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Research Article

Slowly Oscillating Solutions for Differential Equations with Strictly Monotone Operator

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The authors discuss necessary and sufficient conditions for the existence and uniqueness of slowly oscillating solutions for the differential equation $u' + F(u) = h(t)$ with strictly monotone operator. Particularly, the authors give necessary and sufficient conditions for the existence and uniqueness of slowly oscillating solutions for the differential equation $u' + \nabla\Phi(u) = h(t)$, where $\nabla\Phi$ denotes the gradient of the convex function Φ on $\mathbb{R}^N$.

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1. Introduction

In this paper, we will consider the following differential equation:

$$u' + F(u) = h(t), \tag{1.1}$$

where the maps $h : \mathbb{R} \to \mathbb{R}^N$ and $F : \mathbb{R}^N \to \mathbb{R}^N$ are continuous. A special class, of the dissipative equation (1.1), is the case where the field F is derived from a convex potential Φ:

$$u' + \nabla\Phi(u) = h(t). \tag{1.2}$$

For the dissipative equation (1.1), Biroli [1], Dafermos [2], Haraux [3], Huang [4], and Ishii [5] have given important contributions to the question of almost periodic solutions which are valid even for the abstract evolution equations. In [6], Philippe Cieutat gives necessary and sufficient conditions for the existence and uniqueness of the bounded (resp., almost periodic) solution of (1.2) when the forcing term $h(t)$ is bounded (resp., almost periodic). In the scalar case $N = 1$, Slyusarchuk established similar results in [7]. But the conditions which are established in [6] for (1.2) do not hold for (1.1), even in the linear case. So in [6] Cieutat also gives a sufficient condition, then a necessary condition, for the existence and uniqueness of the bounded (resp., almost periodic) solution of (1.1).

The numerical space $\mathbb{R}^N$ is endowed with its standard inner product $\sum_{k=1}^N x_k y_k$, $|\cdot|$ denotes the associated Euclidian norm. We denote by $BC(\mathbb{R}^N)$ the Banach space of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}^N$ endowed with the norm $\|u\|_\infty := \sup_{t\in\mathbb{R}} |u(t)|$. When k is a positive integer, $BC^k(\mathbb{R}^N)$ is the space of functions in $BC(\mathbb{R}^N)\bigcap C^k(\mathbb{R}^N)$ such that all their derivatives, up to order k, are bounded functions. When $u \in BC^1(\mathbb{R}^N)$, we set $\|u\|_{c^1} = \|u\|_\infty + \|u'\|_\infty$ and when $u \in BC^2(\mathbb{R}^N)$, we set $\|u\|_{c^2} = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$.

In 1984, Sarason in [8] extended almost periodic functions and introduced the definition of remotely periodic functions. The space of remotely periodic functions, as a C^*-subalgebra of $BC(\mathbb{R}^N)$, is generated by almost periodic functions and slowly oscillating functions which are defined as following.

Definition 1.1 [8]. A function $f \in BC(\mathbb{R}^N)$ is said to be slowly oscillating if

$$\lim_{|t|\to+\infty} |f(t+a) - f(t)| = 0, \quad \text{for each } a \in \mathbb{R}, \tag{1.3}$$

the set of all these functions is denoted by $SO(\mathbb{R}^N)$.

Comparing with the space $AP(\mathbb{R})$ of almost periodic functions, the space of slowly oscillating functions is quite large. In fact, $AP(\mathbb{R}) = \overline{\text{span}}\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$, where the closure is taken in $BC(\mathbb{R})$ (e.g., see [9] for details). $SO(\mathbb{R})$ not only contains such space as $C_0(\mathbb{R})$ which consists of all the functions f such that $f(t) \to 0$ as $|t| \to \infty$, but also properly contains $X = \overline{\text{span}}\{e^{i\lambda t^\alpha} : \lambda \in \mathbb{R},\ 0 < \alpha < 1\}$ (see [10–12] for details). The only functions in $AP(\mathbb{R}) \cap SO(\mathbb{R})$ are the constant functions on $\mathbb{R}$. We also point out that the slowly oscillating functions $SO(\mathbb{R}^N)$ studied here form a strict subset of the slowly oscillating functions studied on [13, page 250, Definition 4.2.1]. Thus, all functions in $SO(\mathbb{R}^N)$ are uniformly continuous.

