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Research Article

Schur-Convexity of Two Types of One-Parameter Mean Values in n Variables

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We establish Schur-convexities of two types of one-parameter mean values in n variables. As applications, Schur-convexities of some well-known functions involving the complete elementary symmetric functions are obtained.

1. Introduction

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ denotes the set of strictly positive real numbers. Let $n \geq 2$, $n \in \mathbb{N}$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, and $\mathbf{x}^{1/r} = (x_1^{1/r}, x_2^{1/r}, \dots, x_n^{1/r})$, where $r \in \mathbb{R}$, $r \neq 0$; let $E_{n-1} \subset \mathbb{R}^{n-1}$ be the simplex

$$E_{n-1} = \left\{ (u_1, \dots, u_{n-1}) : u_i > 0 (1 \leq i \leq n-1), \sum_{i=1}^{n-1} u_i \leq 1 \right\}, \tag{1.1}$$

and let $d\mu = du_1, \dots, du_{n-1}$ be the differential of the volume in E_{n-1}.

The weighted arithmetic mean $A(\mathbf{x}, \mathbf{u})$ and the power mean $M_r(\mathbf{x}, \mathbf{u})$ of order r with respect to the numbers $x_1, x_2, \dots, x_n$ and the positive weights $u_1, u_2, \dots, u_n$ with $\sum_{i=1}^n u_i = 1$ are defined, respectively, as $A(\mathbf{x}, \mathbf{u}) = \sum_{i=1}^n u_i x_i$, $M_r(\mathbf{x}, \mathbf{u}) = (\sum_{i=1}^n u_i x_i^r)^{1/r}$ for $r \neq 0$, and $M_0(\mathbf{x}, \mathbf{u}) = \prod_{i=1}^n x_i^{u_i}$. For $\mathbf{u} = (1/n, 1/n, \dots, 1/n)$, we denote $A(\mathbf{x}, \mathbf{u}) \overset{\Delta}{=} A(\mathbf{x})$, $M_r(\mathbf{x}, \mathbf{u}) \overset{\Delta}{=} M_r(\mathbf{x})$.

The well-known logarithmic mean $L(x_1, x_2)$ of two positive numbers x_1 and x_2 is

$$L(x_1, x_2) = \begin{cases} \dfrac{x_1 - x_2}{\ln x_1 - \ln x_2}, & x_1 \neq x_2, \\ x_1, & x_1 = x_2. \end{cases} \tag{1.2}$$

As further generalization of $L(x_1,x_2)$, Stolarsky [1] studied the one-parameter mean, that is,

$$L_r(x_1,x_2)=\begin{cases}\left(\dfrac{x_1^{r+1}-x_2^{r+1}}{(r+1)(x_1-x_2)}\right)^{1/r}, & r\neq-1,0,\ x_1\neq x_2,\\[2ex] \dfrac{x_1-x_2}{\ln x_1-\ln x_2}, & r=-1,\ x_1\neq x_2,\\[2ex] \dfrac{1}{e}\left(\dfrac{x_1^{x_1}}{x_2^{x_2}}\right)^{1/(x_1-x_2)}, & r=0,\ x_1\neq x_2,\\[2ex] x_1, & x_1=x_2.\end{cases}\tag{1.3}$$

Alzer [2, 3] obtained another form of one-parameter mean, that is,

$$F_r(x_1,x_2)=\begin{cases}\dfrac{r}{r+1}\cdot\dfrac{x_1^{r+1}-x_2^{r+1}}{x_1^r-x_2^r}, & r\neq-1,0,\ x_1\neq x_2,\\[2ex] x_1x_2\cdot\dfrac{\ln x_1-\ln x_2}{x_1-x_2}, & r=-1,\ x_1\neq x_2,\\[2ex] \dfrac{x_1-x_2}{\ln x_1-\ln x_2}, & r=0,\ x_1\neq x_2,\\[2ex] x_1, & x_1=x_2.\end{cases}\tag{1.4}$$

