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Research Article

Some Properties of Pythagorean Modulus

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We consider two Pythagorean modulus introduced by Gao (2005, 2006) recently. The exact values concerning these modulus for some classical Banach spaces are determined. Some applications in geometry of Banach spaces are also obtained.

1. Introduction

Recently, Gao introduced some moduli from Pythagorean theorem. In terms of these moduli, he got some sufficient conditions for a Banach space X to have uniform normal structure, which plays an import role in fixed-point theory.

In this paper, we mainly discuss the moduli $E_\epsilon(X)$ and $f_\epsilon(X)$. Let X be a Banach space. By S_X and B_X we will denote the unit sphere and unit ball of X, respectively. For every nonnegative number ϵ, the Pythagorean moduli are given by [1, 2]

$$\begin{aligned} E_\epsilon(X) &= \sup\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : x, y \in S_X\}, \\ f_\epsilon(X) &= \inf\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : x, y \in S_X\}. \end{aligned} \tag{1.1}$$

For simplicity, we will write $E(\epsilon)$ and $f(\epsilon)$ for $E_\epsilon(X)$ and $f_\epsilon(X)$ provided no confusion occurs. It is clear that $2 \le f(\epsilon) \le 2(1+\epsilon^2) \le E(\epsilon) \le 2(1+\epsilon)^2$. It is also worth noting that the first moduli $E_\epsilon(X)$ has been proved to be very useful in the study of the well-known von Neumann-Jordan constant (see e.g., [3, 4]).

Following Gao, we study the further properties concerning the Pythagorean moduli. We find that these moduli are connected with some geometric properties. They enable us to distinguish several important classes of spaces such as uniformly convex, uniformly smooth, or uniformly nonsquare.

2. Pythagorean modulus

We can replace S_X by B_X in the definition of $E(\epsilon)$ by [4, Proposition 2.2]. Analogously, we can deduce an alternative definition for the modulus $f(\epsilon)$.

Proposition 2.1. *Let* $\epsilon \geq 0$, *then*

$$f(\epsilon) = \inf\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : \|x\|, \|y\| \geq 1\}. \tag{2.1}$$

Proof. First, consider the elements $x,\ y$ of X to be fixed, and let $\varphi(t) := \|x+ty\|^2 + \|x-ty\|^2$ whenever $t \in \mathbb{R}$. Obviously $\varphi(t)$ is convex, even, $\varphi(0) = 2\|x\|^2$ and $\varphi(1) = \varphi(-1) \geq 2\|x\|^2$. This immediately yields $\varphi(t) \geq \varphi(1)$ for every $t \geq 1$, that is,

$$\|x+ty\|^2 + \|x-ty\|^2 \geq \|x+y\|^2 + \|x-y\|^2. \tag{2.2}$$

Taking $x, y \in X$ with $\min(\|x\|, \|y\|) \geq 1$, we may assume without loss of generality that $1 \leq \|x\| \leq \|y\|$. By the inequality (2.2),

$$\begin{aligned} \|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 &= \|x\|^2\left(\left\|\frac{x}{\|x\|} + \frac{\|y\|}{\|x\|}\frac{\epsilon y}{\|y\|}\right\|^2 + \left\|\frac{x}{\|x\|} - \frac{\|y\|}{\|x\|}\frac{\epsilon y}{\|y\|}\right\|^2\right) \\ &\geq \left\|\frac{x}{\|x\|} + \frac{\epsilon y}{\|y\|}\right\|^2 + \left\|\frac{x}{\|x\|} - \frac{\epsilon y}{\|y\|}\right\|^2 \geq f(\epsilon), \end{aligned} \tag{2.3}$$

and the arbitrariness of $x,\ y$ yields

$$\inf\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : \|x\|, \|y\| \geq 1\} \geq f(\epsilon). \tag{2.4}$$

This completes the proof since the converse inequality holds obviously. □

Proposition 2.2. *Both* $\sqrt{E(\epsilon)/2}$ *and* $\sqrt{f(\epsilon)/2}$ *are convex on* $[0, +\infty)$.