To our knowledge, nobody has investigated the existence and uniqueness of slowly oscillating solutions for the differential equation (1.1). So in this paper, we give a sufficient, then a necessary condition for the existence and uniqueness of slowly oscillating solutions for the differential equation (1.1). We will give sufficient and necessary conditions for the existence and uniqueness of slowly oscillating solutions for differential equation (1.2).

To show the main results of the paper, we need the following definition and lemma.

Definition 1.2. A function $F : \mathbb{R}^N \to \mathbb{R}^N$ is said to be strictly monotone on $\mathbb{R}^N$ if $(F(x_1) - F(x_2), x_1 - x_2) > 0$ for all $x_1, x_2 \in \mathbb{R}^N$ such that $x_1 \neq x_2$.

Lemma 1.3 [6]. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous and strictly monotone map. Then for every compact subset K of $\mathbb{R}^N$ and for every $\varepsilon > 0$, there exists $c > 0$ such that*

$$(F(x_1) - F(x_2), x_1 - x_2) > c|x_1 - x_2|^2 \tag{1.4}$$

for all $x_1, x_2 \in K$ such that

$$|x_1 - x_2| \geq \varepsilon. \tag{1.5}$$

2. Main results

For each $u \in BC^1(\mathbb{R}^N)$, the function $t \to u'(t) + F(u(t))$ belongs to $BC(\mathbb{R}^N)$, so we can define the operator $\mathcal{F}_1 : BC^1(\mathbb{R}^N) \to BC^0(\mathbb{R}^N)$ with $\mathcal{F}_1(u)(t) := u'(t) + F(u(t))$ for all $u \in BC^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$. Let $SO^1(\mathbb{R}^N) = SO(\mathbb{R}^N) \bigcap C^1(\mathbb{R}^N)$ and $\|u\|_{c^1} = \|u\|_\infty + \|u'\|_\infty$ for $u \in SO^1(\mathbb{R}^N)$. Set $\mathcal{F}_2 = \mathcal{F}_1|_{SO^1(\mathbb{R}^N)}$.

Consider the following assertions:

(A) F is a strictly monotone map on $\mathbb{R}^N$ such that

$$\lim_{|x|\to\infty} \frac{(F(x), x)}{|x|} = +\infty; \tag{2.1}$$

(B) $\mathcal{F}_2 : (SO^1(\mathbb{R}^N), \|\cdot\|_{C^1}) \to (SO(\mathbb{R}^N), \|\cdot\|_\infty)$ is a homeomorphism;

(C) $F : (\mathbb{R}^N, |\cdot|) \to (\mathbb{R}^N, |\cdot|)$ is a homeomorphism.

Theorem 2.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous map. Then the following implications hold: (A)⇒(B)⇒(C).*

Proof. By [6], (A) implies that $\mathcal{F}_1 : (BC^1(\mathbb{R}^N), \|\cdot\|_{C^1}) \to (BC(\mathbb{R}^N), \|\cdot\|_\infty)$ is a homeomorphism. That is, (1.1) has for each h from $SO(\mathbb{R}^N)$ a unique solution $u(t)$ from $SO^1(\mathbb{R}^N)$, which depends continuously on h. To show (A)⇒(B), it remains to show $u \in SO(\mathbb{R}^N)$ if $h \in SO(\mathbb{R}^N)$.

Suppose, by the way of contradiction, $u(t) \notin SO(\mathbb{R}^N)$. Then there exist $a_0, \varepsilon_0 > 0$ and sequence $t_n \to \infty$ such that

$$|u(t_n + a_0) - u(t_n)| \geq \varepsilon_0. \tag{2.2}$$

Without loss of generality, we can assume $t_n - t_{n-1} \to +\infty$.