These two means can be written also as

$$\begin{gathered}L_r(x_1,x_2)=\begin{cases}\left(\displaystyle\int_0^1(x_1u+x_2(1-u))^r\,du\right)^{1/r}, & r\neq0,\\[2ex] \exp\left(\displaystyle\int_0^1\ln(x_1u+x_2(1-u))\,du\right), & r=0,\end{cases}\\ F_r(x_1,x_2)=\begin{cases}\displaystyle\int_0^1(x_1^ru+x_2^r(1-u))^{1/r}\,du, & r\neq0,\\[2ex] \displaystyle\int_0^1x_1^ux_2^{1-u}\,du, & r=0.\end{cases}\end{gathered}\tag{1.5}$$

Correspondingly, Pittenger [4] and Pearce et al. [5] investigated the means above in n variables, respectively,

$$\begin{gathered}L_r(\mathbf{x})=\begin{cases}\left((n-1)!\displaystyle\int_{E_{n-1}}(A(\mathbf{x},\mathbf{u}))^r\,d\mu\right)^{1/r}, & r\neq0,\\[2ex] \exp\left((n-1)!\displaystyle\int_{E_{n-1}}\ln A(\mathbf{x},\mathbf{u})\,d\mu\right), & r=0,\end{cases}\\ F_r(\mathbf{x})=(n-1)!\int_{E_{n-1}}M_r(\mathbf{x},\mathbf{u})\,d\mu,\end{gathered}\tag{1.6}$$

where $u_n=1-\sum_{i=1}^{n-1}u_i$.

Expressions (1.3) and (1.4) can be also written by using 2-order determinants, that is,

$$L_r(x_1,x_2)=\begin{cases}\left(\dfrac{1}{r+1}\cdot\begin{vmatrix}1 & x_2^{r+1}\\ 1 & x_1^{r+1}\end{vmatrix}\Big/\begin{vmatrix}1 & x_2\\ 1 & x_1\end{vmatrix}\right)^{1/r}, & r\neq -1,0,\ x_1\neq x_2,\\ \begin{vmatrix}1 & x_2\\ 1 & x_1\end{vmatrix}\Big/\begin{vmatrix}1 & \ln x_2\\ 1 & \ln x_1\end{vmatrix}, & r=-1,\ x_1\neq x_2,\\ \exp\left\{\left(\begin{vmatrix}1 & x_2\ln x_2\\ 1 & x_1\ln x_1\end{vmatrix}\Big/\begin{vmatrix}1 & x_2\\ 1 & x_1\end{vmatrix}\right)-1\right\}, & r=0,\ x_1\neq x_2,\\ x_1, & x_1=x_2,\end{cases}$$
$$F_r(x_1,x_2)=\begin{cases}\left(r\begin{vmatrix}1 & x_2^{r+1}\\ 1 & x_1^{r+1}\end{vmatrix}\right)\Big/\left((r+1)\begin{vmatrix}1 & x_2^{r}\\ 1 & x_1^{r}\end{vmatrix}\right), & r\neq -1,0,\ x_1\neq x_2,\\ x_1x_2\begin{vmatrix}1 & \ln x_2\\ 1 & \ln x_1\end{vmatrix}\Big/\begin{vmatrix}1 & x_2\\ 1 & x_1\end{vmatrix}, & r=-1,\ x_1\neq x_2,\\ \begin{vmatrix}1 & x_2\\ 1 & x_1\end{vmatrix}\Big/\begin{vmatrix}1 & \ln x_2\\ 1 & \ln x_1\end{vmatrix}, & r=0,\ x_1\neq x_2,\\ x_1, & x_1=x_2.\end{cases}\tag{1.7}$$

Utilizing higher-order generalized Vandermonde determinants, Xiao et al. [8, 7, 6, 9] gave the analogous definitions of $L_r(\mathbf{x})$ and $F_r(\mathbf{x})$.

Obviously, $L_r(\mathbf{x})$ and $F_r(\mathbf{x})$ are symmetric with respect to $x_1,x_2,\dots,x_n$, $r\mapsto L_r(\mathbf{x})$ and $r\mapsto F_r(\mathbf{x})$ are continuous for any $\mathbf{x}\in\mathbb{R}_+^n$.

In [4, 5, 10, 11], the authors studied the Schur-convexities of $L_r(x_1,x_2)$ and $F_r(x_1,x_2)$. In this paper, we establish the Schur-convexities of two types of one-parameter mean values $L_r(\mathbf{x})$ and $F_r(\mathbf{x})$ for several positive numbers. As applications, Schur-convexities of some well-known functions involving the complete elementary symmetric functions are obtained.

