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Research Article New Strengthened Carleman's Inequality and Hardy's Inequality

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In this note, new upper bounds for Carleman's inequality and Hardy's inequality are established.

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1. Introduction

The following Carleman's inequality and Hardy's inequality are well known.

THEOREM 1.1 (see [1, Theorem 334]). Let $a_n \ge 0 (n \in N)$ and $0 < \sum_{n=1}^{\infty} a_n < +\infty$, then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$
(1.1)

THEOREM 1.2 (see [1, Theorem 349]). Let $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^n \lambda_m$, $a_n \ge 0 (n \in N)$ and $0 < \sum_{n=1}^\infty \lambda_n a_n < +\infty$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n.$$
(1.2)

In [2–16], some refined work on Carleman's inequality and Hardy's inequality had been gained. It is observing that in [3] the authors obtained the following inequalities

$$\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1/5}\right)^{1/2} < e < \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1/6}\right)^{1/2}.$$
(1.3)

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From the inequality above, [3, 4] extended Theorems A and B to the following new results.

THEOREM 1.3 (see [3, Theorem 1]). Let $a_n \ge 0 (n \in N)$ and $0 < \sum_{n=1}^{\infty} a_n < +\infty$, then

$$\sum_{n=1}^{\infty} \left(a_1 a_2 \cdots a_n \right)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n+1/5} \right)^{-1/2} a_n.$$
(1.4)

THEOREM 1.4 (see [4, Theorem]). Let $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^n \lambda_n$, $a_n \ge 0$ $(n \in N)$ and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n \left(1 + \frac{1}{\Lambda_n / \lambda_n + 1/5} \right)^{-1/2} a_n.$$
(1.5)

In this note, Carleman's inequality and Hardy's inequality are strengthened as follows. THEOREM 1.5. Let $a_n \ge 0$ $(n \in N)$, $0 < \sum_{n=1}^{\infty} a_n < +\infty$, and $c \ge \sqrt{6}/4$. Then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2cn + 4c/3 + 1/2} \right)^c a_n.$$
(1.6)

THEOREM 1.6. Let $c \ge \sqrt{6}/4$, $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^n \lambda_m$, $a_n \ge 0$ $(n \in N)$, and $0 < \sum_{n=1}^\infty \lambda_n a_n < +\infty$. Then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2c\Lambda_n + (4c/3 + 1/2)\lambda_n} \right)^c \lambda_n a_n.$$
(1.7)

In order to prove two theorems mentioned above, we need introduce several lemmas first.

2. Lemmas

LEMMA 2.1. Let x > 0 and $c \ge \sqrt{6}/4$. Then inequality

$$\left(1 + \frac{1}{x}\right)^{x} \left(1 + \frac{1}{2cx + 4c/3 - 1/2}\right)^{c} < e$$
(2.1)

or

$$\left(1 + \frac{1}{x}\right)^{x} < e \left(1 - \frac{1}{2cx + 4c/3 + 1/2}\right)^{c}$$
(2.2)

holds. Furthermore, 4c/3 - 1/2 is the best constant in inequality (2.1) or 4c/3 + 1/2 is the best constant in inequality (2.2).

Proof. (i) We construct a function as

$$f(x) = x \ln\left(1 + \frac{1}{x}\right) + c \ln\left(1 + \frac{1}{2cx + b}\right) - 1,$$
(2.3)

where $x \in (0, +\infty)$ and b = 4c/3 - 1/2. It is obvious that the existence of Lemma 2.1 can be ensured when proving f(x) < 0. We simply compute

$$f'(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1} + 2c^2\left(\frac{1}{2cx+b+1} - \frac{1}{2cx+b}\right),$$

$$f''(x) = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} + 4c^3\left(\frac{1}{(2cx+b)^2} - \frac{1}{(2cx+b+1)^2}\right)$$