Since $u(t) \in BC^1(\mathbb{R}^N)$, $u(t)$ is uniformly continuous on $\mathbb{R}$. Then there exists $\delta > 0$ such that

$$|u(t + a_0) - u(t)| \geq \frac{\varepsilon_0}{2}, \quad \forall t \in p_n', \tag{2.3}$$

where $P_n' = (t_n - \delta, t_n + \delta)$.

Set

$$P_n = (t_n - \delta, t_n), \qquad I_n = [t_{n-1}, t_n],$$
$$C_n = \left\{s \in I_n : |u(s + a_0) - u(s)| \geq \frac{\varepsilon_0}{2}\right\}, \qquad C_n' = \left\{s \in I_n : |u(s + a_0) - u(s)| < \frac{\varepsilon_0}{2}\right\} \tag{2.4}$$

and put

$$\Phi(t) = u(t + a_0) - u(t). \tag{2.5}$$

Obviously $P_n \subset C_n$.

Let $K=\overline{u(\mathbb{R})}$. Note that $u(t)$ is bounded on $\mathbb{R}^N$ and therefore, K is a compact subset of $\mathbb{R}^N$. By Lemma 1.3 there exists c_o such that

$$(F(u(s+a_o))-F(u(s)),u(s+a_o)-u(s))\geq c_o|u(s+a_o)-u(s)|^2 \tag{2.6}$$

for each $s\in C_n$.

Note

$$\int_{t_n-1}^{t_n}\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)ds=\frac{1}{2}|\Phi(t_n)|^2\cdot e^{2c_0t_n}-\frac{1}{2}|\Phi(t_{n-1})|^2\cdot e^{2c_0t_{n-1}}. \tag{2.7}$$

At the same time, we also have

$$\int_{t_n-1}^{t_n}\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)ds=\int_{C_n}\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)ds+\int_{C_n'}\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)ds. \tag{2.8}$$

Case 1. $s\in C_n$, that is,

$$|u(s+a_0)-u(s)|\geq\frac{\varepsilon_0}{2}. \tag{2.9}$$

We can get

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2\right)&=(\Phi(s)',\Phi(s))\\&=(h(s+a_0)-h(s),\Phi(s))-(F(u(s+a_0))-F(u(s)),\Phi(s)).\end{aligned} \tag{2.10}$$

By (2.6), we can obtain

$$\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2\right)\leq|h(s+a_o)-h(s)|\cdot|\Phi(s)|-c_0|\Phi(s)|^2. \tag{2.11}$$

Also we can see

$$\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)=(\Phi'(s),\Phi(s))\cdot e^{2c_0s}+c_0e^{2c_0s}\cdot|\Phi(s)|^2. \tag{2.12}$$

By (2.11), we deduce

$$\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)\leq|h(s+a_o)-h(s)|\cdot|\Phi(s)|\cdot e^{2c_0s}. \tag{2.13}$$

Case 2. $s\in C_n'$, that is,

$$|u(s+a_0)-u(s)|<\frac{\varepsilon_0}{2}. \tag{2.14}$$

Moreover, one has

$$\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2e^{2c_0s}\right)=(\Phi'(s),\Phi(s))\cdot e^{2c_0s}+c_0e^{2c_0s}\cdot|\Phi(s)|^2, \tag{2.15}$$

$$(\Phi(s)',\Phi(s))=(h(s+a_0)-h(s),\Phi(s))-(F(u(s+a_0))-F(u(s)),\Phi(s)). \tag{2.16}$$

For F is strictly monotone on $\mathbb{R}^N$, we can deduce

$$(F(u(s+a_0)) - F(u(s)), \Phi(s)) > 0. \tag{2.17}$$

Moreover, by (2.17) and (2.16) one has

$$(\Phi(s)', \Phi(s)) < |\Phi(s)| \cdot |h(s+a_0) - h(s)| < \frac{\varepsilon_0}{2} |h(s+a_0) - h(s)|. \tag{2.18}$$