2. Some definitions and lemmas

The Schur-convex function was introduced by Schur [12] in 1923, and has many important applications in analytic inequalities. The following definitions can be found in many references such as [12–17].

Definition 2.1. For $\mathbf{u}=(u_1,u_2,\dots,u_n)$, $\mathbf{v}=(v_1,v_2,\dots,v_n)\in\mathbb{R}^n$, without loss of generality, it is assumed that $u_1\geq u_2\geq\cdots\geq u_n$ and $v_1\geq v_2\geq\cdots\geq v_n$. Then $\mathbf{u}$ is said to be majorized by $\mathbf{v}$ (in symbols $\mathbf{u}\prec\mathbf{v}$) if $\sum_{i=1}^k u_i\leq\sum_{i=1}^k v_i$ for $k=1,2,\dots,n-1$ and $\sum_{i=1}^n u_i=\sum_{i=1}^n v_i$.

Definition 2.2. Let $\Omega\subset\mathbb{R}^n$. A function $\varphi:\Omega\mapsto\mathbb{R}$ is said to be a Schur-convex (Schur-concave) function if $\mathbf{u}\prec\mathbf{v}$ implies $\varphi(\mathbf{u})\leq(\geq)\varphi(\mathbf{v})$.

Every Schur-convex function is a symmetric function [18]. But it is not hard to see that not every symmetric function can be a Schur-convex function [15, page 258]. However, we have the following so-called Schur condition.

Lemma 2.3 [12, page 57]. *Suppose that $\Omega \subset \mathbb{R}^n$ is symmetric with respect to permutations and convexset, and has a nonempty interior set Ω^0. Let $\varphi : \Omega \mapsto \mathbb{R}$ be continuous on Ω and continuously differentiable in Ω^0. Then, φ is a Schur-convex (Schur-concave) function if and only if it is symmetric and if*

$$(u_1 - u_2)\left(\frac{\partial \varphi}{\partial u_1} - \frac{\partial \varphi}{\partial u_2}\right) \geq (\leq)0 \tag{2.1}$$

holds for any $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \Omega^0$.

Lemma 2.4. *Let $m \geq 1, n \geq 2$, m, $n \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$, $\phi : \Lambda \times \Omega \mapsto \mathbb{R}$, $\phi(\mathbf{v}, \mathbf{x})$ be continuous with respect to $\mathbf{v} \in \Lambda$ for any $\mathbf{x} \in \Omega$. Let Δ be a set of all $\mathbf{v} \in \Lambda$ such that the function $\mathbf{x} \mapsto \phi(\mathbf{v}, \mathbf{x})$ is a Schur-convex (Schur-concave) function. Then Δ is a closed set of Λ.*

Proof. Let $l \geq 1$, $l \in \mathbb{N}$, $\mathbf{v}_l \in \Delta$, $\mathbf{v}_0 \in \Lambda$, $\mathbf{v}_l \to \mathbf{v}_0$ if $l \to +\infty$. According to Definition 2.2, $\phi(\mathbf{v}_l, \mathbf{y}) \geq (\leq)\phi(\mathbf{v}_l, \mathbf{z})$ holds for any $\mathbf{y}, \mathbf{z} \in \Omega$ and $\mathbf{y} \succ \mathbf{z}$. Let $l \to +\infty$, then we have $\phi(\mathbf{v}_0, \mathbf{y}) \geq (\leq)\phi(\mathbf{v}_0, \mathbf{z})$. Hence $\mathbf{v}_0 \in \Delta$, so Δ is a closed set of Λ. □

3. Main results

Theorem 3.1. *Given $r \in \mathbb{R}$, $L_r(\mathbf{x})$ is Schur-convex if $r \geq 1$ and Schur-concave if $r \leq 1$.*

Proof. Denote $\tilde{\mathbf{u}} = (u_2, u_1, u_3, \dots, u_n)$.

