(c) $1/\|T^{-1}\|_\infty^{\theta+1}\le\beta(T)/\beta(T^{-1})$, $\alpha(T)/\alpha(T^{-1})\le\|T\|_\infty^{1+1/\theta}$.

Proof. For any nonzero $x\in\mathbb{R}^n$, we know

$$x_i\big(T(x)\big)_i\le\|x\|_\infty\cdot\|T(x)\|_\infty\le\|T\|_\infty\cdot\|x\|_\infty^{\theta+1},\quad i=1,2,\dots,n.\tag{2.15}$$

By (2.9), we obtain $\beta(T)\le\|T\|_\infty$. Hence (a) is true.

From (2.2) and Theorem 2.1,

$$||T^{-1}||_\infty=\sup_{x\neq0}\frac{||T^{-1}(x)||_\infty}{\|x\|_\infty^{1/\theta}}=\sup_{y\neq0}\frac{\|y\|_\infty}{||T(y)||_\infty^{1/\theta}}=\sup_{y\neq0}\frac{\big(\|y\|_\infty^{1+\theta}\big)^{1/\theta}}{\big(||T(y)||_\infty\cdot\|y\|_\infty\big)^{1/\theta}}.\tag{2.16}$$

Since $\|T(y)\|_\infty\cdot\|y\|_\infty\ge\max_{1\le i\le n}y_i(T(y))_i$, we have

$$\begin{aligned}||T^{-1}||_\infty&\le\sup_{y\neq0}\left[\frac{\|y\|_\infty^{1+\theta}}{\max_{1\le i\le n}y_i\big(T(y)\big)_i}\right]^{1/\theta}\\&=\left[\sup_{y\neq0}\frac{\|y\|_\infty^{1+\theta}}{\max_{1\le i\le n}y_i\big(T(y)\big)_i}\right]^{1/\theta}=\left[\frac{1}{\alpha(T)}\right]^{1/\theta}\end{aligned}\tag{2.17}$$

and so

$$\alpha(T)\le\frac{1}{||T^{-1}||_\infty^\theta}.\tag{2.18}$$

Hence (b) is true.

From (2.8), (2.9), and Theorem 2.1, we know

$$\begin{aligned}
\beta(T^{-1}) &= \sup_{x\neq 0}\frac{\max_{1\le i\le n} x_i\big(T^{-1}(x)\big)_i}{\|x\|_\infty^{1+1/\theta}} = \sup_{y\neq 0}\frac{\max_{1\le i\le n} y_i\big(T(y)\big)_i}{\|T(y)\|_\infty^{1+1/\theta}} \\
&\ge \sup_{y\neq 0}\frac{\max_{1\le i\le n} y_i\big(T(y)\big)_i}{\|T\|_\infty^{1+1/\theta}\cdot\|y\|_\infty^{1+\theta}} = \frac{\beta(T)}{\|T\|_\infty^{1+1/\theta}}, \\
\alpha(T^{-1}) &= \inf_{x\neq 0}\frac{\max_{1\le i\le n} x_i\big(T^{-1}(x)\big)_i}{\|x\|_\infty^{1+1/\theta}} = \inf_{y\neq 0}\frac{\max_{1\le i\le n} y_i\big(T(y)\big)_i}{\|T(y)\|_\infty^{1+1/\theta}} \\
&\ge \inf_{y\neq 0}\frac{\max_{1\le i\le n} y_i\big(T(y)\big)_i}{\|T\|_\infty^{1+1/\theta}\cdot\|y\|_\infty^{1+\theta}} = \frac{\alpha(T)}{\|T\|_\infty^{1+1/\theta}},
\end{aligned} \tag{2.19}$$

which yields the second inequality in (c).

By the same arguments, we can prove

$$\beta(T)\ge\frac{\beta(T^{-1})}{\|T^{-1}\|_\infty^{1+\theta}},\qquad \alpha(T)\ge\frac{\alpha(T^{-1})}{\|T^{-1}\|_\infty^{1+\theta}}, \tag{2.20}$$

which yields the first inequality in (c). This completes the proof. □

Similarly, we can obtain the following results.