By (2.15) and (2.18), we can get

$$\frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2 e^{2c_0 s}\right) < \frac{\varepsilon_0}{2}|h(s+a_0) - h(s)| \cdot e^{2c_0 s} + \frac{\varepsilon_0^2}{4} c_0 e^{2c_0 s}. \tag{2.19}$$

Considering the above two cases, one has

$$\begin{aligned}
&\frac{1}{2}|\Phi(t_n)|^2 \cdot e^{2c_0 t_n} - \frac{1}{2}|\Phi(t_{n-1})|^2 \cdot e^{2c_0 t_{n-1}} \\
&\quad = \int_{t_{n-1}}^{t_n} \frac{d}{ds}\left(\frac{1}{2}|\Phi(s)|^2 e^{2c_0 s}\right) ds \\
&\quad < \int_{C_n} |h(s+a_o) - h(s)| \cdot |\Phi(s)| \cdot e^{2c_0 s} ds \\
&\qquad + \int_{C_n'} \left(\frac{\varepsilon_0}{2}|h(s+a_0) - h(s)| + \frac{\varepsilon_0^2}{4} c_0\right) \cdot e^{2c_0 s} ds \\
&\quad \leq \sup_{t \in I_n} |h(t+a_0) - h(t)| \cdot \sup_{t \in I_n} |\Phi(t)| \cdot \int_{C_n} e^{2c_0 s} ds \\
&\qquad + \left(\frac{\varepsilon_0}{2} \sup_{t \in I_n} |h(t+a_0) - h(t)| + \frac{\varepsilon_0^2}{4} c_0\right) \cdot \int_{C_n'} e^{2c_0 s} ds.
\end{aligned} \tag{2.20}$$

Since

$$\begin{gathered}
\int_{C_n} e^{2c_0 s} ds \leq \int_{t_{n-1}}^{t_n} e^{2c_0 s} ds, \\
\int_{C_n'} e^{2c_0 s} ds \leq \int_{t_{n-1}}^{t_n} e^{2c_0 s} ds - \int_{t_n - \delta}^{t_n} e^{2c_0 s} ds,
\end{gathered} \tag{2.21}$$

one has

$$\begin{aligned}
&\frac{1}{2}|\Phi(t_n)|^2 \cdot e^{2c_0 t_n} - \frac{1}{2}|\Phi(t_{n-1})|^2 \cdot e^{2c_0 t_{n-1}} \\
&\quad \leq \sup_{t \in I_n} |h(t+a_0) - h(t)| \cdot \sup_{t \in I_n} |\Phi(t)| \cdot \int_{t_{n-1}}^{t_n} e^{2c_0 s} ds \\
&\qquad + \left(\frac{\varepsilon_0}{2} \sup_{t \in I_n} |h(t+a_0) - h(t)| + \frac{\varepsilon_0^2}{4} c_0\right) \cdot \left(\int_{t_{n-1}}^{t_n} e^{2c_0 s} ds - \int_{t_n - \delta}^{t_n} e^{2c_0 s} ds\right).
\end{aligned} \tag{2.22}$$

So, we can get

$$\begin{aligned}&\frac{1}{2}\left|\Phi(t_n)\right|^2\cdot e^{2c_0t_n}-\frac{1}{2}\left|\Phi(t_{n-1})\right|^2\cdot e^{2c_0t_{n-1}}\\&\quad\leq\sup_{t\in I_n}\left|h(t+a_0)-h(t)\right|\cdot\sup_{t\in I_n}\left|\Phi(t)\right|\cdot\frac{1}{2c_0}\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)\\&\qquad+\left(\frac{\varepsilon_0}{2}\sup_{t\in I_n}\left|h(t+a_0)-h(t)\right|+\frac{\varepsilon_0^2}{4}c_0\right)\cdot\left[\frac{1}{2c_0}\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)-e^{2c_0(t_n-\delta)}\cdot\left(e^{2c_0\delta}-1\right)\cdot\frac{1}{2c_0}\right].\end{aligned}\tag{2.23}$$