Proof. Let $\epsilon_1, \epsilon_2 \geq 0,\ \lambda \in (0,1)$, and $r_1(t) = \operatorname{sgn}(\sin 2\pi t)$ be the Rademacher function. We have, for any $x, y \in S_X$,

$$\begin{aligned} &\left(\int_0^1 \|x + r_1(t)(\lambda\epsilon_1 + (1-\lambda)\epsilon_2)y\|^2 dt\right)^{1/2} \\ &\quad \leq \left(\int_0^1 (\lambda\|x + r_1(t)\epsilon_1 y\| + (1-\lambda)\|x + r_1(t)\epsilon_2 y\|)^2 dt\right)^{1/2} \\ &\quad \leq \lambda\left(\int_0^1 \|x + r_1(t)\epsilon_1 y\|^2 dt\right)^{1/2} + (1-\lambda)\left(\int_0^1 \|x + r_1(t)\epsilon_2 y\|^2 dt\right)^{1/2} \\ &\quad \leq \lambda\sqrt{E(\epsilon_1)/2} + (1-\lambda)\sqrt{E(\epsilon_2)/2}, \end{aligned} \tag{2.5}$$

where we have used, in succession, triangular and Minkowski inequalities. Thus,

$$\sqrt{E(\lambda\epsilon_1 + (1-\lambda)\epsilon_2)/2} \leq \lambda\sqrt{E(\epsilon_1)/2} + (1-\lambda)\sqrt{E(\epsilon_2)/2}. \tag{2.6}$$

The proof for $\sqrt{f(\epsilon)/2}$ is similar to that of $E(\epsilon)$. □

Corollary 2.3. *The following statements hold.*

(1) *Both* $E(\epsilon)$ *and* $f(\epsilon)$ *are nondecreasing on* $(0,+\infty)$.

(2) *Both* $E(\epsilon)$ *and* $f(\epsilon)$ *are continuous on* $(0,+\infty)$.

(3) *Both* $(\sqrt{E(\epsilon)/2}-1)/\epsilon$ *and* $(\sqrt{f(\epsilon)/2}-1)/\epsilon$ *are nondecreasing on* $(0,+\infty)$.

It has been shown in [1, 4] that for the ℓ_p space and $\epsilon \in [0,1]$,

$$E(\epsilon)=2\left(\frac{(1+\epsilon)^p+(1-\epsilon)^p}{2}\right)^{2/p} \tag{2.7}$$

with $p \geq 2$ and

$$f(\epsilon)=2\left(\frac{(1+\epsilon)^p+(1-\epsilon)^p}{2}\right)^{2/p} \tag{2.8}$$

with $1 \leq p \leq 2$. Let us now discuss the remaining cases. The key to compute the Pythagorean modulus is the well-known inequalities of Clarkson [5], in which x and y are elements in $\ell_p(L_p)$:

$$\left(\|x+y\|^{p'}+\|x-y\|^{p'}\right)^{1/p'} \leq 2^{1/p'}\left(\|x\|^p+\|y\|^p\right)^{1/p} \quad \text{for } 1<p\leq 2, \tag{2.9}$$

$$\left(\|x+y\|^{p}+\|x-y\|^{p}\right)^{1/p} \leq 2^{1/p}\left(\|x\|^{p'}+\|y\|^{p'}\right)^{1/p'} \quad \text{for } 2\leq p<\infty. \tag{2.10}$$

Here, as usual, p' is the conjugate number of p. In the cases $2 \leq p < \infty$ and $1 < p \leq 2$, the inequalities in (2.9) and (2.10), respectively, hold in the reversed sense.

Theorem 2.4. *Let* $\epsilon \in [0,1]$. *Then for the* ℓ_p *space*

(1) $E(\epsilon)=2(1+\epsilon^p)^{2/p}$ *with* $1<p\leq 2$;

(2) $f(\epsilon)=2(1+\epsilon^p)^{2/p}$ *with* $2\leq p<\infty$.

Proof. (1) Let x,y in X with $\|x\|=1$, $\|y\|=\epsilon$. It follows from Clarkson's inequality (2.9) and Hölder inequality that

$$\left(\frac{\|x+y\|^2+\|x-y\|^2}{2}\right)^{1/2} \leq \left(\frac{\|x+y\|^{p'}+\|x-y\|^{p'}}{2}\right)^{1/p'} \leq \left(\|x\|^p+\|y\|^p\right)^{1/p}, \tag{2.11}$$

which gives that $E(\epsilon) \leq 2(1+\epsilon^p)^{2/p}$.