Theorem 2.6. *Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be an M-type and positively homogeneous function with degree θ and have inverse T^{-1} in $\mathcal{H}$. Then*

(i) $\beta'(T)\le 1/\|T^{-1}\|_\infty^{\theta}$;

(ii) $1/\|T^{-1}\|_\infty^{\theta+1}\le\beta'(T)/\beta'(T^{-1})$, $\alpha'(T)/\alpha'(T^{-1})\le\|T\|_\infty^{1+1/\theta}$.

Theorem 2.7. *Let $T,S:\mathbb{R}^n\to\mathbb{R}^n$ be two positively homogeneous functions with the same degree θ. Then the following conclusions hold:*

(1) *if both T and S are P-type, then $\beta(T+S)\le\beta(T)+\beta(S)$;*

(2) *if T is P-type and S is M-type, then $\alpha(T+S)\ge\alpha(T)$, $\beta(T+S)\ge\beta(T)$;*

(3) *if both T and S are M-type, then*

$$\begin{gathered}
\beta'(T+S)\ge\beta'(T)+\beta'(S),\qquad \alpha'(T+S)\ge\max\{\alpha'(T),\alpha'(S)\}, \\
\beta'(T+S)\ge\max\{\beta'(T),\beta'(S)\}.
\end{gathered} \tag{2.21}$$

Proof. The facts directly follow from the definitions of α, β, α', β', and simple arguments. □

Remarks 2.8. Theorems 2.5–2.7 generalize partly Theorems 2.1 and 2.5 of Xiu and Zhang [6].

3. Extensions

In this section, we introduce the definitions of P-property and M-property for a pair $\{T,S\}$ and generalize some results for a function T in Section 2 to a pair $\{T,S\}$.

Definition 3.1. Let $T,S:\mathbb{R}^n \to \mathbb{R}^n$ be two functions. Say that $\{T,S\}$ has

(1) P-property if for any nonzero $x \in \mathbb{R}^n$,

$$\max_{1\le i\le n}(T(x))_i(S(x))_i > 0; \tag{3.1}$$

(ii) M-property if for any nonzero $x \in \mathbb{R}^n$,

$$\min_{1\le i\le n}(T(x))_i(S(x))_i > 0. \tag{3.2}$$

Let $T,S \in \mathcal{H}$ with positively homogeneous degrees θ and ρ, respectively, and $\{T,S\}$ have P-property. Define $\alpha\{T,S\}$ and $\beta\{T,S\}$ as follows:

$$\alpha\{T,S\} = \min_{\|x\|_\infty=1}\max_{1\le i\le n}(T(x))_i(S(x))_i = \inf_{x\neq 0}\frac{\max_{1\le i\le n}(T(x))_i(S(x))_i}{\|x\|_\infty^{\theta+\rho}}, \tag{3.3}$$

$$\beta\{T,S\} = \max_{\|x\|_\infty=1}\max_{1\le i\le n}(T(x))_i(S(x))_i = \sup_{x\neq 0}\frac{\max_{1\le i\le n}(T(x))_i(S(x))_i}{\|x\|_\infty^{\theta+\rho}}. \tag{3.4}$$

Remarks 3.2. The definitions of $\alpha\{T,S\}$, $\beta\{T,S\}$ associated with a positively homogeneous function pair $\{T,S\}$ having P-property generalize the definitions of $\alpha\{M,N\}$, $\beta\{M,N\}$ associated with a matrix pair having ν-column P-property in [2, 6, 7].