That is,

$$\begin{aligned}&\frac{1}{2}\left|\Phi(t_n)\right|^2\cdot e^{2c_0t_n}-\frac{1}{2}\left|\Phi(t_{n-1})\right|^2\cdot e^{2c_0t_{n-1}}\\&\quad\leq\sup_{t\in I_n}\left|h(t+a_0)-h(t)\right|\\&\qquad\cdot\left[\frac{e^{2c_0t_n}-e^{2c_0t_{n-1}}}{2c_0}\cdot\sup_{t\in I_n}\left|\Phi(t)\right|+\frac{\varepsilon_0}{2}\cdot\left(\frac{e^{2c_0t_n}-e^{2c_0t_{n-1}}}{2c_0}-\frac{\left(e^{2c_0\delta}-1\right)}{2c_0}\cdot e^{2c_0(t_n-\delta)}\right)\right]\\&\qquad+\left(\frac{\varepsilon_0^2\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)}{8}-\frac{\varepsilon_0^2\left(e^{2c_0\delta}-1\right)}{8}\cdot e^{2c_0(t_n-\delta)}\right).\end{aligned}\tag{2.24}$$

Thus,

$$\begin{aligned}&\sup_{t\in I_n}\left|h(t+a_0)-h(t)\right|\\&\geq\frac{\frac{1}{2}\left|\Phi(t_n)\right|^2\cdot e^{2c_0t_n}-\frac{1}{2}\left|\Phi(t_{n-1})\right|^2\cdot e^{2c_0t_{n-1}}-\frac{\varepsilon_0^2\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)}{8}+\frac{\varepsilon_0^2\left(e^{2c_0\delta}-1\right)}{8}\cdot e^{2c_0(t_n-\delta)}}{\sup_{t\in I_n}\left|\Phi(t)\right|\cdot\frac{1}{2c_0}\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)+\frac{\varepsilon_0}{2}\cdot\left[\frac{1}{2c_0}\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)-e^{2c_0(t_n-\delta)}\cdot\left(e^{2c_0\delta}-1\right)\cdot\frac{1}{2c_0}\right]}\\&=\frac{\left(4c_0\left|\Phi(t_n)\right|^2-\varepsilon_0^2c_0\right)\cdot e^{2c_0t_n}-\left(4c_0\left|\Phi(t_{n-1})\right|^2-\varepsilon_0^2c_0\right)\cdot e^{2c_0t_{n-1}}+\varepsilon_0^2c_0\left(e^{2c_0\delta}-1\right)\cdot e^{2c_0(t_n-\delta)}}{\left(4\sup_{t\in I_n}\left|\Phi(t)\right|+2\varepsilon_0\right)\cdot\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)-2\varepsilon_0\left(e^{2c_0\delta}-1\right)\cdot e^{2c_0(t_n-\delta)}}\\&\geq\frac{\left(4c_0\left|\Phi(t_n)\right|^2-\varepsilon_0^2c_0\right)\cdot e^{2c_0t_n}-\left(4c_0\left|\Phi(t_{n-1})\right|^2-\varepsilon_0^2c_0\right)\cdot e^{2c_0t_{n-1}}+\varepsilon_0^2c_0\left(e^{2c_0\delta}-1\right)\cdot e^{2c_0(t_n-\delta)}}{\left(4\sup_{t\in I_n}\left|\Phi(t)\right|+2\varepsilon_0\right)\cdot\left(e^{2c_0t_n}-e^{2c_0t_{n-1}}\right)}\\&=\frac{\left(4c_0\left|\Phi(t_n)\right|^2-\varepsilon_0^2c_0\right)-\left(4c_0\left|\Phi(t_{n-1})\right|^2-\varepsilon_0^2c_0\right)\cdot e^{2c_0(t_{n-1}-t_n)}+\varepsilon_0^2c_0\left(e^{2c_0\delta}-1\right)\cdot e^{-2c_0\delta}}{\left(4\sup_{t\in I_n}\left|\Phi(t)\right|+2\varepsilon_0\right)\cdot\left(1-e^{2c_0(t_{n-1}-t_n)}\right)}.\end{aligned}\tag{2.25}$$