On the other hand, let us put $x_0=(1,0,\dots)$, $y_0=(0,\epsilon,0,\dots)$. It is clear that $\|x_0\|=1$, $\|y_0\|=\epsilon$, and $\|x_0+y_0\|=\|x_0-y_0\|=(1+\epsilon^p)^{1/p}$. This, together with the preceding inequality, yields the equality as desired.

(2) By replacing x with $x+y$ and y with $x-y$, we get an equivalent form of Clarkson's inequality (2.10), that is,

$$\left(\|x+y\|^{p'}+\|x-y\|^{p'}\right)^{1/p'} \geq 2^{1/p'}\left(\|x\|^p+\|y\|^p\right)^{1/p}. \tag{2.12}$$

The rest proof is similar to that of (2.2). □

The inequality (2.9) is called, by Takahashi and Kato, the (p,p') Clarkson inequality. It is obvious that these inequalities (2.9) and (2.10) are equivalent. Moreover, Takahashi and

Kato [6, Proposition 2] proved that the (p,p') Clarkson inequality holds in X if and only if it holds in the dual space X^*. Thus, we can generalize Theorem 2.4 as the following.

Theorem 2.5. *Assume that X contains an isometric copy of ℓ_p^2 with $1<p\le 2$. If the (p,p') Clarkson inequality holds, then $E_\epsilon(X)=2(1+\epsilon^p)^{2/p}$ and $f_\epsilon(X^*)=2(1+\epsilon^{p'})^{2/p'}$.*

3. Geometric properties

The concepts of uniform convexity and its dual property, uniform smoothness, play an important role in analysis. Recall that a Banach space X is called *uniformly convex* if and only if $d_X(\epsilon)>0$ for any $0<\epsilon\le 1$ (see e.g., [7]), where the function

$$d_X(\epsilon)=\inf\{\max(\|x+\epsilon y\|,\|x-\epsilon y\|)-1 : x,y\in S_X\} \tag{3.1}$$

is Milman's modulus of convexity defined in [8]. A Banach space X is called *uniformly smooth* if and only if $\lim_{\epsilon\to 0}\rho_X(\epsilon)/\epsilon=0$, where the function $\rho_X(\epsilon)$ is Lindenstrauss's modulus of smoothness defined by [9]

$$\rho_X(\epsilon)=\sup\left\{\frac{\|x+\epsilon y\|+\|x-\epsilon y\|}{2}-1 : x,y\in S_X\right\}. \tag{3.2}$$

It is convenient for us to assume that X is a Banach space of finite dimension through the rest proofs of this paper. The extension of the results to the general case is immediate, depending only on the formula

$$E_\epsilon(X)=\sup\{E_\epsilon(Y) : Y \text{ subspace of } X,\ \dim Y=2\}. \tag{3.3}$$

The case for the modulus $f(\epsilon)$ is similar.

Theorem 3.1. *X is uniformly convex if and only if $f(\epsilon)>2$ for any $0<\epsilon\le 1$.*

Proof. Since $\sqrt{f(\epsilon)/2}-1\le d(\epsilon)$, it suffices to show that uniform convexity implies $\sqrt{f(\epsilon)/2}>1$ for any $0<\epsilon\le 1$. Suppose conversely that there is an $\epsilon\in(0,1]$ such that $\sqrt{f(\epsilon)/2}=1$. Thus, we can find two vectors $x,\ y$ in S_X such that $\|x+\epsilon y\|^2+\|x-\epsilon y\|^2=2$. Therefore,

$$1\le\frac{\|x+\epsilon y\|+\|x-\epsilon y\|}{2}\le\sqrt{\frac{\|x+\epsilon y\|^2+\|x-\epsilon y\|^2}{2}}=1. \tag{3.4}$$

It follows that the equality in (3.4) can occur only when $\|x+\epsilon y\|=\|x-\epsilon y\|=1$. This immediately yields $d_X(\epsilon)=0$, a contradiction. □

Now, let us turn to the modulus $E(\epsilon)$, we will show that this modulus is actually a kind of modulus of smoothness.