In addition, if $\{T,S\}$ has M-property, we can define $\alpha'\{T,S\}$ and $\beta'\{T,S\}$ by

$$\alpha'\{T,S\} = \max_{\|x\|_\infty=1}\min_{1\le i\le n}(T(x))_i(S(x))_i = \sup_{x\neq 0}\frac{\min_{1\le i\le n}(T(x))_i(S(x))_i}{\|x\|_\infty^{\theta+\rho}}, \tag{3.5}$$

$$\beta'\{T,S\} = \min_{\|x\|_\infty=1}\min_{1\le i\le n}(T(x))_i(S(x))_i = \inf_{x\neq 0}\frac{\min_{1\le i\le n}(T(x))_i(S(x))_i}{\|x\|_\infty^{\theta+\rho}}. \tag{3.6}$$

By the definitions of $\alpha\{T,S\}$, $\beta\{T,S\}$, $\alpha'\{T,S\}$, and $\beta'\{T,S\}$, we can obtain the following proposition.

Proposition 3.3. *Let $T,S:\mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Then the following conclusions hold:*

(i) *if $\{T,S\}$ has P-property, then*

$$\alpha\{T,S\}\|x\|_\infty^{\theta+\rho} \le \max_{1\le i\le n}(T(x))_i(S(x))_i \le \beta\{T,S\}\|x\|_\infty^{\theta+\rho}; \tag{3.7}$$

(ii) *if $\{T,S\}$ has M-property, then*

$$\beta'\{T,S\}\|x\|_\infty^{\theta+\rho} \le \min_{1\le i\le n}(T(x))_i(S(x))_i \le \alpha'\{T,S\}\|x\|_\infty^{\theta+\rho}, \tag{3.8}$$

$$\beta'\{T,S\} \le \alpha'\{T,S\} \le \alpha\{T,S\} \le \beta\{T,S\}. \tag{3.9}$$

Note that, if T^{-1} exists, then the condition that $\{T,S\}$ has P-property (M-property) is equivalent to the condition that ST^{-1} is P-type (M-type).

Theorem 3.4. *Let $T,S:\mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Suppose that $\{T,S\}$ has P-property and T has inverse T^{-1} in $\mathcal{H}$. Then* *the following conclusions hold:*

(a) $\beta\{T,S\} \leq \|T\|_\infty \cdot \|S\|_\infty$;

(b) $\alpha\{T,S\} \leq \|S\|_\infty / \|T^{-1}\|_\infty^{\theta}$;

(c) $1/\|T^{-1}\|_\infty^{\theta+\rho} \leq \beta\{T,S\}/\beta(ST^{-1})$, $\alpha\{T,S\}/\alpha(ST^{-1}) \leq \|T\|_\infty^{1+(\rho/\theta)}$.

Proof. (a) For any nonzero $x \in \mathbb{R}^n$, it follows from (i) of Theorem 2.1 that

$$\big(T(x)\big)_i\big(S(x)\big)_i \leq \|T(x)\|_\infty \cdot \|S(x)\|_\infty \leq \|T\|_\infty \cdot \|S\|_\infty \cdot \|x\|_\infty^{\theta+\rho}, \quad i = 1,2,\ldots,n. \tag{3.10}$$

By (3.4),

$$\beta\{T,S\} \leq \|T\|_\infty \cdot \|S\|_\infty. \tag{3.11}$$

(b) From (2.2) and (i) of Theorem 2.1,

$$\left\|T^{-1}\right\|_\infty = \sup_{x\neq 0} \frac{\left\|T^{-1}(x)\right\|_\infty}{\|x\|_\infty^{1/\theta}} = \sup_{y\neq 0} \frac{\|y\|_\infty}{\left\|T(y)\right\|_\infty^{1/\theta}} = \sup_{y\neq 0} \frac{\left(\|y\|_\infty^{\theta} \cdot \left\|S(y)\right\|_\infty\right)^{1/\theta}}{\left(\left\|T(y)\right\|_\infty \cdot \left\|S(y)\right\|_\infty\right)^{1/\theta}}. \tag{3.12}$$