Since $\Phi(t)=u(t+a_0)-u(t)$ and the solution $u(t)$ is bounded, then we can assume $\exists M>0$, such that

$$\left|\Phi(t)\right|<M,\quad\text{for each } t\in\mathbb{R}.\tag{2.26}$$

Also we have

$$|\Phi(t_n)|^2 \geq \frac{\varepsilon_0^2}{4}, \quad \text{for } t_n \in C_n. \tag{2.27}$$

Then

$$\sup_{t\in I_n} |h(t+a_0)-h(t)| \geq \frac{-(4c_0M^2-\varepsilon_0^2c_0)\cdot e^{2c_0(t_{n-1}-t_n)}+\varepsilon_0^2c_0(e^{2c_0\delta}-1)\cdot e^{-2c_0\delta}}{(4M+2\varepsilon_0)\cdot(1-e^{2c_0(t_{n-1}-t_n)})}. \tag{2.28}$$

When $n \to +\infty$, one has

$$t_n - t_{n-1} \longrightarrow +\infty, \qquad e^{2c_0(t_{n-1}-t_n)} \longrightarrow 0. \tag{2.29}$$

So

$$\lim_{t\to+\infty} |h(t+a_0)-h(t)| > \frac{1}{2}\cdot\frac{\varepsilon_0^2c_0(e^{2c_0\delta}-1)\cdot e^{-2c_0\delta}}{4M+2\varepsilon_0} = \frac{\varepsilon_0^2c_0(e^{2c_0\delta}-1)}{2e^{2c_0\delta}(4M+2\varepsilon_0)} > 0. \tag{2.30}$$

This contradicts the fact $h(t) \in SO(\mathbb{R}^N)$. We must have $u(t) \in SO(\mathbb{R}^N)$.

Finally we show that

$$(\mathrm{B}) \Longrightarrow (\mathrm{C}). \tag{2.31}$$

If we denote by $\mathscr{C}$ the set of constant mapping from $\mathbb{R}$ to $\mathbb{R}^N$, one has $\mathscr{C} \subset SO^1(\mathbb{R}^N)$ and for $u \in \mathscr{C}$, the function $\mathscr{F}_3(u) \in SO(\mathbb{R}^N)(\mathscr{F}_3(u) = F(u(0))$, for all $t \in \mathbb{R})$, so we can define the restriction operator of $\mathscr{F}_3$ to $\mathscr{C}$ by $\mathscr{F}_4 : \mathscr{C} \to \mathscr{C}$ with $\mathscr{F}_4(u) = F(u(0))$ for all $u \in \mathscr{C}$ and all $t \in \mathbb{R}$. For $u \in \mathscr{C}$, one has $\|u\|_{C^1} = |u(0)|$ and $\|\mathscr{F}_3(u)\|_\infty = |F(u(0))|$; then it is equivalent to prove $\mathscr{F}_4$ or F is a homeomorphism. It remains to prove that $\mathscr{F}_4$ is surjective. Let $h \in \mathscr{C}$. By hypothesis, there exists $u \in SO^1(\mathbb{R}^N)$ such that $\mathscr{F}_3(u) = h$. we want to prove that $u \in \mathscr{C}$. For that we denote by $u_a(t) = u(t+a)$ for all t and $a \in \mathbb{R}$. Note that $\mathscr{F}_3(u_a) = h$ for all $a \in \mathbb{R}$. By injectivity of $\mathscr{F}_3$, we deduce that $u_a(t) = u(t)$ for all $a \in \mathbb{R}$, therefore $u \in \mathscr{C}$.