Theorem 3.2. *X is uniformly smooth if and only if $\lim_{\epsilon\to 0}(\sqrt{E(\epsilon)/2}-1)/\epsilon=0$.*

Proof. The sufficiency is trivial since $\sqrt{E(\epsilon)/2}-1\ge\rho(\epsilon)$ holds for any $\epsilon\ge 0$. To see the necessity, suppose, to get a contradiction, that $\lim_{\epsilon\to 0}(\sqrt{E(\epsilon)/2}-1)/\epsilon>0$. Corollary 2.3

shows that there is a $c \in (0,1)$ such that $\sqrt{E(\epsilon)/2} - 1 \geq c\epsilon$ for any $\epsilon > 0$. In particular, let $0 < \epsilon < 2c/(1-c^2)$ and choose $x,\ y$ with $\|x\| = 1,\ \|y\| = \epsilon$ such that

$$\|x+y\|^2 + \|x-y\|^2 = E(\epsilon) \geq 2(1+c\epsilon)^2. \tag{3.5}$$

Assume without loss of generality that $\min(\|x+y\|, \|x-y\|) = \|x-y\| = t$, and so $t \in [1-\epsilon, 1+c\epsilon]$. It follows from the inequality (3.5) that

$$\|x+y\| + \|x-y\| \geq t + \sqrt{2(1+c\epsilon)^2 - t^2} =: \varphi(t). \tag{3.6}$$

Note that $\varphi(t)$ attains its minimum at $t = 1-\epsilon$, or equivalently that

$$\|x+y\| + \|x-y\| \geq \varphi(1-\epsilon) \tag{3.7}$$

which in view of the definition of $\rho(\epsilon)$ implies that

$$\frac{\rho(\epsilon)}{\epsilon} \geq \frac{\varphi(1-\epsilon) - 2}{2\epsilon} = \frac{2c - (1-c^2)\epsilon}{\sqrt{2(1+c\epsilon)^2 - (1-\epsilon)^2} + 1 + \epsilon}. \tag{3.8}$$

Letting $\epsilon \to 0$, we get

$$\lim_{\epsilon \to 0} \rho(\epsilon)/\epsilon \geq c > 0 \tag{3.9}$$

which contradicts our hypothesis. □

Recall that a Banach space X is called *uniformly nonsquare* if there exists $\delta > 0$, such that if $x, y \in S_X$, then $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$. In [1], Gao proved that X is uniformly nonsquare provided there is an $\epsilon \in (0,1)$ such that $f(\epsilon) > 2$. The following is an improvement of such assertion.

Theorem 3.3. *The following statements are equivalent.*

(a) *X is uniformly nonsquare.*

(b) $f(1) > 2$.

(c) *There is an $\epsilon \in (0,1)$ such that $f(\epsilon) > 2$.*

Proof. Since (b)⇒(c) follows directly from the continuity of $f(\epsilon)$ at $\epsilon = 1$ and (c)⇒(a) is proven by Gao in [1, Theorem 1], it suffices to show that (a)⇒(b).

(a)⇒(b) Suppose on the contrary that $f(1) = 2$ and choose two elements $x, y \in S_X$ such that

$$\sqrt{\frac{\|x+y\|^2 + \|x-y\|^2}{2}} - 1 = 0. \tag{3.10}$$

Therefore,

$$1 \leq \frac{\|x+y\| + \|x-y\|}{2} \leq \sqrt{\frac{\|x+y\|^2 + \|x-y\|^2}{2}} = 1. \tag{3.11}$$

It follows that the equality in (3.11) can occur only when $\|x+y\| = \|x-y\| = 1$. Let $u = x+y,\ v = x-y$. Clearly, $u,v \in S_X$ and $\|u+v\| = \|u-v\| = 2$, which contradicts our hypothesis. □

Remark 3.4. For the modulus $S(\epsilon, X)$ [10], we can also obtain that X is uniformly non-square if and only if there is an $\epsilon \in (0,1)$ such that $S(\epsilon, X) > 1$.

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