$$= -\frac{1}{x(x+1)^2} + \frac{4c^3(4cx+2b+1)}{(2cx+b)^2(2cx+b+1)^2}$$

$$= -\frac{p(x)}{x(x+1)^2(2cx+b)^2(2cx+b+1)^2},$$

(2.4)

where $p(x) = (24b^2c^2 + 24bc^2 + 4c^2 - 16c^4 - 16bc^3 - 8c^3)x^2 + (8b^3c + 4b^2c + 4bc - 8bc^3 - 4c^3)x + b^2(b+1)^2$. Since $x \in (0, +\infty)$, b = 4c/3 - 1/2, and $c \ge \sqrt{6}/4$, we have

$$24b^{2}c^{2} + 24bc^{2} + 4c^{2} - 16c^{4} - 16bc^{3} - 8c^{3} \ge 0,$$

$$8b^{3}c + 4b^{2}c + 4bc - 8bc^{3} - 4c^{3} > 0,$$

$$b^{2}(b+1)^{2} > 0.$$
(2.5)

From the above analysis, we easily get that f''(x) < 0 and f'(x) is decreasing on $(0, +\infty)$. Meanwhile $f'(x) > \lim_{x \to +\infty} f'(x) = 0$ for $x \in (0, +\infty)$. Thus, f(x) is increasing on $(0, +\infty)$, and $f(x) < \lim_{x \to +\infty} f(x) = 0$ for $x \in (0, +\infty)$.

(ii) The inequality (2.2) is equivalent to

$$\frac{e^{1/c}}{e^{1/c} - (1+1/x)^{x/c}} - 2cx < \frac{4}{3}c + \frac{1}{2}, \quad x > 0.$$
(2.6)

Let $g(t) = (1+t)^{1/(ct)}$ and t > 0. Then

$$g'(0^{+}) = \lim_{t \to 0^{+}} \frac{(1+t)^{1/(ct)}}{c} \left[\frac{1}{t(1+t)} - \frac{\log(1+t)}{t^{2}} \right] = -\frac{e^{1/c}}{2c},$$

$$g''(0^{+}) = \lim_{t \to 0^{+}} \frac{(1+t)^{1/(ct)}}{c^{2}} \left[\frac{1}{t(1+t)} - \frac{\log(1+t)}{t^{2}} \right]^{2}$$

$$+ \lim_{t \to 0^{+}} \frac{(1+t)^{1/(ct)} \left[-3t^{2} - 2t + 2(1+t^{2})\log(1+t) \right]}{ct^{3}(1+t)^{2}}$$

$$= \left(\frac{1}{4c^{2}} + \frac{2}{3c} \right) e^{1/c}.$$
(2.7)

Using Taylor's formula, we have

$$g(t) = e^{1/c} - \frac{e^{1/c}}{2c}t + \frac{1}{2}\left(\frac{1}{4c^2} + \frac{2}{3c}\right)e^{1/c}t^2 + o(t^2).$$
(2.8)

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When letting x = 1/t and using (2.8) we find that

$$\lim_{x \to +\infty} \left[\frac{e^{1/c}}{e^{1/c} - (1+1/x)^{x/c}} - 2cx \right] = \lim_{t \to 0^+} \frac{te^{1/c} - 2c[e^{1/c} - (1+t)^{1/(ct)}]}{t[e^{1/c} - (1+t)^{1/(ct)}]}$$
$$= \lim_{t \to 0^+} \frac{(1/(4c) + 2/3)e^{1/c}t^2 + o(t^2)}{e^{1/c}t^2/(2c) + o(t^2)}$$
$$= \frac{4}{3}c + \frac{1}{2}.$$
 (2.9)

Therefore, 4c/3 + 1/2 is the best constant in (2.2).

LEMMA 2.2. The inequality

$$\left(1 + \frac{1}{n+1/5}\right)^{1/2} < \left(1 + \frac{2}{3n+1}\right)^{3/4} \tag{2.10}$$

 \square

 \square

holds for every positive integer n.