If $r \neq 0$, owing to the symmetry of $L_r(x)$ with respect to $x_1, x_2, \dots, x_n$, we have

$$g_r(\mathbf{x}) \triangleq \int_{E_{n-1}} (A(\mathbf{x}, \mathbf{u}))^r d\mu = \int_{E_{n-1}} (A(\mathbf{x}, \tilde{\mathbf{u}}))^r d\mu. \tag{3.1}$$

Therefore,

$$\begin{aligned} \frac{\partial g_r}{\partial x_1} &= r\int_{E_{n-1}} u_1 (A(\mathbf{x}, \mathbf{u}))^{r-1} d\mu = r\int_{E_{n-1}} u_2 (A(\mathbf{x}, \tilde{\mathbf{u}}))^{r-1} d\mu, \\ \frac{\partial g_r}{\partial x_2} &= r\int_{E_{n-1}} u_1 (A(\mathbf{x}, \tilde{\mathbf{u}}))^{r-1} d\mu = r\int_{E_{n-1}} u_2 (A(\mathbf{x}, \mathbf{u}))^{r-1} d\mu. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned} \frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} &= r\int_{E_{n-1}} u_1 \left[(A(\mathbf{x}, \mathbf{u}))^{r-1} - (A(\mathbf{x}, \tilde{\mathbf{u}}))^{r-1} \right] d\mu, \\ \frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} &= r\int_{E_{n-1}} u_2 \left[(A(\mathbf{x}, \tilde{\mathbf{u}}))^{r-1} - (A(\mathbf{x}, \mathbf{u}))^{r-1} \right] d\mu. \end{aligned} \tag{3.3}$$

By combining (3.3) with (3.2), we have

$$\frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} = \frac{r}{2}\int_{E_{n-1}} (u_1 - u_2)\left[(A(\mathbf{x}, \mathbf{u}))^{r-1} - (A(\mathbf{x}, \tilde{\mathbf{u}}))^{r-1} \right] d\mu. \tag{3.4}$$

By Lagrange's mean value theorem, we find that

$$\begin{aligned}(A(\mathbf{x},\mathbf{u}))^{r-1}-(A(\mathbf{x},\tilde{\mathbf{u}}))^{r-1} &= (r-1)(x_1u_1+x_2u_2-x_2u_1-x_1u_2)\left(\xi+\sum_{i=3}^{n}u_ix_i\right)^{r-2} \\ &= (r-1)(u_1-u_2)(x_1-x_2)\left(\xi+\sum_{i=3}^{n}u_ix_i\right)^{r-2},\end{aligned} \tag{3.5}$$

where ξ is between $x_1u_1+x_2u_2$ and $x_2u_1+x_1u_2$.

From (3.4) and (3.5), we have

$$(x_1-x_2)\left(\frac{\partial g_r}{\partial x_1}-\frac{\partial g_r}{\partial x_2}\right)=\frac{r(r-1)}{2}(x_1-x_2)^2S_r(\mathbf{x}), \tag{3.6}$$

where

$$S_r(\mathbf{x})=\int_{E_{n-1}}(u_1-u_2)^2\left(\xi+\sum_{i=3}^{n}u_ix_i\right)^{r-2}d\mu\geq 0. \tag{3.7}$$

Hence, for $r\neq 0$, we get

$$\begin{aligned}(x_1-x_2)\left(\frac{\partial L_r}{\partial x_1}-\frac{\partial L_r}{\partial x_2}\right) &= (n-1)!\cdot\frac{1}{r}\cdot(L_r)^{1-r}\cdot(x_1-x_2)\left(\frac{\partial g_r}{\partial x_1}-\frac{\partial g_r}{\partial x_2}\right) \\ &= (n-1)!\cdot\frac{r-1}{2}\cdot(L_r)^{1-r}\cdot(x_1-x_2)^2S_r(\mathbf{x}).\end{aligned} \tag{3.8}$$

From Lemma 2.3, it is clear that L_r is Schur-convex for $r>1$ and Schur-concave for $r<1$ and $r\neq 0$.