Since $\|T(y)\|_\infty \cdot \|y\|_\infty \geq \max_{1\leq i\leq n} y_i(T(y))_i$ and $\|S(y)\|_\infty \leq \|S\|_\infty \cdot \|y\|_\infty^{\rho}$, we have

$$\begin{aligned}\left\|T^{-1}\right\|_\infty &\leq \sup_{y\neq 0}\left[\frac{\|S\|_\infty \cdot \|y\|_\infty^{\rho+\theta}}{\max_{1\leq i\leq n}\big(T(y)\big)_i\big(S(y)\big)_i}\right]^{1/\theta} \\ &= \|S\|_\infty^{1/\theta} \cdot \left[\sup_{y\neq 0}\frac{\|y\|_\infty^{\rho+\theta}}{\max_{1\leq i\leq n}\big(T(y)\big)_i\big(S(y)\big)_i}\right]^{1/\theta} = \left[\frac{\|S\|_\infty}{\alpha\{T,S\}}\right]^{1/\theta}.\end{aligned} \tag{3.13}$$

This implies that

$$\alpha\{T,S\} \leq \frac{\|S\|_\infty}{\left\|T^{-1}\right\|_\infty^{\theta}}. \tag{3.14}$$

(c) It follows from (2.9) that

$$\begin{aligned}\beta(ST^{-1}) &= \sup_{x\neq 0}\frac{\max_{1\leq i\leq n} x_i\big((ST^{-1})(x)\big)_i}{\|x\|_\infty^{1+\rho/\theta}} = \sup_{y\neq 0}\frac{\max_{1\leq i\leq n}\big(T(y)\big)_i\big(S(y)\big)_i}{\left\|T(y)\right\|_\infty^{1+\rho/\theta}} \\ &\geq \sup_{y\neq 0}\frac{\max_{1\leq i\leq n}\big(T(y)\big)_i\big(S(y)\big)_i}{\|T\|_\infty^{1+\rho/\theta}\cdot\|y\|_\infty^{\rho+\theta}} = \frac{\beta\{T,S\}}{\|T\|_\infty^{1+\rho/\theta}}.\end{aligned} \tag{3.15}$$

By (3.4) and Theorem 2.1,

$$\begin{gathered}\left\|T^{-1}(y)\right\|_\infty \leq \left\|T^{-1}\right\|_\infty \cdot \|y\|_\infty^{1/\theta}, \\ \beta\{T,S\} = \sup_{x\neq 0}\frac{\max_{1\leq i\leq n}\big(T(x)\big)_i\big(S(x)\big)_i}{\|x\|_\infty^{\rho+\theta}} = \sup_{y\neq 0}\frac{\max_{1\leq i\leq n} y_i\big((ST^{-1})(y)\big)_i}{\left\|T^{-1}(y)\right\|_\infty^{\rho+\theta}}.\end{gathered} \tag{3.16}$$

It follows that

$$\beta\{T,S\} \geq \sup_{y\neq 0} \frac{\max_{1\leq i\leq n} y_i\big((ST^{-1})(y)\big)_i}{\|T^{-1}\|_\infty^{\theta+\rho} \cdot \|y\|_\infty^{1+\rho/\theta}} = \frac{\beta(ST^{-1})}{\|T^{-1}\|_\infty^{\theta+\rho}}. \tag{3.17}$$

Hence

$$\frac{1}{\|T^{-1}\|_\infty^{\theta+\rho}} \leq \frac{\beta\{T,S\}}{\beta(ST^{-1})} \leq \|T\|_\infty^{1+\rho/\theta}. \tag{3.18}$$

By similar arguments, we can prove that

$$\frac{1}{\|T^{-1}\|_\infty^{\theta+\rho}} \leq \frac{\alpha\{T,S\}}{\alpha(ST^{-1})} \leq \|T\|_\infty^{1+\rho/\theta}. \tag{3.19}$$

This completes the proof. □

Remarks 3.5. Theorem 3.4 generalizes and improves Theorem 2.7 of Xiu and Zhang [6].

Similarly, we can obtain the following result.