Proof. Let

$$h(x) = \frac{1}{2} \ln\left(1 + \frac{1}{x + 1/5}\right) - \frac{3}{4} \ln\left(1 + \frac{2}{3x + 1}\right)$$
(2.11)

for $x \in [1, +\infty)$, then

$$h'(x) = \frac{x/5 - 7/25}{2(x+6/5)(x+1/5)(x+1)(3x+1)}.$$
(2.12)

Thus, h(x) is decreasing on [1,7/5). Since for h(1) < 0, we have h(x) < 0 on [1,7/5). At the same time, h(x) is increasing on [7/5,+ ∞), and we have $h(x) < \lim_{x \to +\infty} h(x) = 0$ on [7/5,+ ∞). Hence h(x) < 0 on [1,+ ∞). By the definition of h(x), it turns out that the inequality (2.10) is accrate.

In the same way we can prove the following result.

LEMMA 2.3. *The inequality*

$$\left(1 + \frac{2}{3n+1}\right)^{3/4} < \left(1 + \frac{1}{(5/4)n+1/3}\right)^{5/8}$$
(2.13)

holds for every positive integer n.

Combining Lemmas 2.1, 2.2, and 2.3 gives

LEMMA 2.4. The inequality

$$\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1/5}\right)^{1/2} < \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{2}{3n+1}\right)^{3/4} < \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{(5/4)n+1/3}\right)^{5/8} < e$$
(2.14)

holds for every positive integer n.

3. Proof of Theorem 1.5

By the virtue of the proof of article [3], we can testify Theorem 1.5. Assume that $c_n > 0$ for $n \in N$. Then applying the arithmetic-geometric average inequality, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} = \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n}$$
$$\leq \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/n} \frac{1}{n} \sum_{m=1}^n c_m a_m$$
$$= \sum_{m=1}^{\infty} c_m a_m \sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}.$$
(3.1)

Setting $c_m = (m+1)^m / m^{m-1}$, we have $c_1 c_2 \cdots c_n = (n+1)^n$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{\infty} c_m a_m \sum_{n=m}^{\infty} \frac{1}{n(n+1)}$$
$$= \sum_{m=1}^{\infty} \frac{1}{m} c_m a_m$$
$$= \sum_{m=1}^{\infty} \left(1 + \frac{1}{m}\right)^m a_m.$$
(3.2)

By (3.2) and (2.2), we obtain

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2cn + 4c/3 + 1/2} \right)^c a_n.$$
(3.3)

Thus, Theorem 1.5 is proved.

4. Proof of Theorem 1.6

Now, processing the proof of Theorem 1.6. Assume that $c_n > 0$ for $n \in N$. Using the arithmetic-geometric average inequality we obtain

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} = \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \Big[(c_1 a_1)^{\lambda_1} (c_2 a_2)^{\lambda_2} \cdots (c_n a_n)^{\lambda_n} \Big]^{1/\Lambda_n} \\ \leq \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m c_m a_m \\ = \sum_{m=1}^{\infty} \lambda_m c_m a_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}}.$$
(4.1)

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Choosing $c_n = (1 + \lambda_{n+1}/\Lambda_n)^{\Lambda_n/\lambda_n} \Lambda_n$, we get that

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \leq \sum_{m=1}^{\infty} \left(1 + \frac{\lambda_{m+1}}{\Lambda_m} \right)^{\Lambda_m/\lambda_m} \lambda_m a_m$$

$$\leq \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m} \right)^{\Lambda_m/\lambda_m} \lambda_m a_m$$

$$< e \sum_{m=1}^{\infty} \left(1 - \frac{1}{2c(\Lambda_m/\lambda_m) + 4c/3 + 1/2} \right)^c \lambda_m a_m$$

$$= e \sum_{m=1}^{\infty} \left(1 - \frac{\lambda_m}{2c\Lambda_m + (4c/3 + 1/2)\lambda_m} \right)^c \lambda_m a_m,$$
(4.2)

from (4.1) and (2.2).

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