According to Lemma 2.4 and the continuity of $r\mapsto L_r(\mathbf{x})$, let $r\to 0, 1-$, or $1+$ in $L_r(\mathbf{x})$, we know that $L_0(\mathbf{x})$ is a Schur-concave function, and $L_1(\mathbf{x})$ is both a Schur-concave function and a Schur-convex function. □

Theorem 3.2. *Given $r\in\mathbb{R}$, $F_r(\mathbf{x})$ is Schur-convex if $r\geq 1$ and Schur-concave if $r\leq 1$.*

Proof. Denote $\tilde{\mathbf{u}}=(u_2,u_1,u_3,\ldots,u_n)$. For $r\neq 0$,

$$F_r(\mathbf{x})=(n-1)!\int_{E_{n-1}}M_r(\mathbf{x},\mathbf{u})d\mu=(n-1)!\int_{E_{n-1}}M_r(\mathbf{x},\tilde{\mathbf{u}})d\mu, \tag{3.9}$$

$$\frac{\partial F_r}{\partial x_1}=(n-1)!\int_{E_{n-1}}x_1^{r-1}u_1(M_r(\mathbf{x},\mathbf{u}))^{1-r}d\mu=(n-1)!\int_{E_{n-1}}u_1\left[\frac{M_r(\mathbf{x},\mathbf{u})}{x_1}\right]^{1-r}d\mu, \tag{3.10}$$

$$\frac{\partial F_r}{\partial x_2}=(n-1)!\int_{E_{n-1}}x_2^{r-1}u_1(M_r(\mathbf{x},\tilde{\mathbf{u}}))^{1-r}d\mu=(n-1)!\int_{E_{n-1}}u_1\left[\frac{M_r(\mathbf{x},\tilde{\mathbf{u}})}{x_2}\right]^{1-r}d\mu. \tag{3.11}$$

Combination of (3.10) with (3.11) yields

$$\frac{\partial F_r}{\partial x_1}-\frac{\partial F_r}{\partial x_2}=(n-1)!\int_{E_{n-1}}u_1\left\{\left[\frac{M_r(\mathbf{x},\mathbf{u})}{x_1}\right]^{1-r}-\left[\frac{M_r(\mathbf{x},\tilde{\mathbf{u}})}{x_2}\right]^{1-r}\right\}d\mu. \tag{3.12}$$

By using the mean value theorem, we find

$$\begin{aligned}
&\left[\frac{M_r(\mathbf{x},\mathbf{u})}{x_1}\right]^{1-r}-\left[\frac{M_r(\mathbf{x},\tilde{\mathbf{u}})}{x_2}\right]^{1-r}\\
&\quad=\left(u_1+\frac{u_2x_2^r+\sum_{i=3}^n u_ix_i^r}{x_1^r}\right)^{(1-r)/r}-\left(u_1+\frac{u_2x_1^r+\sum_{i=3}^n u_ix_i^r}{x_2^r}\right)^{(1-r)/r}\\
&\quad=\frac{1-r}{r}\left(\frac{u_2x_2^r+\sum_{i=3}^n u_ix_i^r}{x_1^r}-\frac{u_2x_1^r+\sum_{i=3}^n u_ix_i^r}{x_2^r}\right)(u_1+\theta_1)^{(1-2r)/r}\\
&\quad=\frac{1-r}{r}\cdot\frac{u_2x_2^{2r}+x_2^r\sum_{i=3}^n u_ix_i^r-u_2x_1^{2r}-x_1^r\sum_{i=3}^n u_ix_i^r}{x_1^rx_2^r}\cdot(u_1+\theta_1)^{(1-2r)/r}\\
&\quad=(1-r)(x_2-x_1)(u_1+\theta_1)^{(1-2r)/r}T(\mathbf{x},\mathbf{u};\theta_2),
\end{aligned} \tag{3.13}$$

where θ_1 is between $(u_2x_2^r+\sum_{i=3}^n u_ix_i^r)/x_1^r$ and $(u_2x_1^r+\sum_{i=3}^n u_ix_i^r)/x_2^r$, θ_2 is between x_1 and x_2, and $T(\mathbf{x},\mathbf{u};\theta_2)=(2u_2\theta_2^{2r-1}+\theta_2^{r-1}\sum_{i=3}^n u_ix_i^r)/x_1^rx_2^r\geq 0$.

From (3.12) and (3.13), we have

$$\begin{aligned}
&(x_1-x_2)\left(\frac{\partial F_r}{\partial x_1}-\frac{\partial F_r}{\partial x_2}\right)\\
&\quad=(r-1)(x_1-x_2)^2(n-1)!\int_{E_{n-1}}u_1(u_1+\theta_1)^{(1-2r)/r}T(\mathbf{x},\mathbf{u};\theta_2)\,d\mu.
\end{aligned} \tag{3.14}$$

It follows that F_r is Schur-convex for $r>1$ and Schur-concave for $r<1$ and $r\neq 0$ by Lemma 2.3.