THEOREM 3.6. *Let $T,S:\mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Suppose that $\{T,S\}$ has M-property and T has inverse T^{-1} in $\mathcal{H}$. Then the following conclusions hold:*

(1) $\beta'\{T,S\} \leq \|S\|_\infty/\|T^{-1}\|_\infty^{\theta}$;

(2) $1/\|T^{-1}\|_\infty^{\theta+\rho} \leq \alpha'\{T,S\}/\alpha'(ST^{-1}),\ \beta'\{T,S\}/\beta'(ST^{-1}) \leq \|T\|_\infty^{1+(\rho/\theta)}$.

4. An application

In this section, we give a bound for the solution of the homogeneous complementarity problem, denoted by HCP(T,q), which consists of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0, \qquad T(x)+q \geq 0, \qquad x^T\big(T(x)+q\big) = 0, \tag{4.1}$$

where $T:\mathbb{R}^n \to \mathbb{R}^n$ is a P-type and positively homogenous function and $q \in \mathbb{R}^n$.

THEOREM 4.1. *Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a P-type and positively homogeneous function with degree θ. Suppose that T has inverse T^{-1} in $\mathcal{H}$ and x is the unique solution of* HCP(T,q). *Then*

$$\big[\alpha(T^{-1})\big]^\theta \|(-q)_+\|_\infty \leq \|x\|_\infty^\theta \leq \frac{\|(-q)_+\|_\infty}{\alpha(T)}, \tag{4.2}$$

where $(-q)_+$ denotes the nonnegative part of $-q$.

Proof. If $x=0$, then $(-q)_+=0$. The conclusion holds trivially. In the sequel we always suppose that $x \neq 0$, equivalently, q is not nonnegative. Since x solves HCP(T,q), by Proposition 2.4, one has

$$\begin{aligned}\alpha(T)\|x\|_\infty^{\theta+1} &\leq \max_{1\leq i\leq n} x_i\big(T(x)\big)_i = \max_{1\leq i\leq n} x_i(-q)_i \\ &\leq \max_{1\leq i\leq n} x_i\big((-q)_+\big)_i \leq \|x\|_\infty \cdot \|(-q)_+\|_\infty.\end{aligned} \tag{4.3}$$

This implies that

$$\|x\|_\infty^\theta \le \frac{\|(-q)_+\|_\infty}{\alpha(T)}. \tag{4.4}$$

It follows from (2.12) that

$$\alpha(T^{-1})\|y\|_\infty^{1+1/\theta} \le \max_{1\le i\le n} y_i\left(T^{-1}(y)\right)_i. \tag{4.5}$$

Thus we have

$$\alpha(T^{-1})\|T(x)\|_\infty^{1+1/\theta} \le \max_{1\le i\le n} x_i\left(T(x)\right)_i. \tag{4.6}$$

Since $T(x) \ge -q$, we know that $|T(x)| \ge (T(x))_+ \ge (-q)_+$ and so

$$\|T(x)\|_\infty \ge \|(-q)_+\|_\infty. \tag{4.7}$$

By (4.6), (4.7), and the fact that $x_i(T(x)+q)_i = 0$, we know

$$\begin{aligned}\alpha(T^{-1})\|(-q)_+\|_\infty^{1+1/\theta} &\le \alpha(T^{-1})\|T(x)\|_\infty^{1+1/\theta} \le \max_{1\le i\le n} x_i\left(T(x)\right)_i \\ &= \max_{1\le i\le n} x_i(-q)_i \le \|x\|_\infty \cdot \|(-q)_+\|_\infty.\end{aligned} \tag{4.8}$$

Hence

$$\left[\alpha(T^{-1})\right]^\theta \|(-q)_+\|_\infty \le \|x\|_\infty^\theta. \tag{4.9}$$

This completes the proof. □

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Ya-Ping Fang: Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China
Email address: fabhcn@yahoo.com.cn

Nan-Jing Huang: Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China
Email address: nanjinghunag@hotmail.com

Yeol Je Cho: Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea
Email addresses: yjcho@gsnu.ac.kr; mathyjcho@yahoo.com