Remark 2.2. The following example constructed in [6] can be used to show that Assertion (A) is not a necessary condition for the existence or the uniqueness of a bounded or slowly oscillating solution of (1.1). Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x_1,x_2) = Bx = (-x_2, x_1 + x_2)$. The map F is monotone and does not satisfy (A). However, the eigenvalues of B are conjugate and their real parts are equal to 1/2, therefore the linear system $u' + Bu = 0$ has an exponential dichotomy: namely, there exists $k > 0$ such that $\|\exp(-Bt)\|_{L(\mathbb{R})} \leq k\exp(-t/2)$ for all $t \geq 0$. As a consequence, the system $u' + Bu = 0$ has precisely one bounded solution on $\mathbb{R}$: $u = 0$; this implies the injectivity of $\mathscr{F}_1$. Moreover, the following function $u(t) := \int_{-\infty}^t \exp(-B(t-s))h(s)ds$ is a solution of $u' + Bu = h$ for $h \in BC(\mathbb{R}^N)$ and satisfies $|u(t)| \leq 2k\|h\|_\infty$ for all $t \in \mathbb{R}$; this implies the surjectivity of $\mathscr{F}_1$. Since $\mathscr{F}_1$ is a bounded linear map between Banach spaces, which is bijective, then $\mathscr{F}_1$ is an isomorphism between $BC^1(\mathbb{R}^2)$ and $BC(\mathbb{R}^N)$. To show that (B) holds in this case, it remains to show that $u \in SO(\mathbb{R}^2)$ if $h \in SO(\mathbb{R}^2)$. In fact, for $a \in \mathbb{R}$

$$\begin{aligned} |u(t+a)-u(t)| &= \left| \int_{-\infty}^{t+a} \exp\left(-B(t+a-s)\right)h(s)ds - \int_{-\infty}^{t} \exp\left(-B(t-s)\right)h(s)ds \right| \\ &\le \int_{-\infty}^{t} \exp\left(-B(t-s)\right) |h(s+a)-h(s)| ds. \end{aligned} \tag{2.32}$$

It follows that $|u(t+a)-u(t)| \to 0$ as $t \to -\infty$. Since $h \in SO(\mathbb{R}^2)$, for $\varepsilon > 0$ there exists $t_0 > 0$ such that $|h(t+a)-h(t)| < \varepsilon$ for all $t > t_0$. Now

$$\begin{aligned} |u(t+a)-u(t)| &\le \left(\int_{-\infty}^{t_0} + \int_{t_0}^{t} \right) \exp\left(-B(t-s)\right) |h(s+a)-h(s)| ds \\ &\le 2\|h\|_\infty \int_{-\infty}^{t_0} \exp\left(-B(t-s)\right) ds + \varepsilon \int_{t_0}^{t} \exp\left(-B(t-s)\right) ds \end{aligned} \tag{2.33}$$

and therefore, $u(t+a)-u(t) \to 0$ as $t \to +\infty$. This shows that $u \in SO(\mathbb{R}^2)$.

Remark 2.3. The following example constructed also in [6] can be used to show that (C) is not a sufficient condition for the existence of slowly oscillating solution of (1.1) even when F is a linear monotone map. Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x_1, x_2) := Ax = (-x_2, x_1)$. F is a homeomorphism and a monotone map. Let $v = (\sin t, \cos t)$. Then $v' + Av = 0$. Let f be any continuously differentiable function on $\mathbb{R}$ such that $f(t) = t^{1/3}$ for $|t| > 1$ and let $h(t) = f'(t)v(t)$. Since $h(t) \to 0$ as $|t| \to \infty$, $h \in SO(\mathbb{R}^2)$. The equation $u' + Au = h$ has no bounded solution, because $u(t) = f(t)v(t)$ is an unbounded solution, therefore $\mathcal{F}_2$ is not surjective.

Nevertheless, for (1.2) we have the following result.

Theorem 2.4. *Let Φ be a convex and continuously differentiable function on $\mathbb{R}^N$. Assume that $F = \nabla\Phi$. Then (A), (B), and (C) are equivalent.*

We have already shown that (A)⇒(B)⇒(C) in Theorem 2.1. The equivalence of (A) and (C) is [6, Theorem 1.1]. □

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