According to Lemma 2.4 and the continuity of $r\mapsto F_r(\mathbf{x})$, let $r\to 0, 1-$, or $1+$ in $F_r(\mathbf{x})$. We know that $F_0(\mathbf{x})$ is a Schur-concave function, and $F_1(\mathbf{x})$ is both a Schur-concave function and a Schur-convex function. □

Theorem 3.3. *$L_r(\mathbf{x}^{1/r})$ and $F_r(\mathbf{x}^{1/r})$ are Schur-concave functions if $r\geq 1$, and Schur-convex functions if $r\leq 1$ and $r\neq 0$.*

Proof. We can easily obtain that

$$L_r(\mathbf{x}^{1/r}) = \left[(n-1)!\int_{E_{n-1}} M_{1/r}(\mathbf{x},\mathbf{u})d\mu\right]^{1/r} = F_{1/r}^{1/r}(\mathbf{x}),$$
$$F_r(\mathbf{x}^{1/r}) = (n-1)!\int_{E_{n-1}} [A(\mathbf{x},\mathbf{u})]^{1/r} d\mu = L_{1/r}^{r}(\mathbf{x}), \tag{3.15}$$

$$(x_1-x_2)\left(\frac{\partial L_r(\mathbf{x}^{1/r})}{\partial x_1} - \frac{\partial L_r(\mathbf{x}^{1/r})}{\partial x_2}\right) = \frac{1}{r}(x_1-x_2)\left(\frac{\partial F_{1/r}(\mathbf{x})}{\partial x_1} - \frac{\partial F_{1/r}(\mathbf{x})}{\partial x_2}\right)\cdot F_{1/r}^{(1-r)/r}(\mathbf{x}),$$
$$(x_1-x_2)\left(\frac{\partial F_r(\mathbf{x}^{1/r})}{\partial x_1} - \frac{\partial F_r(\mathbf{x}^{1/r})}{\partial x_2}\right) = r(x_1-x_2)\left(\frac{\partial L_{1/r}(\mathbf{x})}{\partial x_1} - \frac{\partial L_{1/r}(\mathbf{x})}{\partial x_2}\right)\cdot L_{1/r}^{r-1}(\mathbf{x}). \tag{3.16}$$

From Theorems 3.1 and 3.2, we know that both $L_{1/r}(\mathbf{x})$ and $F_{1/r}(\mathbf{x})$ are Schur-concave functions if $r \geq 1$ and Schur-convex functions if $0 < r \leq 1$ or $r < 0$. According to Lemma 2.3 and (3.16), the required result of Theorem 3.3 is proved. □

4. Applications

As applications of the theorems above, we have the following corollaries.

Corollary 4.1 (See [19, Theorem 3.1] and [12, page 82]). *For $r \geq 1$, $r \in \mathbb{N}$, the complete elementary symmetric function*

$$C_r(\mathbf{x}) = \sum_{\substack{i_1+i_2+\cdots+i_n=r,\\ i_1,\dots,i_n\geq 0 \text{ are integers}}} x_1^{i_1}x_2^{i_2},\dots,x_n^{i_n} \tag{4.1}$$

is Schur-convex.

Proof. If $r \geq 1$, $r \in \mathbb{N}$, then (see [20, page 164])

$$C_r(\mathbf{x}) = \binom{n-1+r}{r} L_r^r(\mathbf{x}). \tag{4.2}$$

By Theorem 3.1 and Lemma 2.3, it is easy to see that $L_r^r(\mathbf{x})$ is a Schur-convex function. Therefore, $C_r(\mathbf{x})$ is a Schur-convex function. □

Corollary 4.2. *The complete symmetric function of the first degree:*

$$D_r(\mathbf{x}) = \sum_{\substack{i_1+i_2+\cdots+i_n=r,\\ i_1,\dots,i_n\geq 0 \text{ are integers}}} \left(x_1^{i_1}x_2^{i_2},\dots,x_n^{i_n}\right)^{1/r} \tag{4.3}$$

(see [6, Theorem 5] and [9]), is Schur-concave for $r \geq 1$, $r \in \mathbb{N}$.

Proof. If $r \geq 1$, $r \in \mathbb{N}$, then we have (see [6, Theorem 5])

$$D_r(\mathbf{x}) = \binom{n-1+r}{r} F_{1/r}(\mathbf{x}). \tag{4.4}$$

By considering Theorem 3.2, we prove the required result. □

Corollary 4.3. *Let $r \neq 0$, $\mathbf{x},\mathbf{y} \in \mathbb{R}_+^n$, $\mathbf{x}^r \succ \mathbf{y}^r$. Then $L_r(\mathbf{x}) \leq L_r(\mathbf{y})$ and $F_r(\mathbf{x}) \leq F_r(\mathbf{y})$ if $r \geq 1$. They are reversed if $r \leq 1$ and $r \neq 0$.*

Proof. Suppose $r \geq 1 (r \leq 1, r \neq 0)$. $L_r(\mathbf{x}^{1/r})$ is a Schur-concave (Schur-convex) function by Theorem 3.3. Then

$$L_r\left(\left(\mathbf{x}^r\right)^{1/r}\right) \leq (\geq) L_r\left(\left(\mathbf{y}^r\right)^{1/r}\right), \quad L_r(\mathbf{x}) \leq (\geq) L_r(\mathbf{y}). \tag{4.5}$$

For $F_r(\mathbf{x}^{1/r})$, the proof is similar; we omit the details. □

Corollary 4.4. *If $r \geq 1$, then*

$$\begin{aligned} A(\mathbf{x}) &\leq L_r(\mathbf{x}) \leq M_r(\mathbf{x}), \\ A(\mathbf{x}) &\leq F_r(\mathbf{x}) \leq M_r(\mathbf{x}). \end{aligned} \tag{4.6}$$

Inequalities (4.6) are reversed if $r \leq 1$.

Proof. If $r \geq 1$, owing to Theorem 3.1 and

$$(x_1,x_2,\ldots,x_n) \succ (A(\mathbf{x}),A(\mathbf{x}),\ldots,A(\mathbf{x})) \triangleq \overline{A}(\mathbf{x}), \tag{4.7}$$

we have

$$\begin{aligned} L_r(\mathbf{x}) \geq L_r(\overline{A}(\mathbf{x})) &= \left((n-1)!\int_{E_{n-1}}\left(\sum_{i=1}^{n} A(\mathbf{x})u_i\right)^r d\mu\right)^{1/r} \\ &= A(\mathbf{x})\left((n-1)!\int_{E_{n-1}}\left(\sum_{i=1}^{n} u_i\right)^r d\mu\right)^{1/r} = A(\mathbf{x}). \end{aligned} \tag{4.8}$$

Obviously, if $r \leq 1$, $r \neq 0$, inequality (4.8) is reversed by Theorem 3.1. For $r = 0$, because of the continuity of $r \mapsto L_r(\mathbf{x})$, we have $L_0(\mathbf{x}) \leq A(\mathbf{x})$.

By the same way, we find that $F_r(\mathbf{x}) \geq A(\mathbf{x})$ if $r \geq 1$, and $F_r(\mathbf{x}) \leq A(\mathbf{x})$ if $r \leq 1$. In addition,

$$\begin{aligned} \mathbf{x}^r = (x_1^r,x_2^r,\ldots,x_n^r) &\succ (M_r^r(\mathbf{x}),M_r^r(\mathbf{x}),\ldots,M_r^r(\mathbf{x})) \\ &\triangleq (M_r(\mathbf{x}),M_r(\mathbf{x}),\ldots,M_r(\mathbf{x}))^r \triangleq (\overline{M}_r(\mathbf{x}))^r. \end{aligned} \tag{4.9}$$

If $r \geq 1$, according to Corollary 4.3, we get

$$L_r(\mathbf{x}) \leq L_r(\overline{M}_r(\mathbf{x})) = \left((n-1)!\int_{E_{n-1}} \left(\sum_{i=1}^{n} M_r(\mathbf{x})u_i\right)^r d\mu\right)^{1/r} = M_r(\mathbf{x}). \tag{4.10}$$

If $r \leq 1$, inequality (4.10) is obviously reversed by Corollary 4.3 again.

Similarly, we have $F_r(\mathbf{x}) \leq M_r(\mathbf{x})$ if $r \geq 1$, and $F_r(\mathbf{x}) \geq M_r(\mathbf{x})$ if $r \leq 1$. □

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